NOTE
Cutting a Set of Disks by a Line with Leaving Many Intact Disks in Both Sides
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Cut by a line the union of given disjoint disks in the plane so that both sides of the line contain many intact disks. At least how many intact disks can we leave in either side? It is proved that there is a family of infinitely many disjoint disks in the plane for which every line has a side that contains at most one intact disk. On the other hand, for any family of \( n \) disjoint disks, there is a circle \( C \) such that both the interior and the exterior of \( C \) contain \( n/4 - o(n) \) intact disks.

1. INTRODUCTION

Let \( \mathcal{F} \) be a family of \( n \) disjoint disks, and \( L \) be a line in the plane. The line \( L \) may cut several disks in \( \mathcal{F} \). Let \( a(L) \) be the number of intact disks contained in one side of \( L \) and \( b(L) \) be the number in the other side of \( L \). Then \( a(L) + b(L) \) is probably less than \( n \). We want to maximize \( \min(a(L), b(L)) \) by varying \( L \). How large can \( \min(a(L), b(L)) \) become generally? If all disks are of the same size then we have the next.

**Theorem 1.** For any family \( \mathcal{F} \) of \( n \) disjoint unit disks, there is a line \( L \) such that \( \min(a(L), b(L)) = n/2 - o(n) \).

If the sizes of disks may be different, then the following holds.

**Theorem 2.** There is a family \( \mathcal{F} \) of infinitely many disjoint disks in the plane such that for every line \( L \), \( \min(a(L), b(L)) \leq 1 \).

On the other hand we have the following.
Theorem 3. For any family $F$ of $n$ disjoint disks in the plane, there is a circle $C$ such that its interior contains more than $\frac{1}{4}n - o(n)$ disks and its exterior contains more than $\frac{1}{4}n - o(n)$ disks.

Open Problem. What is the maximum value $c$ that replaces $\frac{1}{4}$ in the above theorem.

Kupitz [1, 2] considered the separation of a finite point-set $S$ in $\mathbb{R}^d$ by a hyperplane $H$ spanned by a subset of $S$. Among others, Kupitz proved [1] that if $S$ spans $\mathbb{R}^3$ and $\#S > 4k$, then there is a plane $H$ spanned by a subset of $S$ such that either open side of $H$ contains at least $k$ points of $S$, and the condition $\#S > 4k$ cannot be relaxed generally.

2. PROOF OF THEOREM 1

For a family $F$ of $n$ disjoint disks, a line $L$ is called a halving line of $F$ if it divides the union of disks in $F$ into two sets with equal area.

Let us prove the following assertion. For any $\varepsilon > 0$, there is an $n_0 = n_0(\varepsilon)$ such that if $F$ is a family of $n (> n_0)$ disjoint unit disks, then there is a line $L$ satisfying

$$\min(a(L), b(L)) \geq \frac{n}{2} - \varepsilon n.$$

Suppose that there is an $\varepsilon_0 > 0$ for which this assertion does not hold. Then for any $n_0$, there is a family $F$ of $n (> n_0)$ disjoint unit disks such that for every line $L$, $\min(a(L), b(L)) < n/2 - \varepsilon_0 n$. Then, every halving line of $F$ cuts at least $\varepsilon_0 n$ disks of $F$. Let $k$ be the smallest positive integer that satisfies $1/k < \varepsilon_0$ and let $\theta = \pi/2k$. Then, for each $i = 1, 2, \ldots, 2k$, there is the having line $L_i$ of $F$ with slope $\tan \theta$ by the intermediate value theorem. Each $L_i$ cuts at least $\varepsilon_0 n$ disks of $F$.

We show that if $i \neq j$ then the number of unit disks in $F$ that are cut by both $L_i, L_j$ is less than some constant depending only $\varepsilon_0$. Suppose $i < j$ and let $p_i, q_i$ be the two points on $L_i$ that are at distance 2 from $L_j$. Then the midpoint of the line-segment $p_iq_i$ is the intersection point of $L_i, L_j$, and since the angle between $L_i, L_j$ is equal to $(j - i) \theta$, it follows that $\|p_i - q_i\| = 2(2/\sin (j - i) \theta)$. If a unit disk is cut by both $L_i, L_j$, then the disk must intersect the line-segment $p_iq_i$. The number of disjoint unit disks that intersect the line-segment $p_iq_i$ is clearly less than

$$\frac{2(2/\sin (j - i) \theta + 2) \times 4}{\pi} \leq K := \frac{2(2/\sin \theta + 2) \times 4}{\pi}.$$
Thus, if \( i \neq j \), then the number of disks that are cut by both \( L_i, L_j \) is less than \( K \). Hence the total number of disks cut by some \( L_i, i = 1, 2, \ldots, 2k \), is at least

\[
2k\varepsilon_0 n - \left( \frac{2k}{2} \right) K \geq 2n - \left( \frac{2k}{2} \right) K.
\]

Here, \( k \) and \( K \) depend only on \( \varepsilon_0 \), independent from \( n \). Thus, if \( n_0 > \left( \frac{2k}{2} \right) K \), then the number of disks cut by some \( L_i \) is greater than \( n \) (the total number of disks in \( \mathcal{F} \)), a contradiction. Hence the assertion is true.

3. PROOF OF THEOREM 2

We use the following lemma, whose proof will be clear.

**Lemma 1.** Let \( \mathcal{F} \) be a finite family of disjoint disks, and let \( A \) be an angular region (with angle \( < \pi \)) in the plane. Then there is a disk \( D \) inscribed in \( A \) that intersects no disk in \( \mathcal{F} \).

Let us identify the plane with the complex numbers \( \mathbb{C} \). Let \( f: \mathbb{C} \to \mathbb{C} \) be the map defined by \( f(z) = -z \). For each positive integer \( j \), let \( A_j \) denote the angular region defined by

\[
A_j = \left\{ z \in \mathbb{C} \mid \pi \left( 1 - \frac{1}{2^j} \right) + \frac{\pi}{32} \leq \arg z \leq \pi \left( 1 - \frac{1}{2^j} \right) + \pi \right\}.
\]

We choose disks \( D_i, i = 1, 2, 3, \ldots \), in the following way:

1. \( D_1 \) is a disk inscribed in \( A_1 \).
2. \( D_2 \) is a disk inscribed in \( A_2 \) and disjoint from \( D_1, f(D_1) \).
3. \( D_3 \) is a disk inscribed in \( A_3 \) and disjoint from \( D_2, f(D_2) \) \((i = 1, 2)\).
4. \( D_4 \) is a disk inscribed in \( A_4 \) and disjoint from \( D_3, f(D_3) \) \((i = 1, 2, 3)\).
5. Repeat the same process infinitely.

By Lemma 1, such disks \( D_i, i = 1, 2, 3, \ldots \), indeed exist. Let

\[
\mathcal{F} = \{ D_i \mid i = 1, 2, 3, \ldots \} \cup \{ f(D_i) \mid i = 1, 2, 3, \ldots \}.
\]

Then the disks in \( \mathcal{F} \) are mutually disjoint. Cut the family \( \mathcal{F} \) by a line \( L \). How large \( \min(a(L), b(L)) \) is? Let \( L_0 \) be the line passing through the origin and parallel to \( L \). Since \( \mathcal{F} \) is symmetric with respect to the origin, it follows easily that \( \min(a(L), b(L)) \leq \min(a(L_0), b(L_0)) \). How many disks
does $L_0$ leave intact? Let $z_0 \neq 0$ be a point on $L_0$ with $0 \leq \arg z_0 < \pi$. Then

$$L_0 \cap A_j = \emptyset \iff \pi \left(1 - \frac{1}{2} \right) < \arg z_0 < \pi \left(1 - \frac{1}{2} \right) + \frac{\pi}{3j + 1}.$$ 

Since

$$\pi \left(1 - \frac{1}{2} \right) + \frac{\pi}{3j + 1} \leq \pi \left(1 - \frac{1}{2} \right) + \frac{\pi}{2j + 1} = \pi \left(1 - \frac{1}{2j + 1} \right),$$

$L_0 \cap A_j = \emptyset$ implies that $L_0$ cuts all other $A_i, i \neq j$. That is, if $L_0$ does not cut $D_j$, then $L_0$ cuts all other $D_i, i \neq j$. Note that $L_0$ cuts $D_i$ if and only if $L_0$ cuts $f(D_i)$. Therefore, $L_0$ misses at most two disks, and hence $\min(a(L_0), b(L_0)) \leq 1$. Hence we have the theorem.

4. PROOF OF THEOREM 3

**Lemma 2.** Let $V$ be a disk of radius $(1 + \varepsilon) r$ containing a fixed $k$-point-set $S$. If $0 < \varepsilon < 0.3642$, then there is a disk of radius $(1 - \varepsilon^2) r$ that contains at least $k(1 - \varepsilon^2)3$ points of $S$.

**Proof.** Let $S = \{x_1, x_2, \ldots, x_k\}$, and consider a random disk $D$ of radius $(1 - \varepsilon^2) r$ whose center is uniformly distributed on the circle $C$ of radius $(1 - \varepsilon^2) r$ concentric with $V$. For each $i = 1, \ldots, k$, define the random variable $X_i$ by

$$X_i = \begin{cases} 1 & \text{if } x_i \in D \\ 0 & \text{if } x_i \notin D. \end{cases}$$

Then, $\Pr(X_i = 1) > (1 - \varepsilon)/3$ for $0 < \varepsilon < 0.3642$. This can be seen as follows. Let $O$ be the center of $V$, and $\lambda r$ be the distance between $O$ and $x_i$. Let $P_1, P_2$ be the two intersection points of the boundary $\partial D$ of $D$ and the circle of radius $\lambda r$ centered at $O$, and let $2\theta = \angle P_1 OP_2$. Then $\Pr(X_i = 1) = \theta/\pi$. Since $\cos \theta = \frac{\lambda r}{(1 - \varepsilon^2)}$, we have

$$\Pr(X_i = 1) = \frac{\theta}{\pi} = \pi^{-1} \left( \frac{\lambda}{2(1 - \varepsilon^2)} \right) 
\geq \frac{1}{\pi} \cos^{-1} \left( \frac{1 + \varepsilon}{2(1 - \varepsilon^2)} \right) = \frac{1}{\pi} \cos^{-1} \left( \frac{1 + \varepsilon}{2(1 - \varepsilon^2)} \right).$$
Since \( \cos(\pi(1-\varepsilon)/3) > 1/(2(1-\varepsilon)) \) for \( 0 < \varepsilon < 0.3642 \), we have
\[
\Pr(X_i = 1) = \frac{\theta}{\pi} \geq \frac{1}{\pi} \cos^{-1}\left(\frac{1}{2(1-\varepsilon)}\right) > \frac{1-\varepsilon}{3}.
\]

Now, the expected number of points of \( S \) covered by \( D \) is equal to
\[
E(X_1 + \cdots + X_k) = E(X_1) + \cdots + E(X_k) > \left(\frac{1-\varepsilon}{3}\right) k.
\]
Hence there is a disk of radius \( (1-\varepsilon^2) r \) that contains at least \( k(1-\varepsilon)/3 \) points of \( S \).

**Proof of Theorem 3.** Let \( \mathcal{F} \) be a family of \( n \) disjoint disks in the plane and fix a number \( \varepsilon, 0 < \varepsilon < 0.3642 \). Let \( U \) be the disk of minimum radius that contains at least \( pn := n(1-\varepsilon)/4 \) centers of the disks in \( \mathcal{F} \). (Such a disk exists.) Let \( r \) be the radius of \( U \) and let \( V \) be the disk of radius \( r(1+\varepsilon) \) concentric with \( U \). Then at least \( pn \) centers of the disks in \( \mathcal{F} \) lie outside of \( V \). To see this, suppose that \( V \) contains more than \( (1-\rho) n \) centers. Then, by Lemma 2, there is a disk of radius \( r(1-\varepsilon^2) \) that contains \( (1-\rho) n(1-\varepsilon)/3 > pn \), which contradicts the minimality of \( U \).

Let \( W \) be the disk with radius \( r(1+\varepsilon/2) \) concentric with \( U \), and \( C = \partial W \) be its boundary. Let us consider how many disks of \( \mathcal{F} \) can intersect \( C \). Note that if a disk \( D \in \mathcal{F} \) whose center lies outside \( V \) intersects \( C \), then \( D \) contains a disk of diameter \( r\varepsilon/2 = r(1+\varepsilon) - r(1+\varepsilon/2) \) lying in \( V - W \). Hence
\[
\text{area}(D \cap (V - W)) > \frac{\pi r^2 \varepsilon^2}{4},
\]
where \( \text{area}(\cdot) \) denotes the area of \( \cdot \). Suppose that \( m \) disks \( D_1, \ldots, D_m \) of \( \mathcal{F} \) with centers outside \( V \) intersect \( C \). Then we have
\[
\text{area}(V - W) > \sum_{i=1}^{m} \text{area}(D_i \cap (V - W)) > \frac{mnr^2 \varepsilon^2}{16}.
\]
Hence
\[
m < \frac{16 \text{area}(V - W)}{\pi r^2 \varepsilon^2} = 8 \left(\frac{2}{\varepsilon} + \frac{3}{2}\right).
\]
Thus, at least \( pn - 8(2/\varepsilon + 3/2) \) disks lie outside of \( C \). Hence, if \( n > 64/(3\varepsilon^2) + 16/\varepsilon \), at least \( n^4 - pn \) disks lie outside \( C \). Similarly it follows that the number of disks in \( \mathcal{F} \) with centers in \( U \) that intersect \( C \) is less than \( 8(2/\varepsilon + 1/2) \). Hence, if \( n > 64/(3\varepsilon^2) + 16/(3\varepsilon) \), at least \( n^4 - pn \) disks lie inside \( C \).
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REFERENCES