A LEFSCHETZ THEOREM FOR FOLIATED MANIFOLDS

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1. INTRODUCTION

In this paper we prove a Lefschetz Theorem for endomorphisms of Dirac complexes defined along the leaves of a compact oriented foliated manifold. This theorem is a generalization of the classical Lefschetz Theorem for elliptic complexes on a compact manifold [2] I and III, and has as a special case the Atiyah–Singer Index Theorem for the classical elliptic complexes. If the endomorphism is the identity, we obtain an index theorem for Dirac complexes defined along the leaves of a foliation. By standard arguments, this index theorem may be extended to an index theorem for general elliptic complexes defined along the leaves of a foliation of a compact oriented manifold. Note that Connes [7], [8] has proven an index theorem for such complexes lifted to the holonomy groupoid of the foliation, while we work on the foliated manifold itself.

We now briefly describe our main result. Denote by \( F \) a foliation of a compact oriented manifold \( M \) and by \( v \) an invariant transverse measure. Let \( 0 \rightarrow E_0 \xrightarrow{d_0} E_1 \xrightarrow{d_1} \ldots \xrightarrow{d_{n-1}} E_k \rightarrow 0 \) be a Dirac complex on \( M \) along \( F \), and denote it by \( (E, \partial) \). Let \( f: M \rightarrow M \) be a smooth diffeomorphism such that for each leaf \( L \) of \( F \), \( f(L) = L \). For simplicity, assume that on each leaf the fixed points of \( f \) are isolated and non-degenerate. At each fixed point \( p \) there is a well defined complex number \( a(p) \), see [2–1], the local index for \( f \) at \( p \). We assume that the fixed point set of \( f \) is a transverse submanifold, denoted by \( N \). We may integrate \( a(p) \) over \( N \) against the measure which \( v \) induces on \( N \) to obtain our local Lefschetz number.

To obtain a global Lefschetz number we use globally defined metrics on \( M \) to construct Laplacians \( \Delta_f \) on the \( L^2 \) sections of the bundle \( E_1 \) restricted to the leaf \( L \). Assume that \( f \) is a geometric endomorphism which covers the map \( f \) (see [2–1]). Denote by \( T_f \) the compression of \( (f|L)^* \) to \( \ker(\Delta_f) \). The Schwartz kernel \( k_{f,1}(x, y) \) of \( T_f \) is smooth and \( tr(k_{f,1}(x, y)) \) defines a smooth measure on \( L \). This measure varies measurable in \( L \) and is globally bounded on \( M \). The transverse measure \( v \) combines with this measure to define a measure on \( M \) which we integrate over \( M \) to obtain the complex number \( tr_v(T_f) \). Now set

\[ L_v(f) = \sum_{i=0}^k (-1)^i tr_v(T_f^i). \]

Our Lefschetz Theorem is then:

**THEOREM:**

\[ L_v(f) = \int_N a(p) dv. \]

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The theorem we prove is, in fact, more general. In particular (1) the fixed point set must be a closed transverse submanifold, but there is no restriction on its codimension; (2) $df$ restricted to the normal bundle of the fixed point set must be non-degenerate (a condition satisfied by the identity map).

Our proof uses the heat equation method (see [3], and [14], [15]) combined with recent results of Roe [20]. In particular, we consider for $t > 0$,

$$
\sum \frac{(-1)^i}{I^i} tr_e(f^i e^{-t\Delta_i}).
$$

We show that it is independent of $t$ and then show that as $t \to \infty$ it converges to $L_v(f)$ and that as $t \to 0$ it is asymptotic to $\int_N a(p)dv$. We end the paper with applications of the main theorem.

2. THE LEFSCHETZ NUMBER

2.1 Dirac Complexes along a Foliation.

Let $F$ be a codimension $q$ smooth foliation of a smooth $m = p + q$ dimensional compact oriented manifold $M$. We will assume that the tangent bundle along the leaves of $F$ is oriented. We also assume that $M$ has a Riemannian metric and that each bundle over $M$ comes equipped with a metric. If a bundle is a complex bundle, we assume that the metric is Hermitian. A metric on a bundle over $M$ induces a metric on the restriction of the bundle to a leaf of $F$. These metrics are unique in the sense that any two metrics on a bundle over $M$ are quasi isometric, so the metrics they induce on the restriction of the bundle to a leaf are quasi isometric. In particular, the quasi isometry class of any leaf is uniquely determined.

By a Dirac complex $(E, d)$ along $F$ we mean the following.

(1) $E=(E_0, E_1, \ldots, E_k)$ is a family of smooth finite dimensional complex vector bundles over $M$. We denote the space of smooth sections of $E_i$ by $C^\infty(E_i)$.

(2) $d=(d_0, \ldots, d_{k-1})$ is a family of differential operators, where

$$
d_i: C^\infty(E_i) \to C^\infty(E_{i+1}).
$$

We require that $d_{i+1} d_i = 0$ and that the $d_i$ differentiate only in leaf directions. To be more specific on this last point let $(U, x_1, \ldots, x_n)$ be a coordinate chart for $M$ where $F|_U$ is spanned by the vector fields $\partial/\partial x_1, \ldots, \partial/\partial x_p$. We call such a chart a foliation chart. Then the restriction of $d_i$ to $U$ is given by a matrix of partial differential operators of the form

$$
\sum a_{\alpha}(x)D^\alpha.
$$

For $d_i$ to differentiate only in leaf directions means that $\alpha=(\alpha_1, \ldots, \alpha_p)$ where the $\alpha_i$ are non negative integers and

$$
D^\alpha = \frac{\partial^{\mid \alpha \mid}}{\partial x_1^{\alpha_1} \ldots \partial x_p^{\alpha_p}}
$$

where $|\alpha| = \alpha_1 + \ldots + \alpha_p$.

Of course, the $a_{\alpha}$ are smooth complex valued matrix functions on $U$.

(3) We require the complex $(E, d)$ to be a Dirac complex along the leaves of $F$. Briefly, this means the following. Use the given metrics to construct the adjoints $d_{i-1}^*: C^\infty(E_{i-1}) \to C^\infty(E_i)$, and denote by $D$ the operator

$$
D = \bigoplus (d_i + d_i^*): C^\infty(E) \to C^\infty(E)
$$
where $E = \bigoplus_i E_i$. $E$ is required to be a Clifford bundle over the Clifford algebra of $T^*F$, the cotangent bundle along the leaves. There is a Hermitian connection $\nabla$ on $E$ which is compatible with Clifford multiplication and the Dirac operator $D$ is given by the composition

$$C^\infty(E) \xrightarrow{\nabla} C^\infty(T^*M \otimes E) \rightarrow C^\infty(T^*F \otimes E) \rightarrow C^\infty(E)$$

where $m$ is Clifford multiplication.

Note that a Dirac complex along $F$ when restricted to a leaf is a Dirac complex on the leaf, see [16], and hence an elliptic complex. The simplest and most natural example of such a complex is the DeRham complex of the foliation $F$. Here $E^i = \wedge^i T^*F \otimes C$ is the $i$th exterior power of the complexified cotangent bundle of $F$ and $d_i$ is the usual exterior derivative along the leaves of $F$. In fact all the classical complexes give rise to Dirac complexes along a foliation provided, of course, that the leaves support the necessary geometric structures for the complexes.

Naturally associated to a Dirac complex $(E, d)$ along $F$ are families of Hilbert spaces $H^i(E, d)$, $i = 0, \ldots, k$. Denote by $L$ a leaf of $F$ and by $E^i_L$ the restriction of $E^i$ to $L$. Then $d_i$ induces

$$d_i^L : C^\infty_0(E^i_L) \rightarrow C^\infty_0(E^i_{L+1})$$

where $C^\infty_0(E^i_L)$ denotes the smooth sections with compact support. Extend $d_i^L$ to a densely defined unbounded operator on $L^2(E^i_L)$. A section $s \in L^2(E^i_L)$ is in the domain of $d_i^L$ provided that there is a sequence $s_j \in C^\infty_0(E^i_L)$ converging to $s$ (in the $L^2$ sense) so that $d_i^L s_j$ also converges in $L^2$. We then set $d_i^L s = \lim (d_i^L s_j)$. This is well defined. Then define

$$H^i_L(E, d) = \frac{\ker d_i^L}{(\im d_{i-1} \cap \dom d_i^L)}$$

and

$$H^i(E, d) = \{H^i_L(E, d)\}.$$

It is necessary to mod out by the closure of image $d_{i-1}^L$ in the definition of $H^i_L(E, d)$ as this subspace is in general not a closed subspace of the domain of $d_i^L$.

The adjoint $d^*_i$ of $d_{i-1}$ induces

$$d^*_i : L^2(E^i_L) \rightarrow L^2(E^i_{L+1})$$

where the domain is defined just as it was for $d_i^L$. Next we define

$$\Delta_i^L : C^\infty_0(E^i) \rightarrow C^\infty_0(E^i)$$

by

$$\Delta_i^L = d^*_i d_i^L + d_i d^*_i.$$

Since the operator $D^L = \bigoplus (d_i^L \oplus d^*_i)$ is a Dirac operator and each leaf $L$ is a complete Riemannian manifold, $D^L$ is essentially self adjoint [9], hence $(D^L)^2 = \bigoplus_i \Delta_i^L$ is also essentially self adjoint. Thus each $\Delta_i^L$ has a unique extension to a densely defined unbounded self adjoint operator (also denoted $\Delta_i^L$) on $L^2(E^i_L)$. In addition if $g(x)$ is any bounded Borel function on $[0, \infty)$, the bounded operator $g(\Delta_i^L)$ is uniquely defined by the spectral theorem [19]. Again the domain of $\Delta_i^L$ consists of all $s$ in $L^2(E^i_L)$ such that there is a sequence $s_j$ in $C^\infty_0(E^i_L)$ converging to $s$ with $\Delta_i^L s_j$ also converging. Then $\Delta_i^L s = \lim \Delta_i^L s_j$.

**Theorem 2.1.1 (Hodge theorem for Dirac operators on complete manifolds)**

$$L^2(E^i_L) = \ker \Delta_i^L \oplus \im \Delta_{i-1}^L \oplus \im \Delta_i^L.$$
This theorem for non-compact complete Riemannian manifolds appears to be a folk theorem. We give a proof in the Appendix.

**Corollary 2.1.2.** The natural map $\ker \Delta^*_i \to H^*_i(E, d)$ is an isomorphism.

**Proof.** As a consequence of the Hodge theorem, we know that this map is an injection and that to show surjectivity we need only show that $\ker d^*_i$ is perpendicular to the closure of image $d^*_i$. This follows if we show $\ker d^*_i$ is perpendicular to image $d^*_i$. Let $s \in \ker d^*_i$ and choose a sequence $s_j \in C^*_i(E^i)$ so that $s_j \to s$ and $d^*_i s_j \to 0$. Let $r \in \text{image } d^*_i$ and choose $r_k \in C^*_i(E^i_{i+1})$ so that $\{r_k\}$ converges and $d^*_i r_k \to r$. Denote the $L^2$ inner product by $\langle \cdot , \cdot \rangle$. Then

$$\langle s, r \rangle = \lim_{k \to \infty} \langle s, d^*_i r_k \rangle$$

Since both $s_j$ and $r_k$ are smooth and compactly supported we have

$$\langle s_j, d^*_i r_k \rangle = \langle d^*_i s_j, r_k \rangle \to 0 \text{ as } j \to \infty.$$ 

**Note.** Thus $\langle s, d^*_i r_k \rangle \to 0$ and $\langle s, r \rangle = 0$.

### 2.2 Geometric Endomorphisms.

An endomorphism $T = (T_0, \ldots, T_k)$ of a Dirac complex $(E, d)$ along $F$ is a collection of complex linear maps $T_i: C^\infty(E_i) \to C^\infty(E_i)$ so that $d_i T_i = T_{i+1} d_i$. We are interested in the so-called geometric endomorphisms ([2-1]) given as follows. Let $f: M \to M$ be a smooth diffeomorphism such that for each leaf $L$ of $F$, $f(L) = L$. For each $i = 0, \ldots, k$, let

$$A_i: f^* E_i \to E_i$$

be a smooth bundle map. Define $T_i$ by $(T_i s)(x) = A_i x(s(f(x)))$, for $s \in C^\infty(E_i)$. Here $E_{i, x}$ is the fiber over the point $x$,

$$A_i x: (f^* E_i)_x = E_{i, f(x)} \to E_{i, x}$$

and thus $(T_i s)(x) \in E_{i, x}$. We assume that the $A_i$ are chosen so that $T = (T_0, \ldots, T_k)$ defines an endomorphism of $(E, d)$. We call such an endomorphism the geometric endomorphism of $(E, d)$ determined by $f$ and $A = (A_0, \ldots, A_k)$.

**Example.** Let $(E, d)$ be the DeRham complex of $F$, and $f$ a diffeomorphism of $M$ which takes each leaf to itself. For $A_i$ we take the $i$th exterior power of the dual $df^*$ of the differential $df$ of $f$ restricted to the complexified tangent bundle of $F$. Thus for each $x \in M$.

$$A_{i, x} = \wedge^i df^*: \wedge^i T^* F \otimes C_{f(x)} \to \wedge^i T^* F \otimes C$$

and $T_i$ is just the composition of the map $\hat{f}$ defined by $\hat{f}(s)(x) = s(f(x))$ with $A_{i, x}$. Thus for any leaf $L$, $T_i^L = T_i|_L$ is just the map $(f|_L)^*$ induced by $f$ on the differential forms on $L$. We will denote this endomorphism (and the ones it induces on other classical complexes) by $f^*$.

**Proposition 2.2.1.** A geometric endomorphism $T = (T_0, \ldots, T_k)$ of the complex $(E, d)$ defines maps

$$T_i^*: H^*_i(E, d) \to H^*_i(E, d)$$

for all $i$ and $L$. 

Proof. We first show that $T_i$ induces a bounded operator

$$T_i^*: L^2(E^i_0) \to L^2(E^i_1).$$

Since $f$ is a diffeomorphism of a compact manifold there are positive constants $d_1 < d_2$ so that for each leaf $L$ and each $x$ in $L$, we have $d_1 < \det (df_x)|_L| < d_2$. Denote the inner product on $L^2(E^i_1)$ by $\langle \cdot, \cdot \rangle_L$. Then it follows by the change of variables $y = f(x)$ on $L$ that there is a constant $c$ such that for all leaves, $\langle \tilde{f}(s), \tilde{f}(s) \rangle_L \leq c \langle s, s \rangle_L$ where $s \in L^2(E^i_1)$. Similarly, since $A_i; f^* E_i \to E_i$ is smooth over the compact manifold $M$, there is a positive number $e$ such that for all $x$ in $M$ and $v$ in $(f^* E_i)_x$, $\langle A_i(v), v \rangle_x \leq e \langle v, v \rangle_x$, where $(\cdot, \cdot)$ is the inner product on the fiber of the appropriate bundle. This implies immediately that there is $b > 0$ so that for each leaf $L$, the map $T_i^*$ on $C_0^\infty (E^i_1)$ induced by $I_i$ satisfies $\| T_i^* s \| \leq b \| s \|$, where $\| \cdot \|$ is the $L^2$ norm on $E^i_1$. Thus $T_i$ induces a bounded operator on $L^2(E^i_1)$.

Now suppose $s \in \ker (df)$. Let $s_n \in C_0^\infty (E^i_1)$ be a sequence of sections converging to $s$ with $d_i^* s_n \to 0$. Then $T_i^* s_n \to T_i^* s$. If $u \in C_0^\infty (E^i_1)$, there is a section $u_1 \in C_0^\infty (E^i_0)$ so that $u_1|_L = u$. (Here it is essential that the support be compact). As $d_i^* T_i^* u_1 = T_{i+1}^* d_i^* u_1$, we have that $d_i^* T_i^* u = T_{i+1}^* d_i^* u$. Thus

$$d_i^* T_i^* s_n = T_{i+1}^* d_i^* s_n \to 0 \text{ as } n \to \infty.$$ 

So $T_i^* s \in \ker (df)$.

A similar argument shows that $T_i^*$ maps the closed subspace generated by $\text{image } d_i^* \cap \text{domain } d_i^*$ to itself. This completes the proof.

We note that the requirement that $f$ is a diffeomorphism is essential. Examples show that for relatively simple $f$ which are not diffeomorphisms, the $T_i$ are not bounded operators.

We denote the projection of $L^2(E^i_1)$ onto $\ker (\Delta^i)$ by $P_i^-$, and we denote by $T_i^* -$ the map $P_i^{-} T_i^* P_i^-$. 

**Proposition 2.2.2.** The diagram

$$\begin{array}{ccc}
\ker \Delta^i & \stackrel{T_i^-}{\to} & \ker \Delta^i \\
\downarrow & & \downarrow \\
H^i_1(E, d) & \stackrel{T_i^-}{\to} & H^i_1(E, d)
\end{array}$$

is commutative.

**Proof.** Given $u$ in $\ker \Delta^i$ we denote its class in $H^i_1(E, d)$ by $[u]$. To show that the diagram commutes we must show that for all $u$ in $\ker \Delta^i$,

$$[T_i^* u] = [P_i^* T_i^* u].$$

By the Hodge theorem

$$T_i^* u = P_i^* T_i^* u + a + b$$

where $a \in \text{image } d_{i-1}^*$ and $b \in \text{image } d_i^-$. It is not difficult to see that image $d_i^-*$ is perpendicular to $\ker (df_i^*)$. Since $T_i^* u$ is in $\ker (d_i^*)$, $b$ must be 0. The lemma follows.

**2.3 $tr_\sigma$ and the Lefschetz Number.**

Let $M$ and $F$ be as above. A transversal to $F$ is a Borel subset of $M$ which intersects each leaf in a countable set. A smooth transversal is a proper embedded submanifold of $M$ which is a transversal. The set of transversals forms a $\sigma$-ring. A transverse measure is a measure on this $\sigma$-ring.

There is a particularly nice class of smooth transversals which we can use to illustrate these ideas. Let $\{(U_i, \phi_i)\} i = 1, \ldots, N$ be a finite open cover of $M$ (assumed compact) by
coordinate charts. Here $\phi_i: U_i \to \tilde{U}_i \subset R^m$ is a diffeomorphism. We say that this is a good cover for $F$ provided that

1. $\tilde{U}_i = D^p(1) \times D^q(1) \subset R^p \times R^q = R^m$, where $D^q(r) = \{ x \in R^q : |x| < r \}$.
2. For each $x \in D^q(1)$, $P_x = \phi_i^{-1}(D^p(1) \times \{ x \})$ is contained in a leaf of $F$. $P_x$ is called a plaque of $F$.
3. If $\tilde{U}_i \cap \tilde{U}_j \neq \emptyset$, then $U_i \cap U_j \neq \emptyset$ and $U_i \cap U_j$ is connected.
4. Each $\phi_i$ extends to a diffeomorphism of an open set $V_i$ to $D^q(2) \times D^q(2)$, so that the cover $\{(V_i, \phi_i)\}$ satisfies 1 (with $D^q(1)$ replaced by $D^q(2)$ and 3 above).
5. Each plaque of $V_i$ intersects at most one plaque of $V_j$ and a plaque of $U_i$ intersects a plaque of $U_j$ if and only if the corresponding plaques of $V_i$ and $V_j$ intersect.

Good covers always exist.

Let $\{(U_i, \phi_i)\}$ be a good cover for $F$. For each $i$ set

$$R_i = \phi_i^{-1}(\{0\} \times D^q(1)).$$

Then $R_i$ is a transversal and $R = U_i R_i$ is a complete transversal (it intersects each leaf). Given $x \in R_i$, denote the plaque of $U_i$ through $x$ by $P'_x$. If $U_i \cap U_j \neq \emptyset$, we define a local diffeomorphism $f_{ji}: R_i \to R_j$ by $f_{ji}(x) = y$ provided that $P'_x \cap P'_y$ is not empty. The holonomy pseudo-group acting on $R$ is the pseudo-group generated by the $f_{ji}$. This idea extends to give a pseudo-group of local diffeomorphisms on any smooth transversal, called the holonomy pseudo-group. A transverse measure induces a measure $\nu_i$ on $R_i$, and the measure is an invariant transverse measure if $f_{ji}(\nu_i) = \nu_j$ for every such transversal $R$. It follows that the induced measure on any smooth transversal is invariant under the action of the holonomy pseudo-group.

We fix once and for all a Radon invariant transverse measure $\nu$ for $F$ on $M$ (in the sense of [17] chap. iv). The metric on $M$ restricts to a metric on each leaf which gives a volume form $\lambda^L$ on each leaf. These fit together to yield a smooth tangential measure $\lambda = (\lambda^L)$. The measures $\lambda$ and $\nu$ combine to give a measure on $M$ as follows. Let $R$ be a transversal coming from a good coordinate system. Construct a Borel map $\pi: M \to R$ as follows. Define $\pi(y) = x$ for $y \in P'_x$ if $P'_x$ is a plaque of $U_i$ and $x \in R_i$. Define $\pi(y) = z$ for $y \in P'_x - U_i$ if $P'_x$ is a plaque of $U_j$ and $z \in R_j$. Continue in this manner. Let $A$ be a Borel set with characteristic function $\chi$. Define

$$\mu(A) = \int_R \left( \int_{\pi^{-1}(t)} \chi(x) d\lambda \right) dv(t).$$

It is an easy consequence of the invariance of $\nu$ that $\mu(A)$ does not depend on the transversal $R$ or the map $\pi: M \to R$.

Fix once and for all a good cover of $M$, and let $U$ and $V$ be charts with transversals $R_U$ and $R_V$ respectively. Let $\gamma: [0, 1] \to M$ be a path whose image is contained in a leaf with $\gamma(0) \in R_U$ and $\gamma(1) \in R_V$. Then $\gamma$ induces a local diffeomorphism $\gamma: R_U \to R_V$ with domain $D_\gamma$. Denote by $U \times_\gamma V$ the set

$$\bigcup_{x \in D_\gamma} P'_x \times P'_y,$$

where $P'_x$ is the plaque of $x$ in $U$ and similarly for $P'_y$. $U \times_\gamma V$ is called a graph chart for $F$, and it has a natural structure as a $2p+q$ dimensional manifold. For simplicity, assume $D_\gamma = R_0$. Let $U \approx D^p \times D^q$ with coordinates $(u, t)$ and $V \approx D^p \times D^q$ with coordinates $(u, s)$. Then $\gamma$ induces $\gamma: D^q \to D^q$ and $U \times_\gamma V \approx D^p \times D^p \times D^q$. If $y \in U$ has coordinates $(u, t)$ and $z \in V$ has coordinates $(v, \gamma(t))$, then $(y, z) \in U \times_\gamma V$ has coordinates $(u, v, t)$. A section of a bundle over
Let $E$ be a bundle over $M$ with dual bundle $E^*$. Denote by $\Gamma(F; E)$ the space of all sections $k(x, y)$ of the bundle $E \otimes E^*$ over $M \times M$ such that $k(x, y) = 0$ if $x$ and $y$ are not on the same leaf. We say $k$ is tangentially smooth if for any leaf $L$, $k^L = k^L|_{L \times L}$ is smooth. We say $k$ is measurable if $k$ restricted to any graph chart is measurable. Each graph chart is a subset of $M \times M$ and the bundle $E \otimes E^*$ over $M \times M$ restricts to a smooth bundle over $U \times V$. We denote the smooth sections with compact support of this bundle by $\mathcal{C}_c(U, V, \gamma)$, and we consider it as a subspace of $\Gamma(F; E)$ by setting $k(x, y) = 0$ if $(x, y) \notin U \times V$. Recall that for any leaf $L$, $E^L = E|_L$. If $k \in \mathcal{C}_c(U, V, \gamma)$ define

$$k^L : L^2(E^L) \to L^2(E^L)$$

by

$$k^L(s)(x) = \int_L k(x, y)s(y) \, d\lambda^L(y)$$

for $s \in L^2(E^L)$.

**Theorem 2.3.1.** If $k \in \mathcal{C}_c(U, V, \gamma)$, then $k^L$ defines a bounded smoothing operator on $L^2(E^L)$, whose norm is bounded independently of $L$.

**Proof.** Let $\{P_j\} \geq 1$ be the collection of plaques of $V$ which intersect $L$. Let $P_0 = L - U_jP_j$. For $s \in L^2(E^L)$ let $s_j(x) = \chi_j(x)s(x)$ where $\chi_j$ is the characteristic function of $P_j$. Then $s(x) = \sum s_j(x)$ and $k^Ls_j(x)$ can be non zero only for $x$ on the plaque of $U$ (if such exists) corresponding to $P_j$ under $\gamma$. Hence the $\{k^Ls_j\}$ have disjoint support and thus are orthogonal. It follows that the $L^2$ norm of $k^L(s)$ satisfies

$$\|k^L(s)\|_2 \leq \sum \|k^L(s_j)\|_2^2.$$

Now using coordinates for $U \times V$ which define the differentiable structure and the Cauchy–Schwartz inequality on each $P_j$ it follows that

$$\|k^L(s_j)\|_2 \leq C\|s_j\|_2^2$$

where $C$ depends only on the pointwise bound $\sup_{(x, y) \in M \times M} \|k(x, y)\|$ over $(x, y) \in M \times M$, where here $\|\|$ denotes the smooth fiberwise norm on $E \otimes E^*$, and $\sup \text{vol}(P^\gamma_j)$ over all plaques in $V$. Thus, boundedness independent of $L$ follows.

$k^L(x, y)$ is smooth on $L \times L$ with support contained in a bounded neighborhood of the diagonal. In addition, it and each of its derivatives is uniformly bounded on $L \times L$ since $k \in \mathcal{C}_c(U, V, \gamma)$. Thus it follows that $k^L$ is a bounded smoothing operator.

**Definition.** Let $k(x, y) \in \Gamma(F; E)$. We write $k(x, y) \in \mathcal{C}_c(F; E)$ provided there is a finite collection of graph charts $(U_i, V_i, \gamma_i)$ and elements $k_i(x, y) \in \mathcal{C}_c(U_i, V_i, \gamma_i)$ with

$$k(x, y) = \sum k_i(x, y).$$

Note that in the context of [7] p. 564 that $\mathcal{C}_c(F, E) = C_c(G)$ where $G$ is the holonomy graph of the foliation. However, our $k(x, y)$ act on different Hilbert spaces.

Each $k \in \mathcal{C}_c(F; E)$ defines a family $\{k^L\}$ of bounded smoothing operators on $L^2(E^L)$. The bound on the norms of the $k^L$ is independent of $L$. 
THEOREM 2.3.2. If \( k(x, y) \in C^0_{\text{g}}(F; E) \), then it is bounded, measurable and tangentially smooth. In addition, if \( f \) is a leafwise diffeomorphism as in section 2.1 then \( k(f(x), y) \) is bounded, measurable and tangentially smooth.

Proof. We may assume that \( k \in C^0_{\text{g}}(U, V; \gamma) \) for some graph chart. As \( k \) and \( f \) are bounded and tangentially smooth, we need only prove measurability. Let \( U \times \gamma V \) be any graph chart. We may assume, without any loss of generality that \( U = U_1, V = V_1 \) and \( D_y = D_y \).

Consider the section \( k_1(x, y) \) of \( E \otimes E^* \) over \( U \times \gamma V \) given by

\[
k_1(x, y) = \begin{cases} k(x, y) & \text{if } \gamma(\rho(x)) = \sigma(\rho(x)) \\ 0 & \text{otherwise} \end{cases}
\]

where \( \rho: U \to \mathbb{R} \) is the projection. This is clearly a measurable section and there is an obvious diffeomorphism \( \phi: U \times \gamma V \to U \times \gamma V \) so that \( k_1 \circ \phi = k \mid U \times \gamma V \). Thus \( k \) is measurable.

Now consider \( k(f(x), y) \) on a graph chart \( U_1 \times \gamma V_1 \). Here we may assume that \( f(U_1) = U_1 \) and \( V_1 = V \). Let \( \phi: U_1 \times \gamma V \to U \times \gamma V \) be given by \( \phi(x, y) = (f(x), y') \) where \( \gamma(\rho(f(x))) = \rho_1(y'), \rho_1: V \to \mathbb{R} \) and \( \pi_\gamma(y) = \pi_\gamma(y') \). Here \( \pi_\gamma: V \to D^\gamma \) is the obvious projection. Then

\[
k(f(x), y) \mid U_1 \times \gamma V_1 = k_1 \circ \phi(x, y)
\]

and so is measurable.

THEOREM 2.3.3. Suppose \( k(x, y) \in C^0_{\text{g}}(F; E) \) and that \( T \) is a geometric endomorphism of \( E \) given by \( f \) and \( A: f^*E \to E \). Then \( \text{tr} k(x, x) \) and \( \text{tr} A_{f(x)} \circ k(f(x), x) \) are bounded, tangentially smooth, measurable functions on \( M \).

Proof. For the case \( \text{tr} k(x, x) \) we can assume \( k \in C^0_{\text{g}}(U, U; \gamma) \). Let \( k_1 \) be the restriction of \( k \) to \( C^0_{\text{g}}(U, U, id) \). It is apparent that \( \text{tr} k(x, x) = \text{tr} k_1(x, x) \). By the previous theorem \( k_1(x, y) \) is measurable on \( U \times id U \). The diagonal \( U \to U \times id U \) is a Borel map, hence \( \text{tr} k(x, x) \) is measurable on \( U \). For \( \text{tr} A_{f(x)} \circ k(f(x), x) \) we may assume \( k \in C^0_{\text{g}}(U_1, U; \gamma) \) with \( f(U) = U_1 \).

Let \( k_1(f(x), y) \) be the restriction of \( k(f(x), y) \) to \( C^0_{\text{g}}(U, U, id) \). By the previous theorem \( k_1(f(x), y) \) and hence \( A_{f(x)} \circ k_1(f(x), y) \) are measurable. As above, we restrict to the diagonal and apply \( \text{tr} \) to conclude the result.

Definition 2.3.4. Let \( k(x, y) \in \Gamma(F; E) \) be measurable and bounded. The v trace of \( k, \text{tr}_v(k) \) is given by

\[
\text{tr}_v(k) = \int_M \text{tr} k(x, x) \, d\mu.
\]

If \( k_1 \in C^0(F; E) \) and \( k_2 \in \Gamma(F; E) \) is bounded and measurable, we define \( k_1 \circ k_2 \in \Gamma(F; E) \) by

\[
k_1 \circ k_2(x, y) = \int_L k_1(x, z) \, k_2(z, y) \, d\lambda^L(z)
\]

where \( L \) is the leaf through \( x \). We define \( k_2 \circ k_1 \) in the same way.

THEOREM 2.3.5. \( k_1 \circ k_2 \) and \( k_2 \circ k_1 \) are bounded measurable elements of \( \Gamma(F; E) \).

Proof. For fixed \( x \) (or \( y \)), \( k_1(x, y) \) is smooth on \( L \) with compact support whose diameter is bounded independently of \( x, y \) and \( L \). Since the leaves of \( F \) have bounded geometry [10]
A LEFSCHETZ THEOREM FOR FOLIATED MANIFOLDS 135

(coming from the compact manifold $M$), it follows that the measure of this support is bounded independently of $x, y$ and $L$. From this, boundedness of $k_1 \circ k_2$ and $k_2 \circ k_1$ follows. To show measurability we may assume that $k_1 \in C_0^\infty(U, V, \gamma)$ with $D_\gamma = R_U$, and we need only consider $k_1 \circ k_2$ restricted to $U \times \gamma V_1$ where $D_\gamma = R_U$. Consider

$$X = \{(x, \xi, y) : (x, \xi) \in U \times \gamma V \text{ and } (x, y) \in U \times \gamma V_1\}.$$  

For $(x, \xi, y) \in X$ it follows that $(\xi, y) \in V \times \gamma V_1$ where $x = \sigma^{-1}$. Let $U \times \gamma V \approx D^p \times D^p \times D^q$ with coordinates $(u, v, t)$ and $V \times \gamma V_1 \approx D^p \times D^p \times D^q$ with coordinates $(v, w, t')$. Then $X \approx D^p \times D^p \times D^q \times D^q$ with coordinates $(x, \xi, y) \rightarrow (u, v, w, t')$. Here $\rho(x) = \gamma^{-1}(\rho(\xi))$ and $\rho(x) = -\gamma(\rho(y))$. The coordinates $(x, y) \rightarrow (u, v, w, t')$ give precisely the differentiable structure on $U \times \gamma V_1$.

On $X$ we take the Borel measure given by the product measure which is the product of Lebesgue measure $dudvdw$ and $dv(t)$. This is a finite measure on $X$. On $U \times \gamma V_1$

$$k_1 \circ k_2(x, y) = \int k_1(u, v, t)k_2(v, w, t)\ vol(v, t)dv,$$

where $\ vol(v, t)dt = d\lambda_L((v, t))$ on $V$. We can apply the Tonelli Theorem to the positive and negative parts of $k_1(u, v, t)k_2(v, w, t)\ vol(v, t)$ relative to the above product measure to conclude $k_1 \circ k_2$ is measurable on $U \times \gamma V_1$. A similar argument holds for $k_2 \circ k_1$.

**THEOREM 2.3.6.** Suppose $k_1 \in C_0^\infty(F; E)$ and $k_2 \in \Gamma(F; E)$ is bounded and measurable. Then

$$tr_v(k_1 \circ k_2) = tr_v(k_2 \circ k_1).$$

**Proof.** We may assume $k_1 \in C_0^\infty(U, V, \gamma)$. Now

$$tr_v(k_1 \circ k_2) = \int_M tr(k_1 \circ k_2(x, x))d\mu = \int_M \left[ \int_L tr(k_1(x, y)k_2(y, x))d\lambda_L(y) \right] d\mu$$

where $L$ is the leaf through $x$. Set $z = \rho(x)$. Since $k_1(x, y) = 0$ unless $x \in U$, this last term is

$$\int_U \int_L tr(k_1(x, y)k_2(y, x))d\lambda_L(y)d\mu = \int_{R_U} \int_{R_\gamma} \int_{R_\gamma} \int_{R_\gamma} tr(k_1(x, y)k_2(y, x))d\lambda_L(y)d\lambda_L(x)d\mu.$$

Since $k_1(x, y) = 0$ unless $x \in P_\gamma$ and $y \in P_{\gamma(z)}$ (the plaque containing $\gamma(z)$) and $\nu$ is an invariant measure, this last integral equals

$$\int_{R_\gamma} \int_{R_\gamma} \int_{R_\gamma} \left[ \int_{R_\gamma} \int_{R_\gamma} tr(k_2(y, x)k_1(x, y))d\lambda_L(x) \right] d\lambda_L(y)d\nu(z).$$

As the last term is $tr_v(k_2 \circ k_1)$, we are done.

We now collect some facts about the operators $\Delta^v$ on the bundles $E^v$ coming from the Dirac complex $(E, d)$. As noted above if $g$ is any bounded Borel function on $[0, \infty)$, $g(\Delta^v)$ is a well defined operator on $L^2(E)$.

**THEOREM 2.3.7.** Let $g$ be a bounded Borel function on $R$ whose Fourier transform $\hat{g} \in C_0^\infty(R)$. Then $k_1^v(x, y) \in C_0^\infty(F; E)$.

**Proof.** We may use the proof of Theorem (2.1) of [21] *mutatis mutandis*.

Denote by $S(R^+)$ the space of smooth functions on $[0, \infty)$ which are the restriction of the Schwartz class functions on $R$.


THEOREM 2.3.8. Suppose \( g \in \mathcal{S}(\mathbb{R}^+) \). Then \( \text{tr}_s(g(\Delta_i)) \) exists and is finite. If \( B \) is a differential operator on \( E_i \) along \( F \) with smooth coefficients and \( T=(T_0, \ldots, T_k) \) is a geometric endomorphism of \((E, d)\) given by \( f \) and \( A=(A_0, \ldots, A_k) \), then \( \text{tr}_s(BT_i g(\Delta_i)) \) and \( \text{tr}_s(T_i B g(\Delta_i)) \) exist and are finite.

Before proving this we note the following. Let \( \| \cdot \| \) be the smooth fiberwise norm on \( E_i \otimes E_i^* \) over \( M \times M \) induced by the metric on \( E_i \) over \( M \). Let \( \nabla^i \) be the connection on \( E_i \otimes E_i^* \) induced by the Hermitian connection on \( E_i \) (which is part of the definition of the Dirac complex \((E, d)\)). Denote by \( C^r(F; E_i) \) the space of tangentially smooth elements \( k(x, y) \in \Gamma(F; E_i) \) such that the norm
\[
\| k \|_r = \sup \{ \| \nabla^i_{(x_1, y_1)} \cdots \nabla^i_{(x_r, y_r)} k(x, y) \| \}
\]
is finite, where the sup is taken over all \((x, y) \in M \times M\) and all collections \((X_1, Y_1), \ldots, (X_s, Y_s)\), \( s \leq r \), where \( X_j \in TL_x \), \( Y_j \in TL_y \) are unit vectors. Denote by \( C^\infty(F, E_i) \) the Frechet space \( \cap_r C^r(F, E_i) \) and note that \( C^k(g, E_i) \subset C^\infty(F, E_i) \).

THEOREM 2.3.9. If \( g \in \mathcal{S}(\mathbb{R}^+) \) then \( k^t \in C^\infty(F; E_i) \). In particular the map \( g \rightarrow k^t \) is continuous.

(Note: Compare [20] (2.13))

\textbf{Proof.} For each leaf \( L \) and \( k \in \mathcal{K} \), we have the Sobolev space \( H_k(E_i^t) \) which is the completion of \( C_0^\infty(E_i^t) \) in the norm
\[
\| \sigma \|_k = \| (1 + \Delta^t)^{k/2} \sigma \|_0
\]
where \( \| \cdot \|_0 \) is the \( L^2 \) norm on \( C_0^\infty(E_i^t) \). If \( A : H_j(E_i^t) \rightarrow H_k(E_i^t) \) is a bounded operator, we denote its operator norm by \( \| A \|_{j, k} \). If \( g \in \mathcal{S}(\mathbb{R}^+) \) then
\[
\| g(\Delta^t) \|_{j, k} \leq \| (1 + \Delta^t)^{j/2} g(\Delta^t) \|_{0, 0}
\]
which by the spectral mapping theorem is bounded by \( \sup_{x \geq 0} [(1 + x)^{j/2} g(x)] \) and so is finite. Thus \( g(\Delta^t) \) is a smoothing operator for all \( L \) and \( i \), and the family \( k^t g(x, y) \in \Gamma(F; E_i) \) is tangentially smooth.

For \( x \in L \), unit vectors \( X_1, \ldots, X_s \in TL_x \), and \( v \in E_x \), \( \| v \| = 1 \), let \( \delta^{x, v}_s \) be the distributional section of \( E_i^t \) given by
\[
\langle \delta^{x, v}_s, \sigma \rangle = \langle (\nabla^i_{X_1} \cdots \nabla^i_{X_s} \sigma)(x), v \rangle
\]
for \( \sigma \in C_0^\infty(E_i^t) \). (If \( r = 0 \) we merely write \( \delta^v \).) The \( p \) dimensional manifold consisting of the union of all leaves is a manifold of bounded geometry. In addition, the metric on \( E_i^t \) and \( \nabla_{|L} \) come from global objects on \( M \) and so are globally bounded, i.e. in terms of any local orthonormal framing, their coefficients and all the derivatives of their coefficients to finite order are bounded independently of \( L \). It follows that given \( s \), there is \( k > 0 \), depending on \( s \), so that \( \delta^{x, v}_s \in H_{-k}(E_i^t) \) for all \( x, v, L \) and \( X = (X_1, \ldots, X_s) \). In addition \( \| \delta^{x, v}_s \|_{-k} \) is bounded independently of \( x, v, L \) and \( X_1, \ldots, X_s \).

Now let \( r \) be a non negative integer and \((x, v) \in M \times M \). For \( i = 1, \ldots, r \), \( s_i \leq r \), let \( X_i \in TL_x \), \( Y_i \in TL_y \) be unit vectors and assume \( x \) and \( y \) are on the same leaf \( L \) of \( F \). To show that \( \| k^t \| \) is finite, it is enough to show that \( \| \delta^{x, v}_s \delta^{y, w}_t \| \) is bounded independently of \( x, y, L, v, w \) and \( X = (X_1, \ldots, X_s), Y = (Y_1, \ldots, Y_t) \). But
\[
\langle g(\Delta^t) \delta^{y, v}_s, \delta^{x, w}_t \rangle \leq \| g(\Delta^t) \|_{-k, -k} \| \delta^{x, v}_s \|_{-k} \| \delta^{y, w}_t \|_{-k}
\]
and \( \| g(\Delta^t) \|_{-k, -k} \leq \| g \|_k \) where \( \| g \|_k = \sup_{x \geq 0} [(1 + x)^k g(x)] \). Thus we have
\[
\| k^t \| \leq \sum_{s \geq r} c_s \| g \|_k,
\]
where $c_\ast$ is a constant depending only on $s$, and $k$ depends on $s$. Thus $k^r_\varphi \in C^r(F; E)$ for all $r$ and as the semi norms $\| \cdot \|_k$ define the topology on $S(R^+)$, the map $g \rightarrow k^r_\varphi$ is continuous.

**Proof of 2.3.8.** As $k^r_\varphi \in C^\infty(F; E)$, it is uniformly bounded and tangentially smooth, so we need only prove measurability. Chose $g^r_\ast \in S(R^+)$ with $g^r_\ast \in C^\varphi(R)$ and $g^r_\ast \rightarrow g$. Then $k^r_\varphi \in C_0(F; E_i)$, so it is measurable and by 2.3.9 $k^r_\varphi(x, y)$ converges uniformly to $k_\varphi(x, y)$, which is thus measurable.

To prove measurability of $BT^r_\varphi g(D_i)$ and $T^r_\varphi Bg(D_i)$ we need only prove measurability when restricted to a graph chart. Thus, suppose that locally (on $M$) $B$ is of the form

$$\sum \frac{\partial |\varphi|}{\partial x^i_1} \ldots \frac{\partial |\varphi|}{\partial x^i_{p'}}.$$

As the $A_i$ and $b_\varphi$ are smooth on $M$, $BT^r_\varphi g(D_i)$ and $T^r_\varphi Bg(D_i)$ determine elements in $C^\infty(F, E_i)$, which are

$$\sum \frac{\partial |\varphi|}{\partial x^i_1} \frac{\partial |\varphi|}{\partial x^i_2} \ldots \frac{\partial |\varphi|}{\partial x^i_{p'}}.$$

These are obviously measurable, so the assertions about $tr_\varphi(BT^r_\varphi g(D_i))$ and $tr_\varphi(T^r_\varphi Bg(D_i))$ follow.

**Corollary 2.3.10.** For $i=0, \ldots, k$, and $t > 0$, $tr_\varphi(T^r_\varphi \exp(-t\Delta_i))$ exists and is finite.

Recall that $P^r_\varphi : L^2(E_i) \rightarrow \ker(A^r_\varphi)$ is the projection and that if $T=(T_0, \ldots, T_k)$ is a geometric endomorphism of $(E, d)$, $T^r_\varphi : L^2(E_i) \rightarrow \ker(A^r_\varphi)$ is the composition $P^r_\varphi T_\varphi P^r_\varphi$. This is a smoothing operator since $P^r_\varphi$ is a smoothing operator and hence $T^r_\varphi$ has a smooth Schwartz kernel. The family $T^* = (T^r_\varphi)$ determines an element of $\Gamma(F; E_i)$.

**Theorem 2.3.11.** $tr_\varphi(T^*)$ exists and is finite. In particular

$$tr_\varphi(T^*) = \lim_{t \rightarrow \infty} tr_\varphi(T^r\exp(-t\Delta_i)).$$

**Definition 2.3.12.** Let $M$, $F$, $(E, d)$, $\nu, f$, and $T$ be as above. The $\nu$ Lefschetz number of $T$, $L_\nu(T)$ is

$$L_\nu(T) = \sum_{i=0}^k (-1)^i tr_\varphi(T^r_\varphi).$$

That this is a reasonable definition is the content of Proposition (2.2.2).

To prove (2.3.11) we study the family of operators $\exp(-t\Delta_i)T^r_\varphi \exp(-t\Delta_i) = \{\exp(-t\Delta_i) T^r_\varphi \exp(-t\Delta_i)\}$. Each $\exp(-t\Delta_i) T^r_\varphi \exp(-t\Delta_i)$ is a smoothing operator on $L^2(E_i)$ and so has a smooth Schwartz kernel $k^{r_\nu}_T(x, y)$. This family of kernels determines a tangentially smooth element of $\Gamma(F; E_i)$ denoted by $k^{r_\nu}_T(x, y)$.

**Theorem 2.3.13.** Fix $t_0 > 0$. Then for $t > t_0$, the pointwise norm $\| k^{r_\nu}_T (x, y) \|$ is uniformly bounded on $M \times M$ independently of $t$.

**Proof.** As in the proof of (2.3.9), to show that $\| k^{r_\nu}_T (x, y) \|$ is uniformly bounded, it is enough to show that $| \langle \exp(-t\Delta_i) T^r_\varphi \exp(-t\Delta_i) \delta^\nu, \delta^\nu \rangle |$ is bounded independently of $x, y, \nu, w$ and $L$. But this is bounded by

$$\| \exp(-t\Delta_i) T^r_\varphi \exp(-t\Delta_i) \|_{-k, \varphi} \| \delta^\nu \|_{-k} \| \delta^w \|_{-k}.$$
and
\[ \| \exp(-t\Delta_f) T^\dagger_e \exp(-t\Delta_f) \|_{k,k} = \| \exp(-t\Delta_f) \|_{0,k} \| T^\dagger_e \|_{0,0} \| \exp(-t\Delta_f) \|_{-k,0}. \]

As \( \| T^\dagger_e \|_{0,0} \) is bounded independently of \( L \), we need only show that for fixed \( j \) and \( k \), \( \| \exp(-t\Delta_f) \|_{j,k} \) is bounded independently of \( t \geq t_0 \) and \( L \). Now \( \| \exp(-t\Delta_f) \|_{j,k} = \| (1 + \Delta_f)^{-\frac{j}{2} + \frac{k}{2}} \exp(-t\Delta_f) \|_{0,0} \). By the spectral theorem, this last norm is bounded by \( \sup (1 + x)^{\frac{k}{2} - \frac{j}{2}} \exp(-tx) \) where the sup is taken over \( t \geq t_0 \) and \( x \geq 0 \). It is elementary that this is finite.

**Corollary 2.3.14.** For all \( t > 0 \), \( tr_s(\exp(-t\Delta_i) T_i \exp(-t\Delta_i)) \) exists and is finite. For \( t \geq t_0 > 0 \), \( tr_s(\exp(-t\Delta_i) T_i \exp(-t\Delta_i)) \) is bounded independently of \( t \).

**Theorem 2.3.15** \( tr_s(T_i \exp(-2t\Delta_i)) = tr_s(T_i \exp(-t\Delta_i) \exp(-t\Delta_i)). \)

*Proof.* Choose a sequence \( g_n \in S(R^+) \) with \( g_n \in C_0 \) and \( \lim g_n(x) = \exp(-tx) \). As \( \exp(-tx)g_n(x) \rightarrow \exp(-2tx) \) in \( S(R^+) \), the Schwartz kernels of the \( \exp(-t\Delta_i)g_n(\Delta_i) \) converge to the Schwartz kernel of \( \exp(-2t\Delta_i) \) in \( C^\infty(F; E_i) \). It follows that the Schwartz kernels of the \( T_i \exp(-t\Delta_i)g_n(\Delta_i) \) converge uniformly to the Schwartz kernel of \( T_i \exp(-2t\Delta_i) \). Thus
\[ tr_s(T_i \exp(-2t\Delta_i)) = \lim tr_s(T_i \exp(-t\Delta_i) g_n(\Delta_i)). \]

As \( T_i \exp(-t\Delta_i) \) defines a measurable bounded element of \( \Gamma(F; E_i) \) and \( g_n \in C_0 \), we have by (2.3.6)
\[ tr_s(T_i \exp(-t\Delta_i) g_n(\Delta_i)) \leq tr_s(g_n(\Delta_i) T_i \exp(-t\Delta_i)). \]

**Lemma 2.3.16.** The Schwartz kernels of the \( g_n(\Delta_i) T_i \exp(-t\Delta_i) \) converge uniformly to the Schwartz kernel of \( \exp(-t\Delta_i) T_i \exp(-t\Delta_i) \).

This lemma implies that
\[ \lim tr_s(g_n(\Delta_i) T_i \exp(-t\Delta_i)) = tr_s(\exp(-t\Delta_i) T_i \exp(-t\Delta_i)). \]

*Proof* of 2.3.16. As in the proof of (2.3.13) we need only show that, as \( n \rightarrow \infty \),
\[ \lim \| g_n(\Delta_i) T_i^\dagger e^{-t\Delta_i} - e^{-t\Delta_i} T_i^\dagger e^{-t\Delta_i} \|_{-k,k} = 0 \]

independently of \( I \). Now
\[ \| g_n(\Delta_i) T_i^\dagger e^{-t\Delta_i} - e^{-t\Delta_i} T_i^\dagger e^{-t\Delta_i} \|_{-k,k} = \| g_n(\Delta_i) - e^{-t\Delta_i} \|_{0,k} \| T_i^\dagger \|_{0,0} \| e^{-t\Delta_i} \|_{-k,0}. \]

As \( \lim (1 + x)^j (g_n(x) - e^{-tx}) = 0 \) as \( n \rightarrow \infty \) in \( S(R^+) \) for fixed \( j \) and \( t > 0 \), and \( \| T_i^\dagger \|_{0,0} \) and \( \| \exp(-t\Delta_i) \|_{-k,0} \) are bounded independently of \( L \), we have the lemma.

**Theorem 2.3.17.** \( tr_s(T^*_f) \) exists, is finite and equals
\[ \lim_{t \rightarrow \infty} tr_s(e^{-t\Delta_i} T_i e^{-t\Delta_i}). \]

*Proof.* As the Schwartz kernels of the \( \exp(-t\Delta_i) T_i \exp(-t\Delta_i) \) (thought of as elements of \( \Gamma(F; E_i) \)) are uniformly bounded for \( t \geq t_0 > 0 \), we need only show that they converge pointwise to the Schwartz kernel of \( P_i T_i P_i = T^*_f \). We then have that the Schwartz kernel of \( T^*_f \) is bounded and measurable and an application of the dominated convergence theorem finishes the proof (and the proof of (2.3.11)).
As in the proof of Theorem (2.3.13), we need only show
\[ \left| \langle (e^{-t\Delta^f} T^f_t e^{-t\Delta^f} - P^f_t T^f_t P^f_t) \delta^\nu_x, \delta^\nu_x \rangle \right| \]
goesto zero as \( t \) goes to \( \infty \). But this is bounded by
\[ \left| \langle (e^{-t\Delta^f} T^f_t e^{-t\Delta^f} - P^f_t T^f_t P^f_t) \delta^\nu_x, \delta^\nu_x \rangle \right| + \left| \langle (e^{-t\Delta^f} T^f_t P^f_t - P^f_t T^f_t P^f_t) \delta^\nu_x, \delta^\nu_x \rangle \right| \]
which is in turn bounded by
\[ \| e^{-t\Delta^f} \|_{0,k} \| T^f_t \|_{0,0} \left( \| e^{-t\Delta^f} - P^f_t \| \| \delta^\nu_x \|_{-k} + \| (e^{-t\Delta^f} - P^f_t) (T^f_t P^f_t \delta^\nu_x) \| \| \delta^\nu_x \|_{-k} \right) . \]
Now \( \| T^f_t \|_{0,0} \) and \( \| \delta^\nu_x \|_{-k} \) are constant in \( t \) and \( \| \exp(-t\Delta^f) \|_{0,k} \) is bounded independently of \( t \) for \( t \geq t_0 > 0 \). We have
\[ \| (e^{-t\Delta^f} - P^f_t) \delta^\nu_x \|_0 = \| (e^{-t\Delta^f} - P^f_t)(1 + \Delta^f_{1/2})(1 + \Delta^f_{-1/2}) \delta^\nu_x \|_0 \]
and the spectral theorem says that the operator \( (e^{-t\Delta^f} - P^f_t)(1 + \Delta^f_{1/2}) \) converges strongly to zero as \( t \) goes to \( \infty \). Thus
\[ \lim_{t \to \infty} \| (e^{-t\Delta^f} - P^f_t) \delta^\nu_x \|_n = 0 \]
and the first term goes to zero as \( t \) goes to \( \infty \).
To show that the second term converges to zero, we note that
\[ \| (\exp(-t\Delta^f) - P^f_t) (T^f_t P^f_t \delta^\nu_x) \|_k = \| (1 + \Delta^f_{1/2}) (\exp(-t\Delta^f) - P^f_t) (T^f_t P^f_t \delta^\nu_x) \|_0 , \]
remark that since \( P^f_t \) is a smoothing operator, \( T^f_t P^f_t \delta^\nu_x \in H_0(E^f_t) \), and apply the argument above.

3. FIXED POINT INDICES AND THE LEFSCHETZ THEOREM

Let \( M, F, v, f, \) and \( T \) be as above. In order to state our Lefschetz Theorem, we need to make some restriction on the fixed point set \( N \) of \( f \). In particular, we assume:

\( N \) is a finite disjoint union \( \bigcup N_j \) of closed submanifolds of \( M \) and each \( N_j \) is transverse to the foliation.

We note the following immediate consequences.

1. Each \( N_j \) is compact.
2. For each leaf \( L \), \( N^f_L = N_j \cap L \) is an embedded submanifold of dimension \( n^f_L = \dim N_j - q \).
3. Let \( d_L(\cdot, \cdot) \) be the distance function on \( L \) and for \( \varepsilon > 0 \), let
\[ \mathcal{N}_\varepsilon(N^f_L) = \{ x \in L : d_L(x, N^f_L) < \varepsilon \} . \]

Then there is an \( \varepsilon > 0 \) such that for all \( L \) and \( j \), \( \mathcal{N}_\varepsilon(N^f_L) \) is an embedded normal disk bundle in \( L \) and all the \( \mathcal{N}_\varepsilon(N^f_L) \) are disjoint. We shall henceforth denote \( \bigcup_{j,L} \mathcal{N}_\varepsilon(N^f_L) \) by \( \mathcal{N}_\varepsilon(N) \).

We must also impose a restriction on the map \( f \). We assume \( f \) is non-degenerate on each \( N^f_L \). This means the following. For each \( L \), denote by \( \mathcal{N}^f_L \) the quotient bundle \( (TL[N^f_L])/TN^f_L \). \( df_\mathcal{N}^f \) preserves \( TN^f_L \) so it induces \( df_{\mathcal{N}^f} : \mathcal{N}^f_L \to \mathcal{N}^f_L \) covering the identity map on \( N^f_L \). To say \( f \) is non degenerate on \( N \) means that for all \( x \in N \), \( \det(I_{\varepsilon^f} - df_{\mathcal{N}^f}) \neq 0 \), i.e. \( df_{\mathcal{N}^f} \) does not have 1 as an eigenvalue. Note that the identity map satisfies this condition.

Suppose we are given for each \( N^f_L \) a smooth (on \( N^f_L \)) measure \( a^f_L \). We assume that the \( a^f_L \) are measurable on \( N \), i.e. measurable as we move transversely to \( F \). Denote the family \( \{ a^f_L \} \)
by $a$. We define $\int_N adv$ as follows. Let $\{(U_i, \phi_i)\}$ be a good cover for $F$, and $\{\psi_i\}$ a partition of unity subordinate to this cover. Then

$$\int_N adv = \sum_i \int_{R_i} \left\{ \sum_{N_j \cap P_i} \phi \int_{N_j \cap P_i} \psi(y) a_f(y) \right\} dv_i$$

Since the $N_j$ are compact, for any fixed plaque $P_i$ (being relatively compact in its leaf $I$), only a finite number of the components of $N_j$ have non trivial intersection with $P_i$. Also, it is easy to show that $\int_N adv$ is independent of the choice of good cover and partition of unity. We can alternatively define $\int_N adv$ in a manner analogous to the definition of $\mu$. Let $R$ be a complete transversal in $M$ and $\pi_j: N_j \rightarrow R$ a measurable map such that $\pi_j(x)$ is on the leaf through $x$. Then

$$\int adv = \sum_j \int_R \left( \int_{\pi_j^{-1}(t)} a_f \right) dv(t).$$

It follows from invariance of $v$ that the two definitions are the same. Now we state the main theorem.

**Theorem 3.1.1. (Lefschetz Theorem for Foliated Manifolds)**

Let $M, F, (E, d), v, f, A$ and $T$ be as above. To each $N^f_j$ we can associate a smooth measure $a_f^j$ which depends only on $f, A$, the symbols of the $\Delta_i$, the metrics, and their derivatives to a finite order only on $N^f_j$ so that

$$L(T) = \int_N adv.$$

For the classical complexes, we can identify the $a_f^j$ explicitly. Suppose $(E, d)$ is a classical complex (DeRham, signature, Dolbeault, spin) and suppose $T = f^*$ acting on forms. Then $a_f^j$ is the usual local integrand given by the Atiyah-Singer $G$ index theorem. If $f$ is the identity and we take the codimension zero foliation of $M$ which has one leaf (namely $M$), we recover the Atiyah-Singer Index Theorem [4] for the classical operators. For these same operators, if $f$ is not the identity and we take the zero codimension foliation, we recover the $G$ Index Theorem [5] and the Atiyah-Bott Lefschetz theorem [2] for geometric endomorphisms.

If $(E, d)$ is an arbitrary Dirac complex for any component of $N^f_j$ which is a single point $x$, then

$$a_f^j(x) = \frac{\sum_{i=0}^k (-1)^i \text{tr} A_i \cdot x}{\det(I_x - df_{L,x})}$$

where $I_x - df_{L,x}: TL_x \rightarrow TL_x$. See [2-II]. If we take the codimension zero foliation, we recover the Atiyah-Bott Lefschetz Theorem for Dirac operators and zeroth order geometric endomorphisms. For the classical complexes with $T = f^*$ we may further identify $a_f^j$ in this case.

(a) **The DeRham complex.**

$$a_f^j(x) = \text{sign} \det(I_x - df_{L,x})$$

See [2-II] section 3.

(b) **Signature complex.**

In this case the foliation must be even dimensional (say 2$\nu$) and $f_\nu$ an isometry. $TL_x$ splits into a sum of two dimensional subspaces and $df_{L,x}$ on each subspace is given by rotation.
through an angle $\theta_k$, $k = 1, \ldots, v$. The fixed point index is
\[ a^f_j(x) = i^{-v} \prod_k \cot(\theta_k/2). \]
See [2]. Theorem (6.27). The collection $\{\theta_1, \ldots, \theta_v\}$ is called a coherent set of angles for $df_L$ at $x$.

c) Dolbeault complex.

Here, each $L$ must be a complex manifold of dimension $v$, and $f_L$ a holomorphic map. For each $r = 0, \ldots, v$ we have an elliptic complex (the $\partial$ complex of forms of type $(r, s)$ and a Lefschetz number $L_r(f_{\partial L})$. The real vector space $TL_x$ has a complex structure and $df_L$ preserves the complex structure. Thus we may think of $df_{L_{\text{fix}}}$ as a complex linear map and $\det C(\wedge^r df_{L_{\text{fix}}})$ and $tr_C(\wedge^r df_{L_{\text{fix}}})$ make sense. For the complex of forms of type $(r, s)$, $r$ fixed, $a^f_{j,r}$ is given by
\[ a^f_{j,r}(x) = \frac{tr_C(\wedge^r df_{L_{\text{fix}}})}{\det C(I - df_{L_{\text{fix}}})}. \]

d) Spin complex.

In this case, we assume that tangent bundle along the leaves $TF$ has a spin $(2\nu)$ reduction $P$, $P$ being a principal spin$(2\nu)$ bundle, and the map $f$ is an isometry such that $df$ lifts to a map of $P$. If $\theta_1, \ldots, \theta_v$ is a coherent set of angles for $df_L$ at $x = N^L_f$, then
\[ a^f_j(x) = \pm i^r 2^{-v} \prod_k \cosec(\theta_k/2). \]
See [2-II]. Theorem 8.25 and [15]. Theorem 4.2.5. The ambiguity in $a^f_j$ is due to the fact that if $df$ has a lifting to $P$, then it has a second lifting and at a given fixed point, the value of the $a$'s for the two liftings are related by one being the negative of the other. The sign at a particular fixed point depends on the lifting and the particular fixed point. In practice, one can eliminate this ambiguity by exhibiting a particular lifting.

4. A UNIVERSAL EXAMPLE

We now construct a manifold $M$ with a foliation of codimension 2 and a diffeomorphism $f$ of $M$ preserving the foliation which has non zero Lefschetz numbers for all the classical complexes. The manifold is a flat $T^2$ bundle over $\Sigma_4$, the surface of genus 4. First we give an algebraic construction of $f$ and $M$, then we show how to realize them geometrically.

Let $\Gamma \subset SL_2 \mathbb{R}$ be a subgroup generated by elements $\alpha_j = \theta^{-1}\alpha \theta^j$, $j = 1, \ldots, 7$ where
\[ \alpha = \begin{bmatrix} d & 0 \\ 0 & d^{-1} \end{bmatrix}, \quad d > 0 \]
and $\theta$ is rotation by $\pi/16$. For proper choice of $\alpha$, $\Sigma_4 = \Gamma \backslash SL_2 \mathbb{R}/SO_2$. We take for a fundamental domain of $\Sigma_4$ a regular 16-gon $D$ centered at zero in the Poincaré disc ($\cong SL_2 \mathbb{R}/SO_2$). The action of the generators we have chosen for $\Gamma$ identifies opposite edges of $D$ by translation along the geodesic through the midpoints of the respective edges. The elements $\alpha_j$ satisfy one relation, namely
\[ \alpha_0 \alpha_1^{-1} \alpha_2 \alpha_3^{-1} \alpha_4 \alpha_5^{-1} \alpha_6 \alpha_7^{-1} \alpha_0^{-1} \alpha_1 \alpha_2^{-1} \alpha_3 \alpha_4^{-1} \alpha_5 \alpha_6^{-1} \alpha_7 = 1d. \]
We note that the $SO_2$ bundle $\Gamma/SL_2 \mathbb{R}$ over $\Sigma_4$ is a non trivial double cover of the orthogonal frame bundle $\Gamma/PSL_2 \mathbb{R}$ of $\Sigma_4$ and so defines a spin structure of $\Sigma_4$. 

To determine a flat $T^2$ bundle over $\Sigma_4$, we need only define a homomorphism $h: \pi_1(\Sigma_4) \to \text{Diff}(T^2)$. The bundle

$$M = (SL_2R/SO_2) \times T^2$$

is obtained from $(SL_2R/SO_2) \times T^2$ by identifying $(x, t)$ with $(yx, h(y)t)$ for all $y$ in $\pi_1(\Sigma_4)$. The natural foliation $\tilde{F}$ on $(SL_2R/SO_2) \times T^2$, whose leaves are $(SL_2R/SO_2) \times \{t\}$, then descends to a foliation $F$ on $M$ transverse to the fibers of $M$.

To this end denote by $A$ the element of $\text{Diff}(T^2)$ determined by the affine map of $R^2$ given by $(x, y) \to (-x + s, -y + s)$ where $s$ is any irrational number. Denote by $B$ the element determined by $(x, y) \to (-x, -y)$. Here we set $T^2 = R^2/Z^2$. Then define

$$h: \pi_1(\Sigma_4) \to \text{Diff}(T^2)$$

by

$$h(\alpha_j) = A$$

for $j = 0, 3, 4, 7$ and $h(\alpha_j) = B$ for $j = 1, 2, 5, 6$.

Note that $A^2 = B^2 = \text{Id}$. So $h$ preserves the relation among the $\alpha_j$ and defines a homomorphism of $\pi_1(\Sigma_4) \to \text{Diff}(T^2)$. Also note that $[AB]^n = \text{Id}$ if and only if $n = 0$, since $AB$ is determined by the affine map $(x, y) \to (x + s, y + s)$. This implies that all leaves of $F$ are non-compact.

The diffeomorphisms $A$ and $B$ preserve Lebesgue measure $dt$ on $T^2$. Thus $dt$ determines an invariant transverse measure $\nu$ on $F$. Note that for any fiber $T^2$ of $M$, $\nu(T^2) = 1$. A point in $M$ will be denoted by $[gSO_2, t]$ where $g \in SL_2R$, and $t \in T^2$. Let $r \in SO_2$ be rotation by $\pi/4$. Define $f: M \to M$ by

$$f([gSO_2, t]) = [rgSO_2, t].$$

**Lemma 4.1.** $f$ is well defined and preserves $F$.

**Proof.** If $f$ is well defined, it obviously preserves $F$. To see that $f$ is well defined, note that the action of $r$ on the fundamental domain $D$ is to rotate it about its center by $\pi/2$ (not $\pi/4$).

One then easily checks that $ra_j = \alpha_j r$ or $ra_j = \alpha_j^{-1} r$ for all $j$, where the addition of subscripts is mod 8. Now for each $\alpha_j$ we have

$$f([\alpha_j gSO_2, h(\alpha_j)t]) = [\alpha_j h(gSO_2, h(\alpha_j)t)]$$

$$= [\alpha_j^{-1}rgSO_2, h(\alpha_j)t] = [rgSO_2, h(\alpha_j^{-1}rgSO_2)h(\alpha_j)t]$$

$$= [rgSO_2, t] = f([gSO_2, t])$$

since $h(\alpha_j^{-1}rgSO_2) = \text{Id}$ for all $j$.

As an arbitrary $\gamma \in \Gamma$ can be written as a product of $\alpha_j$'s we have that $f$ is well defined.

In order to determine the fixed point set of $f$, we now give a geometric construction of $M$ and $f$. To construct $M$, we identify points on the boundary of $D \times T^2$ in the following way.

The edge $E_j^1$ is identified to the edge $E_j^2$ by the action of $\alpha_j$ on the Poincaré disk $SL_2R/SO_2$. We identify $E_j^1 \times T^2$ to $E_j^2 \times T^2$ by

$$(e, t) \sim (\alpha_j(e), h(\alpha_j)t).$$

Then $M = D \times T^2/\sim$ where $D$ is the fundamental domain. $D \times T^2$ is foliated by leaves of the form $D \times \{t\}$, and the above identifications respect this foliation, so it induces a foliation on $M$ and this foliation is just $F$. The map $\tilde{f}: D \times T^2 \to D \times T^2$ given by rotation by $\pi/2$ on the $D$ factor and the identity on the $T^2$ factor induces $f$ on $M$.

We write $(d, t)$ for a point in $D \times T^2$ and $[d, t]$ for the point it determines in $M$. It is clear that all points $[c, t]$, where $t \in T^2$, and $c$ is the center of the Poincaré disk, are fixed by $f$, and
that the action of $df$ on $TL_{L_{(v,t)}}$ is rotation by $\pi/2$. The only other possible fixed points are the points $[v, t], t \in T^2, v$ a vertex of $D$. Let $v$ be the common vertex of $E^3_5$ and $E^4_5$. Then $v = x_0^{-1}x_1^{-1}x_2^{-1}x_3^{-1}f(v)$. So

\[ f[v, t] = [f(v), t] = [v, h(a_0^{-1}x_1^{-1}x_2^{-1}x_3^{-1})] = [v, t]. \]

It is easy to see that the action of $df$ on $TL_{L_{(v,t)}}$, for any vertex, is rotation by $\pi/2$.

The metric we put on $M$ is the one induced from $D \times T^2$ by the Poincaré metric on $D$ and the natural metric on $T^2$. The orientation we put on $F$ is the one it receives from the natural orientation on $D$.

The local fixed point indices and Lefschetz numbers $L_* (f)$ for $f^*$ for the classical complexes are given below.

**DeRham Complex.**

As $df_p$ is rotation by $\pi/2$, $\det(I - df_p) = 2$ so $a^*_f(p) = 1$ for all fixed points and we have

\[ L_* (f) = \int_N 1dv = \int_{T^2} 1dt + \int_{T^2} 1dt = 2. \]

Now

\[ L_* (f) = \sum_{i=0}^{2} (-1)^i tr_* (f^*) \]

where, for each leaf $L, f^*_* : H^1_L (L, R) \rightarrow H^1_L (L, R)$. As $L$ is a non-compact complete surface, we have $H^0_L (L, R) = H^2_L (L, R) = 0$. Thus $\text{tr}_*(f^*) = \text{tr}_*(f^*) = 0$ and $\text{tr}_*(f^*) = -2$. This implies that for almost all $L, H^1_L (L, R) \neq 0$, i.e. for almost all $L$, there are non-zero harmonic one forms on $L$. We note that it follows from the $L^2$ covering index theorem [1] that $H^1 (L, R) \neq 0$ for all $L$. 

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**Fig. 1.** The fundamental domain $D$. 

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143
Signature Complex.

For each leaf $L, f_L$ is an isometry so we may consider the action of $f$ on the signature complex of $F$. At each fixed point $p$, $df_p: TL_p \to TL_p$ is the isometry of the oriented two-dimensional space $TL_p$ given by rotation by $\pi/2$. Thus a coherent set of angles for $df_p$ is $\theta_p = \pi/2$, so the fixed point index at each fixed point is $-i$ and the Lefschetz number for $f$ is $L_*(f) = -2i$.

Dolbeault Complex.

The surface $\Sigma_4$ is a complex manifold and this complex structure lifts to a complex structure on each leaf of $F$. The map $f$ covers a holomorphic map on $\Sigma_4$, so $f$ restricted to any leaf is holomorphic. Denote by $\wedge^{p,q}$, the bundle

$$\wedge^{p,q} = \wedge^p T^*F \otimes \wedge^q \bar{T}^*F$$

where $T^*F$ and $\bar{T}^*F$ are respectively the holomorphic and antiholomorphic cotangent bundles of $F$. A section of $\wedge^{p,q}$ is then a form of type $p, q$ on each leaf $L$. Since $f$ is a holomorphic map of each leaf, $f^*$ induces an endomorphism of the Dolbeault complex

$$0 \to C^\infty(\wedge^{k,0}) \to C^\infty(\wedge^{k,1}) \to 0$$

for $k = 0, 1$. We denote the Lefschetz number of $f^*$ in this case by $L_*(f_k)$.

Now $df_p: TL_p \to TL_p$ in our example, considered as a complex linear map, is just multiplication by $i$. Thus for $k = 0$, the local indices are $1/(1 - i)$ and $L_*(f_0) = 2/(1 - i) - 1 + i$, while for $k = 1$ the local indices are $i/(1 - i)$ and $L_*(f_1) = 2i/(1 - i) - i - 1$.

Spin Complex.

The surface $\Sigma_4$ is a spin manifold so that each leaf is also a spin manifold. As we noted above, a spin structure on $\Sigma_4$ is given by

$$\Gamma \setminus SL_2 R \to \Gamma \setminus PSL_2 R$$

where $\Gamma \setminus PSL_2 R$ is the orthogonal frame bundle of $\Sigma_4$. Thus we may exhibit a spin structure on $F$ by

$$SL_2 R \times_k T^2 \to PSL_2 R \times_k T^2$$

where $PSL_2 R \times_k T^2$ is the orthogonal frame bundle of the foliation. In this representation $f: M \to M$ is given by $f([gSO_2, t]) = [rgSO_2, t]$ and $df: PSL_2 R \times_k T^2 \to PSL_2 R \times_k T^2$ is given by $df([\pm g, t]) = ([\pm rg, t])$. Here we indicate the class of $g$ in $PSL_2 R = SL_2 R/\pm I$ by $\pm g$. It is clear that $df$ has two liftings $df^\pm$ to $SL_2 R \times_k T^2$, namely

$$df^+[g, t] = [rg, t]$$

$$df^- [g, t] = [r_\pi g, t]$$

where $r_\pi$ is rotation by $5\pi/4$.

The local fixed point index for the lifting $df^+$ of $df$ is given by $-i/(2)^{1/2}$. Thus the Lefschetz number for $df^+$ is

$$L_*(df^+) = -2i/(2)^{1/2} = -(2)^{1/2}i.$$
For the lifting $d^*_T$, the local index is $i/(2)^{1/2}$ and the Lefschetz number is

$$L_v(d^*_T) = (2)^{1/2}i.$$  

**5. PROOF OF THE LEFSCHETZ THEOREM**

Recall that theorem (2.3.11) says

$$\lim_{t \to \infty} tr_v(T_i e^{-tA_i}) = tr_v(T_i^*).$$

Our next step is to prove

**THEOREM 5.1.** For $t > 0$,

$$\sum_{i=0}^{k} (-1)^i tr_v(T_i e^{-i\Delta_i})$$

is independent of $t$.

**Proof.** We will use the formalism of the super trace as found in [18]. Let $E$ be the bundle $E = \bigoplus E_i$. On $E$ we have operators

$$T = \bigoplus_{i=0}^{k} T_i, \quad d = \bigoplus_{i=0}^{k-1} d_i, \quad d^* = \bigoplus_{i=0}^{k-1} d_i^*, \quad \Delta = \bigoplus_{i=0}^{k} \Delta_i$$

These operators satisfy

$$\Delta = dd^* + d^*d, \quad d\Delta = \Delta d, \quad d^*\Delta = \Delta d^*, \quad \text{and} \quad Td = dT.$$  

Set $E^+ = \bigoplus E_{2i}$ and $E^- = \bigoplus E_{2i-1}$. An operator $A$ on $E$ which takes sections of $E^+$ to sections of $E^+$ and sections of $E^-$ to sections of $E^-$ is called an even operator. If $A$ takes sections of $E^+$ to sections of $E^-$ and sections of $E^-$ to sections of $E^+$ it is called an odd operator. The $v$ super trace applied to an even operator $A$ is defined to be

$$\text{tr}^v(A) = tr_v(A|_{E^+}) - tr_v(A|_{E^-}).$$

If $A$ and $B$ are both even operators, then $\text{tr}^v(AB) = \text{tr}^v(BA)$, and if they are both odd then $\text{tr}^v(AB) = -\text{tr}^v(BA)$.

Note that

$$\sum_{i=0}^{k} (-1)^i tr_v(T_i e^{-i\Delta_i}) = tr_v(T e^{-t\Delta})$$

so we must show that $\text{tr}^v(T \exp(-t\Delta))$ is independent of $t$.

**LEMMA 5.3.** Suppose $\phi(x)$ is an analytic function on $C$ whose restriction to $R^+$ is in $S(R^+)$. Then

$$d\phi(\Delta) = \phi(\Delta)d \quad \text{and} \quad d^*\phi(\Delta) = \phi(\Delta)d^*.$$  

**Proof.** As above, we denote the restriction to a leaf $L$ by superscript $L$. Now, recall that one way to define $\phi(\Delta) = \{\phi(\Delta^L)\}$ is

$$\phi(\Delta^L) = \frac{1}{2\pi i} \int_C \phi(\lambda)(\lambda I - \Delta^L)^{-1} d\lambda$$

where $C$ is a curve in $C$ of the form...
and \( I \) is the identity operator on \( E \). As \( d^L \Delta^L = \Delta^L d^L \) and \( d^L (\lambda I - \Delta^L) = (\lambda I - \Delta^L) d^L \) it follows easily that \((\lambda I - \Delta^L)^{-1} d^L = d^L (\lambda I - \Delta^L)^{-1} \) and hence that \( d \phi(\Delta) = \phi(\Delta) d \). Similarly for \( d^* \).

Let \( s > t > 0 \) and set
\[
\psi(x) = \frac{e^{-tx} - e^{-sx}}{x}
\]
and
\[
\phi(x) = e^{-tx} - e^{-sx}
\]
for \( x \in \mathbb{R}^+ \). Both \( \phi \) and \( \psi \) are in \( S(\mathbb{R}^+) \) and \( \phi(x) = x \psi(x) \). As a consequence of (2.3.8) we have.

**Lemma 5.2.** The operators \( Td^* \psi(\Delta) \) and \( Td^* d \psi(\Delta) \) have finite \( tr^s \).

Now consider the following set of equations.
\[
tr^s(Te^{-tA}) - tr^s(Te^{-sA}) = tr^s(T \phi(\Delta)) = tr^s(T \Delta \psi(\Delta)) = tr^s(Tdd^* \psi(\Delta) + Td^* d \psi(\Delta)) = tr^s(Td^* \psi(\Delta)) + tr^s(Td^* d \psi(\Delta)).
\]
The first three equalities are by definition, and the last follows from Lemma (5.2). Now
\[
tr^s(Td^* \psi(\Delta)) = tr^s(d Td^* \psi(\Delta)) = - tr^s(Td^* d \psi(\Delta)).
\]
The first equality follows from the fact that \( Td = d T \). To justify the second equality, we first prove

**Lemma 5.3.** Suppose \( \psi_n \) converges to \( \psi \) in \( S(\mathbb{R}^+) \) and \( \hat{\psi}_n \in \mathcal{C}_0^\infty(\mathbb{R}) \). Then
\[
\lim_{n \to \infty} tr^s(d Td^* \psi_n(\Delta)) = tr^s(d Td^* \psi(\Delta)).
\]

**Proof.** By (2.3.9) the Schwartz kernels of the \( \psi_n(\Delta) \) converge uniformly together with their derivatives to the Schwartz kernel of \( \psi(\Delta) \). By the proof of (2.3.8), the Schwartz kernels of the \( d Td^* \psi_n(\Delta) \) converge uniformly to the Schwartz kernel of \( d T \psi(\Delta) \). Similarly for the second equality.

Now set \( \psi_1(x) = (1/2) \psi(x/2) \) and \( \psi_2(x) = \exp(-tx/2) + \exp(-sx/2) \). Then \( \psi_1, \psi_2 \in S(\mathbb{R}^+) \). Choose sequences \( \psi_{1,n} \in S(\mathbb{R}^+) \) converging to \( \psi_1 \) with \( \hat{\psi}_{1,n} \in \mathcal{C}_0^\infty(\mathbb{R}) \). Let \( \psi_n = \psi_1 \psi_{1,n} \). Then \( \hat{\psi}_n \in \mathcal{C}_0^\infty(\mathbb{R}) \) and \( \psi_n \) converges to \( \psi \) in \( S(\mathbb{R}^+) \). Note that the Schwartz kernels of \( \psi_{2,n}(\Delta) \) and \( \psi_{2,n}(\Delta)d = d \psi_{2,n}(\Delta) \) are in \( \mathcal{C}_0^\infty(F;E) \) and that
$dT^*\psi_{1,n}(\Delta)$ and $Td^*\psi_{1,n}(\Delta)$ have Schwartz kernels which are globally bounded, measurable and tangentially smooth. Thus

$$tr_\gamma^S(dT^*\psi(\Delta)) = \lim_{n \to -\infty} tr_\gamma^S(dT^*\psi_{1,n}\psi_{2,n}(\Delta))$$

$$= \lim_{n \to -\infty} tr_\gamma^S(dT^*\psi_{1,n}(\Delta)\psi_{2,n}(\Delta))$$

$$= \lim_{n \to -\infty} tr_\gamma^S(\psi_{2,n}(\Delta)dT^*\psi_{1,n}(\Delta))$$

$$= -\lim_{n \to -\infty} tr_\gamma^S(Td^*\psi_{1,n}(\Delta)\psi_{2,n}(\Delta)d)$$

since both $\psi_{2,n}(\Delta)d$ and $Td^*\psi_{1,n}(\Delta)$ are odd operators. This last term equals

$$-\lim_{n \to -\infty} tr_\gamma^S(Td^*\psi_{1,n}\psi_{2,n}(\Delta)d) = -\lim_{n \to -\infty} tr_\gamma^S(Td^*d\psi_{1,n}\psi_{2,n}(\Delta))$$

$$= -tr_\gamma^S(Td^*d\psi(\Delta)).$$

This completes the proof of (5.1).

We call $\beta_0 = tr_\gamma(P_i)$ the $i$th Betti number of the complex $(E, d)$ and we note in passing the following.

**Corollary 5.4.** Let $\mu_i = tr_\gamma(\exp(-t\Delta_i))$. Then

$$\mu_0 \geq \beta_0$$

$$\mu_1 - \mu_0 \geq \beta_1 - \beta_0$$

$$\mu_2 - \mu_1 + \mu_0 \geq \beta_2 - \beta_1 + \beta_0$$

$$\vdots$$

that is, the $\mu_i$ satisfy the Morse inequalities with respect to the $\beta_i$.

**Proof.** Denote by $\pi_i$ the projection of the bundle $E$ onto the subbundle $E_i$. Then

$$\pi_i d = d\pi_{i-1}$$

$$d_{i-1} d_{i-1}^* \psi(\Delta_i) = \pi_i d^* \psi(\Delta) \quad \text{and}$$

$$d_{i-1}^* d_{i-1} \psi(\Delta_{i-1}) = \pi_{i-1} d^* \psi(\Delta).$$

Thus

$$tr_\gamma(d_{i-1} d_{i-1}^* \psi(\Delta_i)) = tr_\gamma(\pi_i d^* \psi(\Delta))$$

$$= -tr_\gamma(\pi_i d^* \psi(\Delta) d)$$

$$= tr_\gamma(\pi_{i-1} d^* d \psi(\Delta_i)) = -tr_\gamma(d_{i-1} d_{i-1}^* \psi(\Delta_{i-1})).$$

Set $\lambda_i = \mu_i - \beta_i$. Then

$$\lambda_i = \lim_{s \to -\infty} tr(e^{-s\Delta_i} - e^{-s\Delta_i})$$

and we wish to show

$$\lambda_k - \lambda_{k-1} + \ldots + (-1)^k \lambda_0 \geq 0.$$
But

\[ \lambda_i = \lim_{s \to \infty} tr_s(d_s^* d_s \psi(\Delta_s) + d_{s-1}^* d_{s-1} \psi(\Delta_{s-1})) \]

\[ = \lim_{s \to \infty} tr_s(d_s^* d_s \psi(\Delta_s) - d_{s-1}^* d_{s-1} \psi(\Delta_{s-1})) \]

by the formula above, so

\[ \lambda_k - \lambda_{k-1} + \ldots + (-1)^k \lambda_0 = \lim_{s \to \infty} tr_s(d_s^* d_s \psi(\Delta_s)) \geq 0. \]

This last inequality follows from the fact that \( d_s^* d_s \psi(\Delta) \geq 0 \) for all \( s \), which implies that its Schwartz kernel \( k(x, y) \) satisfies \( tr k(x, x) \geq 0 \) for all \( x \in M \).

To finish the proof of the Lefschetz Theorem we now compute \( \lim tr_s(T_s \exp(-t \Delta_s)) \) as \( t \to 0 \). Our first step in doing this is to replace the Schwartz kernel of \( T_s \exp(-t \Delta_s) \) by an asymptotic estimate as \( t \to 0 \).

We let \( k_{t,L}(x, y) \) be the Schwartz kernel of \( \exp(-t \Delta^L) \) relative to the metric \( \lambda^L \) (that is \( [\exp(-t \Delta^L) g](x) = \int_L k_{t,L}(x, y) g(y) d\lambda^L(y) \)) and

\[ k_{t,1}(x, y) = \sum (-1)^j k_{t,j}(x, y). \]

The Schwartz kernel \( T \exp(-t \Delta^L) \) will be denoted by \( k_{t,L}(x, y) \) and

\[ k_{t,L}(x, y) = \sum_{i=0}^k (-1)^i k_{t,L}^i(x, y) = \sum_{i=0}^k (-1)^i A_i \cdot k_{t,L}^i(f(x), y) \]

\[ = A \cdot k_{t,L}(f(x), y). \]

Let \( \tilde{L} \) be the \( p \) dimensional manifold which is \( \cup L \) over all leaves \( L \), with the Riemannian metric \( g_L = g|L \) on each connected component \( L \), and metric \( d_L \). Any orthonormal frame at a point \( m \in L \subset \tilde{L} \) determines an isomorphism \( R^p \to TL_m \). The composite map

\[ R^p \to TL_m \xrightarrow{\exp_m} L \]

(also denoted by \( \exp_m \)) is a diffeomorphism of a neighborhood of \( 0 \in R^p \) into \( L \). Recall that \( D^p(r) \) is the ball of radius \( r \) about \( 0 \in R^p \). \( \tilde{L} \) is a manifold with bounded geometry \[101\], so there is a number \( r > 0 \) such that

\[ \exp_m : D^p(r) \to L \]

is a diffeomorphism into for all \( L \) and \( m \). The coordinate system about \( m \) this defines is called a normal coordinate system. We will not distinguish between \( D^p(r) \) and its image under \( \exp_m \). Since \( \tilde{L} \) is oriented we can always choose our normal coordinates in such a way that the determinant of the Jacobian \( d\lambda/dx \) is positive.

For \( m \in L \), let \( s \) be an orthonormal frame of \( E \) at \( m \) extended over the domain of a normal coordinate system by parallel translation along radial geodesics. Such a frame will be called a synchronous frame. Then in normal coordinates and relative to a synchronous frame

\[ \Delta = \sum_{i=0}^p \frac{\partial^2}{\partial x_i^2} + \sum_{i=0}^p a_i \frac{\partial}{\partial x_i} + b \]

where \( a_i \) and \( b \) are \( n \times n \) matrix functions and \( n = \dim E \). The coefficient of each \( \partial^2/\partial x_i^2 \) is understood to be the \( n \times n \) identity matrix. Let \( \phi \) be a bump function on \( R^p \) with \( \phi(x) = 1 \) on
$D^p(r/4)$ and $\phi(x) = 0$ outside $D^p(r/2)$. Let

$$F^m = \sum_{i=0}^{p} \frac{\partial^2}{\partial x_i^2} + \phi(x) \left( \sum_{i=0}^{p} a_i \frac{\partial}{\partial x_i} + b \right)$$

on $\mathbb{R}^p$. For $t > 0$ let $\exp(-t F^m)$ be the fundamental solution of the strongly parabolic system $\partial u/\partial t + F^m u = 0$ (see [22] p. 95). Let $k^m_{t,L}$ be the Schwartz kernel of $\exp(-t F^m)$ relative to the measure $dx_1 \ldots dx_p$ for $x_1, \ldots, x_p$ coordinates on $\mathbb{R}^p$.

Let $F^m = \sum_{i=0}^{p} \frac{\partial^2}{\partial x_i^2} + \phi(x) \left( \sum_{i=0}^{p} a_i \frac{\partial}{\partial x_i} + b \right)$ on $\mathbb{R}^p$. For $t > 0$ let $\exp(-t F^m)$ be the fundamental solution of the strongly parabolic system $\partial u/\partial t + F^m u = 0$ (see [22] p. 95). Let $k^m_{t,L}$ be the Schwartz kernel of $\exp(-t F^m)$ relative to the measure $dx_1 \ldots dx_p$ for $x_1, \ldots, x_p$ coordinates on $\mathbb{R}^p$.

Given $\varepsilon > 0$, there is a $\delta > 0$ such that if $x \in N_\varepsilon(N)$ then $d_L(x, f(x)) < \varepsilon$. Choose numbers $\rho$ and $\sigma$ so that $0 < \rho < \sigma/2 < r/4$ and the following properties are satisfied.

1. For $x \in \mathcal{N}_\rho(N)$, $d_L(x, f(x)) < \sigma/2$.
2. For $m \in N$ and $d_L(x, m) < \sigma$, $x \in \exp_m(D^p(r/4))$.
3. $\det(I - df) \neq 0$ in $\mathcal{N}_\rho(N)$.

Note that 1 implies that for $x \in \mathcal{N}_\rho(N)$, $f(x) \in \mathcal{N}_\rho(N)$.

For $y \in \mathcal{N}_\rho(N)$, $y$ is in the image of $\exp_m$ restricted to the orthogonal complement of $T_{N_m}$ in $T_L$, for a unique $m \in N$. For $y \in \mathcal{N}_\rho(N)$ and $x \in \exp_m(D^p(r))$ set

$$k_{t,L}(x, y) = k^m_{t,L}(x, y)(dx/d\lambda)$$

for the unique $m \in N$ as above.

To compute $tr \psi$ we break up the integral over $M$ which gives $tr \psi$ into integrals over $\mathcal{N}_\rho(N)$ and $M - \mathcal{N}_\rho(N)$. The following theorem tells us how to approximate each of these integrals.

**Theorem 5.5.**

(a) For $x \in \mathcal{N}_\rho(N)$, $k_{t,L}(x, f(x), x)$ is asymptotic to $k_{t,L}(x, f(x), x)$ as $t \to 0$. That is, given $\varepsilon > 0$

$$t^{-\varepsilon} \sup \| k_{t,L}(x, f(x), x) - k_{t,L}(x, f(x), x) \| \to 0$$

as $t \to 0$, where the sup is taken over $x \in \mathcal{N}_\rho(N)$.

(b) For $x \in M - \mathcal{N}_\rho(N)$, $k_{t,L}(x, f(x), x)$ is asymptotic to zero as $t \to 0$, that is

$$t^{-\varepsilon} \sup \| k_{t,L}(x, f(x), x) \| \to 0$$

as $t \to 0$, where the sup is taken over $x \in M - \mathcal{N}_\rho(N)$.

**Proof:** The proof of Proposition 2.16 of [20] says that given $c > 0$ there is a kernel $h_{t,L} \in C^\infty(F; E)$ with the property that $h_{t,L}(x, y) = 0$ for $d_L(x, y) > c$, and $h_{t,L}(x, y)$ is asymptotic to $k_{t,L}(x, y)$ as $t \to 0$. In fact, for any $s$ and $\alpha > 0$.

$$t^{-\alpha}(\partial^\alpha/\partial t^\alpha)(h_{t,L}(x, y) - h_{t,L}(x, y)) \to 0$$

in $C^\alpha(F; E)$ for any $r > 0$. Let $\psi(\lambda)$ be an even function on the reals which is one near zero and zero outside $[-c, c]$. Then $h_{t,L}$ is the smooth Schwartz kernel of the operator

$$(2\pi t)^{-1/2} \int \psi(\lambda) \exp(-\lambda^2/4t) e^{i\lambda D} d\lambda$$

where $D = d + d^*$ is the Dirac operator on $E$. Unit propagation speed for solutions of $D + i\partial/\partial \lambda = 0$ implies that the kernel $k_{t}(x, y)$ of $e^{i\lambda D}$ satisfies $k_{t}(x, y) = 0$ for $d_L(x, y) > \lambda$, which further implies that $h_{t,L}(x, y) = 0$ for $d_L(x, y) > c$.

Now let $h_{t,L}(x, y)$ be the kernel constructed via this method which satisfies $h_{t,L}(x, y) = 0$ for $d_L(x, y) > \sigma/2$. Let $w_{t,L}(x, y) = (\partial/\partial t + \Delta_x)h_{t,L}(x, y)$. Since $h_{t,L}$ is asymptotic to $k_{t,L}$ as
James L. Heitsch and Connor Lazarov

t → 0, it follows that sup \( t^{-s}w_{t,L}(x, y) \) → 0 as t → 0 and, in fact, \( t^{-s} \partial^2/\partial t^2 w_{t,L}(x, y) \) → 0 in \( C^\infty(F; E) \) for all \( r \). Let \( \mathcal{N}_r^+(N^L)_m \) be the fiber of \( \mathcal{N}_r^+(N^L) \) over the point \( m \in N \), and fix \( y \in \mathcal{N}_r^+(N^L)_m \) and \( x \in \mathcal{N}_r^+(N^L)_m \). Write \( h_{t,L} \) and \( w_{t,L} \) in local normal coordinates at \( m \). \( \Delta = F^m \) on \( \mathcal{N}_r^+(N^L)_m \)

\[
(\partial/\partial t + F^m)h_{t,L}(x, y) = (\partial/\partial t + \Delta_x)h_{t,L}(x, y) = w_{t,L}(x, y).
\]

In normal coordinates around \( m \) we want to compare \( h_{t,L}(x, y) \) and \( k_{t,L}(x, y) \) to \( dx/d\lambda \). Notice that \( h_{t,L}(x, y) = 0 \) for \( y \notin \mathcal{N}_r^+(N^L)_m \) and \( x \) outside \( \mathcal{N}_r^+(N^L)_m \). Now apply Duhamel's principle to \( k_{t,L} \) and \( h_{t,L} \) in normal coordinates to get

\[
k_{t,L}(x, y)(dx/d\lambda) - h_{t,L}(x, y) = \int_0^t (e^{(s-t)\Delta_x})^* (w_{t,L}(y, x)) \, ds.
\]

Now the Sobolev norms (in \( R^p \)) \( \| (\exp((s-t)F^m))^* (w_{t,L}(y, x)) \| \) can be estimated in terms of the uniform norms \( \| w_{t,L}(y, x) \|_p \) for \( \beta \leq r \) where the constants involved are independent of \( m \) for \( y \in \mathcal{N}_r^+(N^L)_m \), \( x \in \mathcal{N}_r^+(N^L)_m \) and since \( t^{-s}w_{t,L}(y, x) \) → 0 in the norms \( \| \cdot \|_p \) it follows that

\[
\sup \| t^{-s}k_{t,L}(x, y)(dx/d\lambda) - h_{t,L}(x, y) \| \to 0 \text{ as } t \to 0
\]

for \( y \in \mathcal{N}_r^+(N^L)_m \) and \( x \in \mathcal{N}_r^+(N^L)_m \). From this, part (a) of the theorem follows.

For part (b), choose \( \delta > 0 \) such that for \( x \) outside \( \mathcal{N}_r^+(N) \), \( d_L(f(x), x) > \delta \). Now construct a kernel \( h_{t,L}(x, y) \) as in part (a), with \( h_{t,L}(x, y) \) asymptotic to \( k_{t,L}(x, y) \) and \( h_{t,L}(x, y) = 0 \) for \( d_L(x, y) \geq \delta \). Then for \( x \) outside \( \mathcal{N}_r^+(N) \) it follows that

\[
\sup \| t^{-s}k_{t,L}(f(x), x) \| \to 0 \text{ as } t \to 0.
\]

This proves part (b).

**Theorem 5.6.** To each component \( N_f^j \) of \( N \) there is associated a smooth (on \( N_f^j \)) measure \( a_f^j \) which depends only on \( f, A \), the symbol of \( \Delta \), the metrics, and their derivatives to finite order on \( N_f^j \) so that

\[
tr_f^j(De^{-\beta}) = \int_N adv
\]

where \( a = \{ a_f^j \} \).

**Proof.** Set \( K_{t,L}(x, y) = A_xK_{t,L}(f(x), y) \). As \( A \) and \( f \) are uniformly bounded on \( M \), it follows immediately from Theorem 5.5 that

\[
tr_f^j(De^{-\beta}) = \lim_{t \to 0} \int_{\mathcal{N}_r^+(N)} tr K_{t,L}(x, x) \, d\mu(x) = \lim_{t \to 0} \int_{\mathcal{N}_r^+(N)} tr(k_{t,L}(x, x)) \, d\mu(x) dv.
\]

But \( tr_f^j(De^{-\beta}) \) is independent of \( t \), so to compute it we need only express the right hand side as a Laurent series in \( t \) and take the zeroth order term. Now \( K_{t,L}^j \) is defined in terms of \( k_{t,L}^j \), so we begin by studying the operator \( F^m \) acting on vector valued sections. The symbol of \( F^m \) is given by

\[
\sigma(F^m) = \sum_{i=1}^k \zeta_i^m + \sum_{i=1}^k a^m(x)\zeta_i + b^m(x).
\]

From [14], [15], [20] we know that as \( t \to 0 \), \( k_{t,L}^m(x, y) \) has an asymptotic expansion of the
Here $|\zeta|^2 = \sum_{i=1}^{r} |\zeta_i|^2$ and $r$ is sufficiently large. Each $b_{r,s}^m(x, \zeta)$ is homogeneous of degree $s$ in $\zeta$ and is zero if $u + s$ is odd. If we write

$$b_{r,s}^m(x, \zeta) = \sum_{|\alpha| = s} b_{r,s,\alpha}^m(x) \zeta^\alpha$$

(where $\alpha=(\alpha_1, \ldots, \alpha_p)$, $|\alpha| = \Sigma \alpha_i$, $\zeta^\alpha = \zeta_1^{\alpha_1} \cdots \zeta_r^{\alpha_r}$) then each $b_{r,s}^m(x)$ is given by a canonical polynomial in the $a_\mu^m$ and $b_\mu^m$, the metrics, and their derivatives to a finite order.

To say that $k_t^m(x, y)$ is asymptotic to $K_t^m(x, y)$ means that given $a$, there is a constant $C_a$ so that for sufficiently large $t$

$$\sup \| k_t^m(x, y) - K_t^m(x, y) \| < C_a t^r$$

as $t \to 0$. The constant $C_a$ depends continuously on $a_\mu^m$ and $b_\mu^m$ in the $C^\infty$ topology on the space of matrix functions on $R^p$ (this is implicit in the calculations of [15]). Because of the bounded geometry of $\tilde{L}$ the collections $a_\mu^m$ and $b_\mu^m$ lie in a bounded set in the $C^\infty$ topology, so the $C_a$ and $r$ can be chosen independently of the point $m \in N$. Thus if we define

$$K_{t,L}(x, y) = K_{t,L}^m(x, y)(dx/d\lambda)$$

for $x, y \in L, y \in \mathcal{N}_\rho(N)$, $y$ in the image of $\exp_p$ on the orthogonal complement of $TN_m$ in $TL_m$, $x \in \mathcal{N}_\rho(N)$, then $K_{t,L}(x, y)$ is asymptotic to $K_{t,L}(x, y)$ for these values of $x, y$. Since $A$ and $f$ are globally bounded on $M$, as $t \to 0$, for $x \in \mathcal{N}_\rho(N)$, $K_{t,L}(x, x)$ is asymptotic to

$$K_{t,L}(x, x) = A_x K_{t,L}(f(x), x)$$

$$= \sum_{u=0}^{e} \phi(\mu - p/2) \sum_{s=0}^{e} \int e^{i(f(x)-x)\zeta^{(1/2)}} A_x b_{r,s}^m(f(x), \zeta) \frac{e^{-|\zeta|^2}}{[(s + u + 2)/2]!} d\zeta.$$ 

Note that $tr K_{t,L}(x, x)$ is smooth on $\mathcal{N}_\rho(N)$. It follows that

$$tr \gamma(Tc^{-1}\lambda) = \lim_{t \to 0} \int_{\mathcal{N}_\rho(N)} tr K_{t,L}(x, x) d\mu(x)$$

$$= \lim_{t \to 0} \int_{\mathcal{N}_\rho(N)} tr (K_{t,L}(x, x)) L d\nu.$$ 

We write $c_{r,s}^m(x, \zeta)$ for $A_x b_{r,s}^m(f(x), \zeta)/[(s + u + 2)/2]!$ and we remark that $c_{r,s}^m$ is homogeneous of degree $s$ in $\zeta$ with coefficients given by canonical polynomials in $A,f$, the $a_\mu^m$, the $b_\mu^m$, the metrics, and their derivatives to a finite order.

We consider two cases.

**Case I.** $L \subset N$ so $N^L_1 = L$.

Recall that we have assumed that $f$ is non degenerate along the fixed point set and that the fixed point set is a finite disjoint union of submanifolds transverse to $F$. The first condition implies that if $f$ is the identity on an open set in a leaf, then it is the identity on all of the leaf. The second condition implies that if $f$ is the identity on a single leaf, then it is the identity map on all of $M$.
In this case, \( trK_{T,L}^T(x, x) \) is given by
\[
\sum_{s=0}^r t^{(u-p)/2} \sum_{s=0}^{3u} trC_{u,s}(x, \zeta) e^{-1/2} d\zeta.
\]
If \( p = \dim L \) is odd, there is no zeroth order term and we set \( a_i^0 = 0 \). If \( p \) is even we set
\[
a_i^L(x) = \frac{2^p}{s=0} \int \left( trC_{u,s}(x, \zeta) e^{-1/2} (dx/\lambda) d\zeta \right) \lambda
\]
and we note that \( a_i^L \) is given by a canonical polynomial in \( A, f \) the \( a_u^m \), the metrics, and their derivatives to a finite order.

**Case 2.** \( L \) not contained in \( N, N \cap L = U_j N_j^L \).

By the remarks above, we have that no leaf is contained in \( N \). \( tr(K^T_{T,L}(x, x)) \lambda^L \) is a differential \( p \) form \( \omega \) on \( \mathcal{N}_p(N) \). Integration over the fiber of \( \mathcal{N}_p(N) \), denoted by \( f \), of this form yields a smooth density on each \( N_j^L \).

**Lemma 5.7.**
\[
\int_N \left( \int_{\mathcal{N}_p(N)} tr(K_{T,L}^T(x, x)) \lambda^L \right) dv = \int_{\mathcal{N}_p(N)} tr(K_{T,L}^T(x, x)) \lambda^L dv
\]

**Proof.** Let \( \{(U_i, \phi_i)\} \) be a good coordinate system. Let \( R \) be the complete Borel transversal arising from this coordinate system. Define \( \pi: \mathcal{N}_p(N) \rightarrow R \) as follows. For \( x \in N \), \( x \) lying on the plaque \( P_z \) of \( U_1 \), define \( \pi(\mathcal{N}_p(N)_x) = z \). For \( x \in N \), \( x \in U_2 - U_1 \), lying on the plaque \( p_y \) of \( U_2 \) define \( \pi(\mathcal{N}_p(N)_x) = y \). Continue in this manner. Then for \( z \in R \) we have
\[
\pi^{-1}(z) = \bigcup_{\pi(x) = z} \mathcal{N}_p(N)_x.
\]
Let \( \pi_1 \) be \( \pi \) restricted to \( N \). Then a straightforward argument shows
\[
\int_{\pi^{-1}(z)} tr(K_{T,L}^T(x, x)) \lambda^L = \int_{\pi_1^{-1}(z)} tr(K_{T,L}^T(x, x)) \lambda^L.
\]
Recall that for any differential \( p \) form \( \omega \),
\[
\int_{\mathcal{N}_p(N)} \omega dv = \int_R \int_{\pi^{-1}(z)} \omega dv
\]
and
\[
\int_N \left( \int_{\mathcal{N}_p(N)} \omega \right) dv = \int_R \left( \int_{\pi_1^{-1}(z)} \omega \right) dv.
\]
From this the lemma follows.

Let \( \omega \) be a \( p \) form and let \( z \in N_j^L \subset N \). As \( \mathcal{N}_p(N)_z = \mathcal{N}_p(N_j^L)_z \), to compute \( \int_{\mathcal{N}_p(N)} \omega \) we need only compute \( \int_{\mathcal{N}_p(N_j^L)} \omega \).

**Proposition 5.8.** \( \int_{\mathcal{N}_p(N_j^L)} tr(K_{T,L}^T(x, x)) \lambda^L \) is asymptotic as \( t \rightarrow 0 \) to a finite sum of the form
\[
\sum_{x, z \in U} t^{(u+w-n)/2} d_{u,s}
\]
where \( \dim N_j^L = n \) and \( d_{u,s} \) is a smooth measure (density) on \( N_j^L \) which depends only on \( A, f \), the \( a_u^m \) and \( b_u^m \), the metrics, and their derivatives to a finite order, only on \( N_j^L \). This asymptotic expansion is uniform on \( N \), i.e. given \( \varepsilon \), there is a constant \( C(\varepsilon) \) depending only on \( \varepsilon \) so that for
A LEFSCHETZ THEOREM FOR FOLIATED MANIFOLDS 153

all $N_f$

$$\int_{\gamma_\tau(N_f)} \text{tr}(K^T_{I,L}(x,x)) \lambda^L - \sum_{j,u} t^{(u+n-1)/2} d_{u,j} \leq C(x) t^*$$

as $t \to 0$.

This is essentially the content of pp. 104–105 of [14]. The only part of the proposition not proven there is the fact that $C(x)$ depends only on $a$. However, this is implicit in Gilkey's proof. As his notation is somewhat different from ours, we shall repeat his proof.

On $M_f$, we have the quotient bundle $\mathcal{N}_f = (T_L|N_f)/TN_f$. Since $f$ is non-degenerate along $N_f$, the map $df$, which $df_{x,y}$ induces on $\mathcal{N}_f$, does not have $1$ as an eigenvalue. If we put $df_{x,y}$ into Jordan normal form on $T_L|N_f$, this induces a natural splitting

$$T_L|N_f = TN_f \oplus \mathcal{N}_f$$

where $TN_f$ is the space of eigenvectors of eigenvalue $1$ and $\mathcal{N}_f$ is now the space of generalized eigenvectors for the remaining eigenvalues. We assume that the metric on $M$ is chosen so that the splitting above is orthogonal on $L$ under the induced metric.

Let $x_1, \ldots, x_n$ be local coordinates on $N_f$ and $e_1, \ldots, e_{p-n}$ be an orthonormal frame for $\mathcal{N}_f$. This induces coordinates $(x, z) = (x_1, \ldots, x_n, z_1, \ldots, z_{p-n})$ on $\mathcal{N}_f$ where $(x_1, \ldots, x_n, z_1, \ldots, z_{p-n})$ are the coordinates of $y = \sum z_je_j(x)$ and are chosen so that $(z_1, \ldots, z_{p-n}, x_1, \ldots, x_n)$ is positively oriented relative to the given orientation on $L$.

Using the geodesic flow (on $L$), we identify a neighborhood of the zero section in $N_f$ with $\mathcal{N}_f(N_f)$. Thus $(x, z)$ are the coordinates of the point $\exp(y) \in L$, and $N_f = \{ (x, z) | z = 0 \}$.

On $\mathcal{N}_f(N_f)$, we have $y = (x, z)$ and we decompose $f(y)$ as $(f_1(y), f_2(y))$. Then on $T_L|N_f$

$$df_{x,z,0} = \begin{bmatrix} 1 & 0 \\ 0 & df_{x,y} \end{bmatrix}$$

by definition of $\mathcal{N}_f$. Thus $f_1(y) - x$ vanishes to second order in $z$ along $N_f$. Decompose $\zeta \in T^* R^p$ as $\zeta = (\zeta_1, \zeta_2)$, then at the point $x = (x_1, \ldots, x_n)$ of $N_f$, with $dz \wedge dx = dz_1 \wedge \ldots \wedge dz_{p-n} \wedge dx_1 \wedge \ldots \wedge dx_n$, $\int_{\mathcal{N}_f(N_f)} \text{tr}(K^T_{I,L}(x,x)) \lambda^L$ is an $n$ density on $N_f$ given locally by

$$\sum_{u} d^u \rho/2 \left( \int e^{i(f_1(x,z) - x)\zeta_1/\rho/2} e^{i(f_2(x,z) - z)\zeta_2/\rho/2} \times e^{-|\zeta|^2} c_w^a(x,z,\zeta_1,\zeta_2) g(x,z) d\zeta_1 d\zeta_2 dz \right) dx_1 \wedge dx_2 \wedge \ldots \wedge dx_n$$

where

$$c_w^a(x,z,\zeta_1,\zeta_2) = \sum_{\zeta} \text{tr} c_w^a(x,z,\zeta)$$

and

$$\lambda^L = g(x,z) dz \wedge dx.$$

Since $f$ is non-degenerate, we can change variables in the fiber coordinates by $(x, w) = (x, f_2(x, z) - z)$. The integral transforms to

$$\sum_{u} d^u \rho/2 \left( \int e^{i(f_1(x,w) - x)\zeta_1/\rho/2} e^{i(f_2(x,w) - w)\zeta_2/\rho/2} e^{-|\zeta|^2} \times c_w^a(x,w,\zeta_1,\zeta_2) g(x,w) |\det(I - df_{x,y})|^{-1} (x, w) d\zeta_1 d\zeta_2 dw \right) dx_1 \wedge \ldots \wedge dx_n.$$
We now replace $w$ by $t^{1/2} w$ to obtain

$$
\sum_u t^{(s+n-u)/2} \left( \int e^{i(f((x, t^{1/2} w) - x)/t^{1/2})} e^{i|\zeta|^2} e^{-|\zeta|^2}
\times c_u^w(x, t^{1/2} w, \zeta_1, \zeta_2) g(x, t^{1/2} w)|\det(I - df_x)|^{-1}(x, t^{1/2} w) d\zeta_1 d\zeta_2 dw \right) dx_1 \wedge \ldots \wedge dx_n.
$$

As $t \to 0$, $dw$ ranges over the entire fiber of $\mathcal{N}_\mathcal{F}$ over $\eta$, and $d\zeta_1, d\zeta_2$ ranges over the fiber of $T^* \mathcal{R}^n$ over $x$. Define

$$
a_u(x, w, \zeta, t) = e^{i(f((x, t^{1/2} w) - x)/t^{1/2})} e^{i|\zeta|^2} e^{-|\zeta|^2} \frac{d\zeta_1 d\zeta_2 dw}{dx_1 \wedge \ldots \wedge dx_n}.
$$

Since $f((x, z) - x$ vanishes to second order in $z$, $(f((x, t^{1/2} w) - x)/t^{1/2}$ is smooth in $t^{1/2}$ and vanishes to first order in $t^{1/2}$. If we expand $a_u$ in a Taylor series in $t^{1/2}$, it has the form

$$
\sum_s t^{s+u-n/2} a_{u,s}(x, w, \zeta) \quad \text{where } a_{u,s} \text{ is a polynomial in the (w, \zeta) variables with coefficients which depend on the jets of } f \text{ and } c_w \text{ at } w = 0, \text{ i.e. on } N_{\mathcal{F}}^x.
$$

The difference between $a_u$ and its Taylor polynomial is dominated by a function of the form $b_{u,s}(\zeta, w)$ where $b_{u,s}(\zeta, w)$ is a finite polynomial in $\zeta$ and $w$ with globally bounded (on $\mathcal{N}_{\mathcal{F}}(N)$) coefficients. We can see this by computing the $s$-th derivative of the term $a_u$ with respect to $t^{1/2}$. All the terms which are not powers of $w$ or $\zeta$ are then globally bounded on $\mathcal{N}_{\mathcal{F}}(N)$, so just replace them by the upper bounds of their absolute values.

Now the error in replacing $a_u$ by its Taylor polynomial in the computation of the integral is given by

$$
\sum_u t^{(s+n-u)/2} \int b_{u,s}(\zeta, w) e^{iw_2} e^{-|\zeta|^2} d\zeta dw.
$$

Since $b_{u,s}$ is polynomial in $(w, \zeta)$ the $\zeta$ integral yields a polynomial in $w$ multiplied by $
\exp(-|w|^2)$. Thus the iterated integral is well defined. Since the coefficients of the $b_{u,s}$ are globally bounded on $\mathcal{N}_{\mathcal{F}}(N)$ we have that

$$
\left( \sum_u t^{(s+n-u)/2} \left( \int a_{u,s}(x, w, \zeta) e^{iw_2} e^{-|\zeta|^2} d\zeta dw \right) dx_1 \wedge dx_2 \wedge \ldots \wedge dx_n \right)
$$

is (on $N$) asymptotic as $t \to 0$ to

$$
\left( \sum_u t^{(s+n-u)/2} \left( \int a_{u,s}(x, w, \zeta) e^{iw_2} e^{-|\zeta|^2} d\zeta dw \right) dx_1 \wedge dx_2 \wedge \ldots \wedge dx_n \right)
$$

Since $a_{u,s}(x, w, \zeta)$ is polynomial in $(w, \zeta)$, we may conclude just as we did above that this iterated integral is well defined. One checks easily that $a_{u,s}(x, w, \zeta)$ is of odd order in $(w, \zeta)$ if $u + s$ is odd, so the integrals vanish in this case. Now we set

$$
d_{u,s} = \left( \int a_{u,s}(x, w, \zeta) e^{iw_2} e^{-|\zeta|^2} d\zeta dw \right) dx_1 \wedge \ldots \wedge dx_n.
$$

To complete the proof of the theorem we proceed as follows:

if $\dim N_{\mathcal{F}} = n$ is odd, we set $a^{:7} = 0$ as the zeroth order term is zero in this case. If $n$ is even we set

$$
a^{:7} = \sum_{u,s} d_{u,s}.
$$
Then in both cases we have:
\[ \text{tr}^L(Te^{-tA}) = \int_N a \, dv \]

where \( a = \{a_f\} \) and \( a_f \) depends only on \( f, A, \) the symbol of \( \Delta, \) the metrics and their derivatives to a finite order on \( N_f. \)

To identify the \( a_f \) for the classical complexes, we first note that this is purely a local question on \( L. \) Then we may appeal to [14], [15], and [3] where these calculations are made. Note that for the Dolbeault complex, we must assume that the leaves are holomorphic manifolds and that the induced metric is Kähler to conclude directly that \( a_f \) is the classical integrand. For the general Dolbeault complex, the \( a_f \) and the classical integrand, thought of as densities on the \( N_f, \) differ by a functorial exact form, so their integrals are the same.

### 6. APPLICATIONS.

We close with a few applications of the main theorem (3.1). The first two theorems, (6.1) and (6.2), are implicit in the work of Connes.

**The General Index Theorem.**

If, in our Lefschetz Theorem, we take \( f \) to be the identity map \( I_M \) of \( M \) and \( T = I, \) we obtain an index theorem for leafwise Dirac operators. In particular, if \( Q \) is any Hermitian bundle over \( M, \) then by tensoring the leafwise signature complex on \( M \) with \( Q, \) we obtain a twisted signature complex along \( F \) and this complex is a Dirac complex. Now it is well known [3] that in the classical case, the index theorem for twisted signature complexes leads, by a purely topological proof, to the index theorem for arbitrary elliptic complexes. The same is true in our case. In fact, one may use Connes' proof of this fact, [7], [17] chapter viii essentially without change to prove the following.

**Theorem 6.1.** Let \( M, F, v, \) and \( (E, d) \) be as in Theorem (3.1), but require only that \((E^L, d^L)\) be an elliptic complex for each \( L. \) Set \( I_*(E, d) = L_*(I_M). \) Then
\[ I_*(E, d) = \int_M a \, dv \]

where \( a = \{a^L\} \) and \( a^L \) is the usual local integrand (in terms of characteristic differential forms) for the index of \((E^L, d^L)\) given by the Atiyah-Singer Index Theorem.

**Connes' Index Theorem and an Analogue of the Atiyah \( L^2 \) Covering Index Theorem.**

If, in our proof of the Lefschetz Theorem we take \( f = I_M \) and \( T = I \) and systematically replace the leaves \( L \) by their holonomy coverings, we obtain John Roe's proof of the Connes' Index Theorem for Dirac operators [21]. To extend to all leafwise elliptic complexes, one proceeds as in [7], [17].

Now Connes' Index Theorem expresses the index of the complex \((E, d)\) lifted to the holonomy groupoid as an integral over \( M \) of certain characteristic forms defined on the leaves of \( F. \) These characteristic forms are identical to the ones given in Theorem (6.1). Thus we have the following analogue of the Atiyah \( L^2 \) covering Index Theorem [1].
THEOREM 6.2. Let $M$, $F$, $v$ and $(E, d)$ be as in Theorem (6.1). Denote the Connes Index of $(E, d)$ lifted to the holonomy groupoid by $l_0^c(E, d)$. Then

$$l_0^c(E, d) = l_1(E, d).$$

One would like to prove a Lefschetz Theorem for $f$, $T$ and $(E, d)$ lifted to the holonomy groupoid $G$. If this could be done, it would have as a corollary a Lefschetz $L^2$ covering theorem. Unfortunately, it appears to be necessary to assume that $f$ and $T$ have lifts to $G$ which commute with the action of $G$ on $G$ and this rarely happens.

Foliations by Surfaces and Flat Manifolds.

Let $M$, $F$ and $v$ be as in Theorem (3.1), however we assume in this section that $v$ is a non negative transverse measure. Let $F: M \times [0, 1] \to M$ be a smooth one parameter family of leaf preserving diffeomorphisms. For each $t$, $F_t(x) = F(x, t)$ induces, on each leaf $L$, a map on $L^2$ harmonic $k$-forms which we denote by $F^{L,k}_t$.

LEMMA 6.3. The maps $F^{L,k}_t = F^{L,k}_i$ on $L^2$ harmonic forms.

Proof. Let $I = [0, 1]$, let $\pi: L \times I \to L$ be the projection, and $i_0, i_1$ the inclusions of $L$ into $L \times \{0\}, L \times \{1\}$. For the purposes of this proof only let $f^{**}$ denote the map on differential forms induced by a smooth function $f$.

Let $\pi_*$ be the map given by integration over the fiber $I$. If $\omega = \omega_1 + \omega_0 \wedge dt$, then

$$\pi_*(\omega)(x) = \int_0^1 \omega_0(x, t) dt.$$

The well known formula for integration over the fiber yields

$$\pm (d\pi_* + \pi_* d) = i_{1**} - i_{0**}$$

Thus

$$\pm (d\pi_* F^{**} + \pi_* dF^{**}) = F_{1**} - F_{0**}.$$

Now, if $\omega$ is an $L^2$ form, then a change of variables argument similar to (2.3.1) shows that $F^{**}(\omega)$ is an $L^2$ form and a simple argument using the Cauchy–Schwartz inequality shows $\pi_*$ takes $L^2$ forms to $L^2$ forms. If $\omega$ is harmonic, $2F^{**}\omega = F^{**}d\omega = 0$, so

$$F_{1**}(\omega) - F_{0**}(\omega) = \pm d\sigma$$

where $\sigma$ is an $L^2$ form. Now apply $P_k^L$, the projection onto harmonic $k$-forms and recall that the Hodge theorem (2.1.1) for the DeRham complex tells us that $P_k^L(d\sigma) = 0$. Since

$$F^{L,k}_{i*, k} = P_k^L F_{i**}^L P_k^L$$

for $i = 0, 1$

we have the lemma.

COROLLARY 6.4. $L_*(F_0) = L_*(F_1)$

In view of the definition of $T_k^L$ of (2.2.1) the following definitions are appropriate. Let $(E, d)$ be a Dirac complex and $P_i = \{P_i^L\}$ be the family of projections on $\{\ker(\Delta_i^L)\}$.

Definition 6.5. $\dim_i H^i(E, d) = tr_v(P_i)$.

$$\chi_*(F; E) = \sum_{i=0}^k (-1)^i \dim_i H^i(E, d)$$
When \((E, d)\) is the leafwise DeRham complex we just write \(\chi_\ast(F)\). The corollaries following the next theorem are generalizations of the classical theorem on the finiteness of the automorphism group of a compact, oriented Riemann surface of higher genus.

**Theorem 6.6.** Let \(M\) be a compact connected oriented Reimannian three manifold with an oriented foliation \(F\) by surfaces. Assume that there is a non-negative invariant transverse measure \(v\) for \(F\) with the property that \(\chi_\ast(F) < 0\). Suppose \(f\) is an isometry of \(M\) taking each leaf to itself and \(f\) is not the identity. Then \(f\) is not homotopic to the identity through leaf preserving diffeomorphisms of \(M\).

**Proof.** Assume \(f\) is homotopic to the identity through leaf preserving diffeomorphisms. As \(\chi_\ast(F) < 0\), \(f\) must have fixed points, since if \(f\) had no fixed points we could apply the Lefschetz Theorem to the leafwise DeRham complex to obtain \(L_\ast(f) = 0\). But then (6.4) would imply \(\chi_\ast(F) = L_\ast(1) = L_\ast(f) = 0\).

If \(p\) is a fixed point of \(f\) on the leaf \(L\), \(d(f|L)_p\) is orientation preserving (since \(f = 1\)), and it must be rotation through \(\theta\), with \(\cos \theta \neq 1\). For if not, \(d(f|L)_p\) is a rotation in \(TL_p\), thus \(d(f|L)_p = 1\) on \(TM_p\) and \(f = 1\) as \(M\) is connected. Thus \(p\) is a non-isolated non-degenerate fixed point on \(L\) and sign \((\det(I - d(f|L)_p))\) = 1.

Let \(N\) be the fixed point set of \(f\). The fixed point set of an isometry of a compact manifold \(M\) is a closed submanifold of \(M\). We must show that \(N\) is transverse to \(F\). For \(p \in N\), lying on \(L\), \(d(f|L)_p\) is an oriented isometry of \(TM_p\), which is a rotation in \(TL_p\), thus \(d(f|L)_p\) must be the identity on the orthogonal complement to \(TL_p\). It follows that \(f\) is the identity on an arc at \(p\) transverse to \(L\). Thus \(N\) is transverse to each leaf and \(N\) is the finite union of closed, connected one dimensional submanifolds.

Now if we apply the Lefschetz Theorem to the map \(f\) acting on the leafwise DeRham complex we have

\[ L_\ast(f) = \int_N (+1) \, dv = v(N) \geq 0. \]

But as above \(L_\ast(f) = L_\ast(1) = \chi_\ast(F) < 0\).

**Note.** The same theorem holds if we assume that each leaf is a Riemann surface and \(f\) is holomorphic on each leaf. This is because any non identity holomorphic map of a connected Riemann surface (other than \(S^2\)) must have isolated non-degenerate fixed points with fixed point index +1. and under the hypothesis on \(\chi_\ast(F)\) no leaf can be \(S^2\).

For the statement of the next three corollaries we assume that \(M, F, v\) are as in (6.6).

**Corollary 6.7.** Let \(O(M; F)\) be the group of isometries of \(M\) taking each leaf to itself. Then \(O(M; F)\) is totally pathwise disconnected.

**Corollary 6.8.** No compact connected Lie group can act non-trivially as a group of leaf preserving isometries of \(M\).

**Corollary 6.9.** If all of the leaves of \(M\) are compact, then \(O(M; F)\) is finite.

**Proof.** In this case \(O(M; F)\) is a closed subgroup of the isometry group of \(M\) and the corollary follows from (6.8).
Note. In general it is not true that $O(M; F)$ is a closed subgroup of the isometry group. There are simple examples of foliations of $\Sigma_2 \times S^1$ where $O(M; F)$ is not closed.

We say a set $S$ of leaves is negligible (with respect to $\nu$) if for any transversal $T$, $S \cap T$ has measure zero. As a generalization of (6.6) we have:

Theorem 6.10. Let $M$ be a compact, connected, oriented Riemannian manifold with an oriented foliation $F$ by surfaces with a non-negative invariant transverse measure $\nu$, and $f$ a smooth diffeomorphism of $M$ not equal to the identity which takes each leaf to itself and is an isometry on each leaf. Assume that the set of fixed points $N$ is a closed submanifold which is transverse to $F$. If either

(a) $\chi_\nu(F) < 0$
(b) $\nu(N) > 0$ and the set of compact leaves is negligible

then $f$ is not homotopic to the identity through leaf preserving diffeomorphisms of $M$.

Proof. Assume $f$ is homotopic to the identity through leaf preserving diffeomorphisms, and note that $\chi_\nu(F) < 0$ implies $N \neq \emptyset$. Let $p \in N$ and let $L$ be the leaf through $p$. If $d(f|L)_p = i$ then $f|L = I$ and $L \subset N$. As $N$ is a submanifold transverse to $F$, $N = M$ and $f = I$. Thus $d(f|L)_p$ is rotation through an angle $\theta$ (as $f \sim I$, it is orientation preserving) and $\cos \theta \neq 1$, i.e. each fixed point of $f$ is isolated in its leaf, non-degenerate and sign$(\det(I - df|_L)_p) = 1$. Now by (6.3) $f^* = I$ on harmonic k-forms for $k = 0, 1, 2$. Thus we have for $f$ acting on the leafwise DeRham complex

$$\chi_\nu(F) = L_\nu(f) = \int_N \frac{1}{\nu} \text{sign}(\det(I_p - df_p)) dv = \int_N (+1) dv = \nu(N) \geq 0.$$ 

If $\chi_\nu(F) < 0$ we get an immediate contradiction.

For case $b$, we note that on a complete Riemannian manifold the $L^2$ harmonic functions are constant and so there can be non-zero $L^2$ harmonic functions only if the volume is finite. Now if $L$ is an open leaf in $M$ then the volume of $L$ relative to the metric induced from that on $M$ must be infinite. Thus for an open leaf, there are no $L^2$ harmonic functions and, by using the $\ast$ operator to establish an isomorphism between $L^2$ harmonic 0 and 2 forms, there are no $L^2$ harmonic 2 forms. Hence for the DeRham complex, $H^0(L, d) = H^2(L, d) = 0$. Thus if the set of compact leaves is negligible, it follows that $L_\nu(f) = -\dim H^1(E, d) \leq 0$. But $L_\nu(f) = \nu(N)$ so if $\nu(N) > 0$ we again arrive at a contradiction.

We get a similar result for leaves of all dimensions if we assume that the leaves are Riemannian flat.

Theorem 6.11. Let $M$ be a compact Riemannian manifold which is foliated by leaves which are Riemannian flat. Let $f$ be a smooth diffeomorphism of $M$ which takes each leaf to itself and has isolated non-degenerate fixed points on each leaf with fixed point index $+1$ at each fixed point. Assume that the set of compact leaves is negligible and the transversal $N$ of fixed points is a closed submanifold transverse to $F$. Then $\nu(N) = 0$.

Proof. Again we apply (3.1) to the DeRham complex. As in the previous theorem the right hand side $\int_N \nu dv = \nu(N)$. By [11] Theorem 1, if $L$ is a complete Riemannian manifold with infinite volume and Riemannian curvature identically zero then every $L^2$ harmonic form is zero. As open leaves have infinite volume, and the set of compact leaves is negligible, it follows that the left hand side, $L_\nu(f) = 0$. Thus $\nu(N) = 0$. 


Remark. Instances of this theorem arise by taking a cocompact discrete subgroup $\Gamma$ of the affine maps acting on $\mathbb{R}^2$ and taking associated torus bundles as $M$. The result is a flat torus bundle over $\mathbb{R}^2/\Gamma$ with leaves covered by $\mathbb{R}^2$.

**Stable Fiber Theorem.**

The following is a special case of The Stable Fiber Theorem of [13] (for a generalization of Fuller's theorem see [6]).

**Theorem 6.12.** Let $\alpha: M \to B$ be a smooth flat $S^1$ bundle over a compact manifold $B$, and assume that $M$ is orientable. Assume that the Euler number of $B$, $\chi(B) \neq 0$, that the foliation $F_s$ of $M$ given by the fibration is orientable and that the natural foliation $F$ of $M$ given by the flat structure admits a nontrivial invariant transverse measure $\nu$. Then any smooth perturbation of $F_s$ sufficiently close to $F_s$ has a compact leaf (i.e. some fiber is stable under the perturbation).

**Note.** To construct such an $M$ one need only choose $B$ with $\chi(B) \neq 0$, $\pi_1(B)$ amenable and construct a homomorphism $h$ from $\pi_1(B)$ to the orientation preserving diffeomorphisms of $S^1$. Then set $M = \tilde{B} \times S^1$ where $\tilde{B}$ is the universal cover of $B$.

**Proof.** Let $\tilde{F}$ be the perturbation of $F_s$. As $F_s$ is oriented so is $\tilde{F}$ and $\tilde{F}$ defines a flow on $M$. Denote by $f: M \to M$ the first return map of $\tilde{F}$ with respect to $F$. This is defined as follows. Choose a metric on $B$ and $r > 0$ so that for each $x \in B$, the ball of radius $r$ about $x$, $D_r(x)$, is contained in a coordinate chart of $B$. Then $\pi^{-1}(D_r(x)) \approx D_r(x) \times S^1$ and we may choose the diffeomorphism so that for each $\theta \in S^1$, $D_r(x) \times \{\theta\}$ is contained in the leaf of $F$ through $(x, \theta)$. We require $\tilde{F}$ to be so close to $F_s$ that for each $x \in B$ and $\theta \in S^1$, the positive time trajectory (i.e. $t > 0$) of $\tilde{F}$ through $(x, \theta)$ intersects each $D_r(x) \times \{\theta_1\}, \theta_1 \in S^1$, before (possibly) exiting $\pi^{-1}(D_r(x))$. The value of $f$ at $(x, \theta)$ is the first point on the positive time trajectory of $\tilde{F}$ through $(x, \theta)$ in the set $D_r(x) \times \{\theta\}$. As $\tilde{F}$ is a smooth perturbation of $F_s$ and $M$ is compact, $f$ is a leaf preserving diffeomorphism (for $F$) of $M$ which is uniformly close to the identity map $f_M$ of $M$ and so is smoothly homotopic (through leaf preserving diffeomorphisms) to $f_M$. If $\tilde{F}$ has no compact leaves, $f$ has no fixed points and its $\nu$ Lefschetz number $L_\nu(f)$ (with respect to the leafwise DeRham complex) is zero. However as $f$ is homotopic to $f_M$, $L_\nu(f) = L_\nu(f_M)$ by corollary (6.4).

Let $\{\lambda^z\}$ be the family of Lebesgue measures on the leaves of $F$ induced from Lebesgue measure on $B$. We can construct an open set $U \subset B$ of full measure such that for fixed $x \in U$ and for each leaf $L$ of $F$, 

$$\pi^{-1}(U) \cap L = \bigcup_{x \in \pi^{-1}(x) \cap L} U_x$$

where $U_x$ is open, connected and $\pi(U_x) = U$. Neglecting sets of $\mu$ measure zero,

$$\pi^{-1}(U) \approx \bigcup_{x \in \pi^{-1}(x)} U_x \approx U \times \pi^{-1}(x)$$

as measure spaces. The measure on $\pi^{-1}(U)$ is $\mu$, and the measure on $U \times \pi^{-1}(x)$ is the product of the measure $\nu$ on $\pi^{-1}(x)$ and the family of measures $\lambda^z$ where $\lambda^z$ is the measure on $U \times \{z\}$ induced from $B$. The Atiyah $L^2$ covering index theorem applied to the DeRham
complex of $B$ and the leaf $L$ implies that

$$\int_{U_s} \sum_i (-1)^i \text{tr}(P_i^L) d\lambda^2 = \chi(B).$$

Then

$$L_s(I_M) = \int_M \sum_i (-1)^i \text{tr}(P_i^L) d\mu = \int_{\pi^{-1}(U)} \sum_i (-1)^i \text{tr}(P_i^L) d\mu$$

$$= \int_{\pi^{-1}(x)} \left[ \int_{U(x)} \sum_i (-1)^i \text{tr}(P_i^L) d\lambda^2 \right] dv$$

$$= \int_{\pi^{-1}(x)} \chi(B) dv = \chi(B) v(\pi^{-1}(x)).$$

As $\chi(B) \neq 0$ and $v$ is non trivial, $L_s(I_M) = L_s(f) \neq 0$. The theorem follows.

Note that we may replace $x(B) \neq 0$ by the requirement that there is an elliptic complex $(E, d)$ over $B$ whose index $I(E, d) \neq 0$. We then lift this complex to an elliptic complex along $F$ and proceed as above. Since $L_s(f) = I(E, d)v(S^1) \neq 0$, the conclusion remains valid.

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APPENDIX

Proof of the Hodge Theorem.

We prove the Hodge theorem for a complete Riemannian manifold \( L \) and an elliptic complex \( \{ E_i, d_i \} \) such that \((-1)^{i/2} D \) satisfies Theorem (1.3) of [9]. This implies that the \( \Delta_i \) are essentially self adjoint so the spectral theorem may be applied to \( \Delta_i \). This is certainly the case if \( D \) is a Dirac operator. This condition also holds if \( L \) is a leaf of a foliation of a compact manifold and \( \{ E_i, d_i \} \) is the restriction to \( L \) of a leafwise elliptic complex of first order operators.

**Lemma A.1.** \( \ker(\Delta_i) \subset \ker(d_i) \cap \ker(d_i^{*-1}) \).

**Proof.** Let \( s \in \text{domain}(\Delta_i) \) and \( s_j \in C_0(E_i) \) be a sequence converging to \( s \) with \( \Delta_i s_j \) convergent. If \( z_j \) is a convergent sequence in a Hilbert space and \( z_j = v_j + w_j \) where \( \langle v_k, w_j \rangle = 0 \) for all \( k, j \), then both \( \{ v_j \} \) and \( \{ w_j \} \) are convergent sequences. Now,

\[
\Delta_i s_j \equiv d_i^* d_i s_j + d_{i-1}^* d_{i-1}^{*-1} s_j
\]

as \( s_j \in C_0(E_i) \). As \( d_i d_{i-1} = 0 \) we have that

\[
\langle d_i^* d_i s_j, d_{i-1}^* d_{i-1}^{*-1} s_j \rangle = 0 \quad \text{for all } k, j.
\]

Thus both \( d_i^* d_i s_j \) and \( d_{i-1}^* d_{i-1}^{*-1} s_j \) converge, and

\[
\text{domain}(\Delta_i) \subset \text{domain}(d_i^* d_i) \cap \text{domain}(d_{i-1}^* d_{i-1}^{*-1}).
\]

But if \( \Delta_i s_j \to 0 \), then both \( d_i^* d_i s_j \) and \( d_{i-1}^* d_{i-1}^{*-1} s_j \) converge to 0. Thus

\[
\langle d_i^* d_i s_j, s_j \rangle \to 0.
\]

But

\[
\langle d_i^* d_i s_j, s_j \rangle = \langle d_i s_j, d_i s_j \rangle \text{ so } d_i s_j \to 0. \text{ Similarly for } d_{i-1}^*.
\]

**Lemma A.2.** \( \ker(\Delta_i) \cap \text{image} (d_{i-1}) = 0 \)

**Proof.** Let \( s \) be in this intersection. Let \( s_n \in \text{dom}(d_{i-1}) \) be a sequence with \( d_{i-1} s_n \to s \). For each \( n \) let \( s_{n,j} \in C_0(E_{i-1}) \) be a sequence so that \( s_{n,j} \to s_n \) and \( d_{i-1} s_{n,j} \to d_{i-1} s_n \). Choose a subsequence \( s_j \) of the \( s_{n,j} \) so that \( d_{i-1} s_j \to s \). The \( s_j \) are compactly supported smooth sections. Now

\[
\langle s, s \rangle = \lim \langle d_{i-1} s_j, s \rangle
\]

and

\[
\langle d_{i-1} s_j, s \rangle = \langle s_j, d_{i-1}^* s \rangle = 0.
\]

We may do this since \( s \in \ker(\Delta_i) \subset C_0(E_i) \) and \( s_l \) is smooth and compactly supported. Hence \( \langle s, s \rangle = 0 \) and we have the lemma.

Recall that \( H'(E, d) = \ker d_i / \text{image} d_{i-1} \).
COROLLARY A.3. The map \( \ker(\Delta_i) \to H^i(E, d) \) induced by the inclusion of \( \ker(\Delta_i) \) in \( \ker(d_i) \) is injective.

We shall show presently that this map is surjective.

LEMMA A.4. \( L^2(E_i) = \ker \Delta_i \oplus \text{image } \Delta_i \)

Proof. For each \( t > 0 \), define functions \( \psi_t(x) \) and \( \phi_t(x) \) on \([0, \infty)\) by
\[
\psi_t(x) = \frac{e^{-tx} - 1}{x},
\phi_t(x) = e^{-tx}
\]
Both \( \psi_t \) and \( \phi_t \) are bounded Borel functions on \([0, \infty)\) so \( \psi_t(\Delta_i) \) and \( \phi_t(\Delta_i) \) are bounded operators on \( L^2(E_i) \). Furthermore, the spectral theorem says that as \( t \to \infty \), \( \phi_t(\Delta_i) \) converges strongly to \( P_i \), the projection on the kernel of \( \Delta_i \). If we set \( f(x) = x \), then \( f(x)\psi_t(x) \) is also a bounded function on \([0, \infty)\), so \( f\psi_t(\Delta_i) \) is a bounded operator on \( L^2(E_i) \). Now [12] vol. 2. p. 1199 says that
\[
\text{dom}(f(\Delta_i)\psi_t(\Delta_i)) = \text{dom}(f(\Delta_i)\psi_t(\Delta_i)) \cap \text{dom}(\psi_t(\Delta_i))
\]
so the domain of \( f(\Delta_i)\psi_t(\Delta_i) = \Delta_i \psi_t(\Delta_i) \) is \( L^2(E_i) \) and for all \( s \in L^2(E_i) \) we have \( \psi_t(\Delta_i)s \in \text{dom}(\Delta_i) \). Also, since \( \Delta_i \psi_t(\Delta_i) \) and \( \exp(-t\Delta_i) - I \) have the same domain, namely \( L^2(E_i) \), they are the same operator.

Let \( s \in L^2(E_i) \). Then \( \phi_t(\Delta_i)s - s + \phi_t(\Delta_i)s = s + (\Delta_i \psi_t(\Delta_i))s \). As \( t \to \infty \), \( \phi_t(\Delta_i)s \to P_i s \) so \( \Delta_i(\psi_t(\Delta_i)s) \) also must converge and converges in the closure of \( \text{image}(\Delta_i) \). Thus
\[
L^2(E_i) = \ker \Delta_i + \text{image } \Delta_i
\]
Now let \( s \in \ker(\Delta_i) \) and \( s_i \in \text{dom}(\Delta_i) \). \( \Delta_i \) is self adjoint on its domain so we have
\[
\langle \Delta_is, s \rangle = \langle s, \Delta_is \rangle = 0.
\]
Thus \( \ker(\Delta_i) \) is orthogonal to \( \text{image}(\Delta_i) \) and so \( \ker(\Delta_i) \) is orthogonal to the closure of \( \text{image}(\Delta_i) \), hence
\[
L^2(E_i) = \ker \Delta_i \oplus \text{image } \Delta_i
\]
To finish the proof of the Hodge theorem we have

LEMMA A.5. \( \text{image } \Delta_i = \text{image } d^*_i \oplus \text{image } d_{i-1}^* \).

Proof. If \( s \in C_\infty^0(E_{i+1}), s_j \in C_\infty^0(E_{i-1}), \langle d^*_is, d_{i-1}^*s_j \rangle = 0 \) so we have
\[
\text{image } d^*_i \cap \text{image } d_{i-1}^* = 0
\]
Now suppose that \( s \) is in the closure of \( \text{image}(\Delta_i) \). As above we may choose \( s_j \in C_\infty^0(E_i) \) with \( \Delta_is \to s \). Then
\[
\Delta_is_j = (d^*_id_is_j) + (d_{i-1}^*d_{i-1}^*s_j)
\]
and as above both \( d^*_i(d_is_j) \) and \( d_{i-1}^*(d^*_is_{i-1}j) \) converge. Thus
\[
\Delta_is \in \text{image } d^*_i + \text{image } d_{i-1}^*.
\]
We now show \( \ker(\Delta_i) \) is perpendicular to this subspace. If \( s \in \ker(\Delta_i) \) then \( s \in \ker(d_i) \cap \ker(d_{i-1}^*) \) and \( s \in C_\infty^0(E_i) \). Let \( s_1 \) be in the closure of \( \text{image}(d_i^*) \) and choose a sequence \( s_j \in C_\infty^0(E_{i+1}) \) so that \( d_i^*s_j \to s_1 \). Then since \( s \) is smooth and \( s_1 \) is compactly supported, we have
\[
\langle s, d_i^*s_j \rangle = \langle d_is, s_j \rangle = 0.
\]
Thus \( \langle s, s_1 \rangle = 0 \). A similar argument shows that \( \langle s, s_1 \rangle = 0 \) for \( s_1 \) in the closure of \( \text{image}(d_{i-1}^*) \), and we have the Hodge theorem.