Note

Finite form of the quintuple product identity

William Y.C. Chen, Wenchang Chu, Nancy S.S. Gu

Center for Combinatorics, LPMC, Nankai University, Tianjin 300071, PR China

Received 25 February 2005
Communicated by George Andrews
Available online 26 May 2005

Abstract

The celebrated quintuple product identity follows surprisingly from an almost-trivial algebraic identity, which is the limiting case of the terminating $q$-Dixon formula. © 2005 Elsevier Inc. All rights reserved.

MSC: primary 11F27; secondary 33D05

Keywords: Basic hypergeometric series; The terminating $q$-Dixon formula; The quintuple product identity

The celebrated quintuple product identity discovered by Watson [3] (cf. [2, p. 147] also) states that

$$
\sum_{k=-\infty}^{+\infty} (1 - xq^k)q^{3k^2}(q^3x^3)^k = [q, x, q/x; q]_{\infty}[qx^2, q/x^2; q^2]_{\infty} \quad \text{for} \quad |q| < 1,
$$

where the $q$-shifted factorial is defined by

$$(x; q)_0 = 1 \quad \text{and} \quad (x; q)_n = (1-x)(1-qx)\cdots(1-q^{n-1}x) \quad \text{for} \quad n = 1, 2, \cdots$$

with the following abbreviated multiple parameter notation

$$[x, \beta, \cdots, \gamma; q]_{\infty} = (x; q)_{\infty}(\beta; q)_{\infty}\cdots(\gamma; q)_{\infty}.$$ 

This identity has several important applications in combinatorial analysis, number theory and special functions. For the historical note, we refer the reader to the paper [1]. In this

E-mail addresses: chen@nankai.edu.cn (W.Y.C. Chen), chu.wenchang@unile.it (W. Chu), gu@nankai.edu.cn (N.S.S. Gu).

0097-3165/$ - see front matter © 2005 Elsevier Inc. All rights reserved.
short note, we shall show that identity (1) follows surprisingly from the following algebraic identity.

**Theorem.** (Finite form of the quintuple product identity). For a natural number $m$ and a variable $x$, there holds an algebraic identity:

$$1 \equiv \sum_{k=0}^{m} \frac{(x; q)_{m+1}}{(q^k x^2; q)_{m+1}} x^k q^{k^2}. \quad (2)$$

In fact, performing parameter replacements $m \to m+n$, $x \to -q^{-m}x$ and $k \to k+m$ and then simplifying the result through factorial-fraction relation

$$\frac{(-q^{-m}x; q)_{m+n+1}}{(q^{k-m}x^2; q)_{m+n+1}} = \frac{(-q^{-m}x; q)_{m}(-x; q)_{1+n}}{(q^{k-m}x^2; q)_{m-k}(x^2; q)_{1+n+k}}$$

we may restate the algebraic identity displayed in the theorem as the finite bilateral series identity

$$1 \equiv \sum_{k=-m}^{n} (1 - xq^k) \left[ \frac{(m+n)}{(m+k)} \right] \frac{(-x; q)_{1+n}(-q/x; q)_{m}}{(x^2; q)_{1+n+k}(q/x^2; q)_{m-k}} x^{3k} q^{k^2+(\frac{k}{2})}. \quad (3)$$

Letting $m, n \to \infty$ in this equation and applying the relation

$$(q; q)_\infty \left[ (x^2; q)_{\infty} (q/x^2; q)_{\infty} \right] = [q, x, q/x; q]_{\infty} [q x^2, q/x^2, q^2]_{\infty}$$

we derive immediately the quintuple product identity displayed in (1).

In terms of basic hypergeometric series, we remark that the finite sum identity (2) is just the limiting case $M \to \infty$ of the terminating $q$-Dixon formula (cf. [2, II-14]):

$$4 \phi_3 \left[ \begin{array}{c} x^2, -qx, q^{-m} \\ -x, q^{1+m} x^2, q x^2/M \end{array} \right] q^{1+m} x/M q; q \frac{q^{1+m} x}{M} = \frac{(q x^2; q)_m(q x/M; q)_m}{(q x; q)_m(q x^2/M; q)_m}.$$

**Acknowledgments**

The authors are grateful to the referees for valuable suggestions. This work was done under the auspices of the “973” Project on Mathematical Mechanization, the National Science Foundation, the Ministry of Education, and the Ministry of Science and Technology of China.
References