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Note

Finite form of the quintuple product identity

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Abstract

The celebrated quintuple product identity follows surprisingly from an almost-trivial algebraic identity, which is the limiting case of the terminating q-Dixon formula.

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The celebrated quintuple product identity discovered by Watson [3] (cf. [2, p. 147] also) states that

$$\sum_{k=-\infty}^{+\infty} (1 - xq^k) q^{3\binom{k}{2}} (qx^3)^k = [q, x, q/x; q]_\infty [qx^2, q/x^2; q^2]_\infty \quad \text{for} \quad |q| < 1, \tag{1}$$

where the q-shifted factorial is defined by

$$(x;q)_0 = 1 \quad \text{and} \quad (x;q)_n = (1-x)(1-qx)\cdots(1-q^{n-1}x) \quad \text{for} \quad n = 1, 2, \cdots$$

with the following abbreviated multiple parameter notation

$$[\alpha, \beta, \cdots, \gamma; q]_\infty = (\alpha; q)_\infty (\beta; q)_\infty \cdots (\gamma; q)_\infty.$$

This identity has several important applications in combinatorial analysis, number theory and special functions. For the historical note, we refer the reader to the paper [1]. In this

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short note, we shall show that identity (1) follows surprisingly from the following algebraic identity.

Theorem. (Finite form of the quintuple product identity). *For a natural number m and a variable x, there holds an algebraic identity*:

$$1 \equiv \sum_{k=0}^{m} (1+xq^k) \begin{bmatrix} m \\ k \end{bmatrix} \frac{(x;q)_{m+1}}{(q^k x^2;q)_{m+1}} x^k q^{k^2}. \tag{2}$$

In fact, performing parameter replacements $m \to m+n$, $x \to -q^{-m}x$ and $k \to k+m$ and then simplifying the result through factorial-fraction relation

$$\begin{aligned}\frac{(-q^{-m}x;q)_{m+n+1}}{(q^{k-m}x^2;q)_{m+n+1}} &= \frac{(-q^{-m}x;q)_m(-x;q)_{1+n}}{(q^{k-m}x^2;q)_{m-k}(x^2;q)_{1+n+k}} \\ &= (-1)^{m-k} q^{\binom{k}{2}-mk} x^{2k-m} \\ &\quad \times \frac{(-q/x;q)_m(-x;q)_{1+n}}{(q/x^2;q)_{m-k}(x^2;q)_{1+n+k}}\end{aligned}$$

we may restate the algebraic identity displayed in the theorem as the finite bilateral series identity

$$1 \equiv \sum_{k=-m}^{n} (1-xq^k) \begin{bmatrix} m+n \\ m+k \end{bmatrix} \frac{(-x;q)_{1+n}(-q/x;q)_m}{(x^2;q)_{1+n+k}(q/x^2;q)_{m-k}} x^{3k} q^{k^2+\binom{k}{2}}. \tag{3}$$

Letting $m, n \to \infty$ in this equation and applying the relation

$$(q;q)_\infty \frac{(x^2;q)_\infty (q/x^2;q)_\infty}{(-x;q)_\infty(-q/x;q)_\infty} = [q, x, q/x; q]_\infty [qx^2, q/x^2; q^2]_\infty$$

we derive immediately the quintuple product identity displayed in (1).

In terms of basic hypergeometric series, we remark that the finite sum identity (2) is just the limiting case $M \to \infty$ of the terminating q-Dixon formula (cf. [2, II-14]):

$${}_4\phi_3 \left[\begin{matrix} x^2, & -qx, & q^{-m}, & M \\ & -x, & q^{1+m}x^2, & qx^2/M \end{matrix} \middle| q; \frac{q^{1+m}x}{M} \right] = \frac{(qx^2;q)_m (qx/M;q)_m}{(qx;q)_m (qx^2/M;q)_m}.$$

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