Advances in Applied Mathematics 42 (2009) 392–421

Contents lists available at ScienceDirect

Advances in Applied Mathematics

www.elsevier.com/locate/yaama

Invariants, Kronecker products, and combinatorics of some remarkable Diophantine systems

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ARTICLE INFO

Article history:
Received 3 July 2008
Accepted 17 September 2008
Available online 8 January 2009

MSC:
primary 05A15
secondary 05E05, 34C14, 11D45

Keywords:
Invariant
Kronecker product
Diophantine system
Hilbert series

ABSTRACT

This work lies across three areas (in the title) of investigation that are by themselves of independent interest. A problem that arose in quantum computing led us to a link that tied these areas together. This link consists of a single formal power series with a multifaced interpretation. The deeper exploration of this link yielded results as well as methods for solving some numerical problems in each of these separate areas.

1. Introduction

Since our work may be of interest to audiences of varied background we will try to keep our notation as elementary as possible and entirely self contained.

The problem in invariant theory that was the point of departure in our investigation is best stated in its simplest and most elementary version. Given two matrices $A = \left[\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{smallmatrix}\right]$ and $B = \left[\begin{smallmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{smallmatrix}\right]$ of determinants 1, or equivalently in $SL[2] := SL(2, \mathbb{C})$, we recall that their tensor product may be written in the block form

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1 The fourth named author was supported by the 973 Project, the PCSIRT Project of the Ministry of Education, the Ministry of Science and Technology, and the National Science Foundation of China.

doi:10.1016/j.aam.2008.09.002

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B \\ a_{21}B & a_{22}B \end{bmatrix}. \tag{1}$$

We also recall that the action of a matrix $M = [m_{ij}]_{i,j=1}^{n}$ on a polynomial $P(x)$ in $\mathbf{R}_n := \mathbb{C}[x_1, x_2, \ldots, x_n]$ may be defined by setting

$$T_M P(x) = P(xM), \tag{2}$$

where the symbol xM is to be interpreted as multiplication of a row n-vector by an $n \times n$ matrix. This given, we denote by $\mathbf{R}_4^{SL[2]\otimes SL[2]}$ the ring of polynomials in $\mathbf{R}_4$ that are invariant under the action of $A \otimes B$ for all pairs $A, B \in SL[2]$. In symbols

$$\mathbf{R}_4^{SL[2]\otimes SL[2]} = \bigl\{P \in \mathbf{R}_4\colon\ T_{A\otimes B} P(x) = P(x)\bigr\}. \tag{3}$$

Since the action in (2) preserves degree and homogeneity, $\mathbf{R}_4^{SL[2]\otimes SL[2]}$ is graded, and as a vector space it decomposes into the direct sum

$$\mathbf{R}_4^{SL[2]\otimes SL[2]} = \bigoplus_{m\geqslant 0} \mathcal{H}_m\bigl(\mathbf{R}_4^{SL[2]\otimes SL[2]}\bigr),$$

where the mth direct summand here denotes the subspace consisting of the $SL[2] \otimes SL[2]$-invariants that are homogeneous of degree m. The natural problem then arises to determine the Hilbert series

$$W_2(q) = \sum_{m\geqslant 0} q^m \dim \mathcal{H}_m\bigl(\mathbf{R}_4^{SL[2]\otimes SL[2]}\bigr).$$

Now note that using (1) iteratively we can define the k-fold tensor product $A_1 \otimes A_2 \otimes \cdots \otimes A_k$, and thus extend (3) to its general form

$$\mathbf{R}_{2^k}^{SL[2]\otimes SL[2]\otimes\cdots\otimes SL[2]} = \bigl\{P \in \mathbf{R}_{2^k}\colon\ T_{A_1\otimes A_2\otimes\cdots\otimes A_k} P(x) = P(x)\bigr\}$$

and set

$$W_k(q) = \sum_{m\geqslant 0} q^m \dim \mathcal{H}_m\bigl(\mathbf{R}_{2^k}^{SL[2]\otimes SL[2]\otimes\cdots\otimes SL[2]}\bigr).$$

Remarkably, to this date only the series $W_2(q), W_3(q), W_4(q), W_5(q)$ are known explicitly. Moreover, although the three series $W_2(q), W_3(q), W_4(q)$ may be hand computed, so far $W_5(q)$ has only been obtained by computer.

The third named author, using branching tables calculated in [9], was able to predict the explicit form of $W_5(q)$ by computing a sufficient number of its coefficients. The computation of these tables took approximately 50 hours using an array of 9 computers.

The series $W_4(q), W_5(q)$ first appeared in print in works of Luque and Thibon [5,6] which were motivated by the same problem of quantum computing. We understand that their computation of $W_5(q)$ was carried out by a brute force use of the partial fraction algorithm of the fourth named author, and it required several hours with the computers of that time.

The present work was carried out whilst unaware of the work of Luque–Thibon. Our main goal is to acquire a theoretical understanding of the combinatorics underlying such Hilbert series and give a more direct construction of $W_5(q)$ and perhaps bring $W_6(q)$ within reach of present computers.

Fortunately, as is often the case with a difficult problem, the methods that are developed to solve it may be more significant than the problem itself. This is no exception as we shall see.

Let us recall that the pointwise product of two characters $\chi^{(1)}$ and $\chi^{(2)}$ of the symmetric group S_n is also a character of S_n, and we shall denote it here by $\chi^{(1)} \odot \chi^{(2)}$. This is usually called the *Kronecker* product of $\chi^{(1)}$ and $\chi^{(2)}$. An outstanding yet unsolved problem is to obtain a combinatorial rule for the computation of the integer

$$c^{\lambda}_{\lambda^{(1)},\lambda^{(2)},\dots,\lambda^{(k)}} \tag{4}$$

giving the multiplicity of χ^{λ} in the Kronecker product $\chi^{\lambda^{(1)}} \odot \chi^{\lambda^{(2)}} \odot \cdots \odot \chi^{\lambda^{(k)}}$. Here χ^{λ} and each $\chi^{\lambda^{(i)}}$ are irreducible Young characters of S_n. Using the Frobenius map $\mathbf{F}$ that sends the irreducible character χ^{λ} onto the Schur function S_{λ}, we can define the Kronecker product of two homogeneous symmetric functions of the same degree f and g by setting

$$f \odot g = \mathbf{F}\big(\big(\mathbf{F}^{-1} f\big) \odot \big(\mathbf{F}^{-1} g\big)\big).$$

With this notation the coefficient in (4) may also be written in the form

$$c^{\lambda}_{\lambda^{(1)},\lambda^{(2)},\dots,\lambda^{(k)}} = \langle s_{\lambda^{(1)}} \odot s_{\lambda^{(2)}} \odot \cdots \odot s_{\lambda^{(k)}}, s_{\lambda}\rangle,$$

where $\langle\,,\rangle$ denotes the customary Hall scalar product of symmetric polynomials. The relevancy of all this to the previous problem is a consequence of the following identity.

Theorem 1.1.

$$W_k(q) = \sum_{d\geqslant 0} q^{2d} \langle s_{d,d} \odot s_{d,d} \odot \cdots \odot s_{d,d}, s_{2d}\rangle \tag{5}$$

where, in each term, the Kronecker product has k factors.

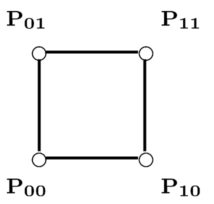
For this reason, we will often refer to the task of constructing $W_k(q)$ as the *Sdd Problem*. Using this connection and some auxiliary results on the Kronecker product of symmetric functions we derived in [3] that

$$W_2(q) = \frac{1}{1-q^2}, \qquad W_3(q) = \frac{1}{1-q^4}, \qquad W_4(q) = \frac{1}{(1-q^2)(1-q^4)^2(1-q^6)}. \tag{6}$$

Although this approach is worth pursuing (see [3]), the present investigation led us to another surprising facet of this problem.

Let us start with a special case. We are asked to place (nonnegative) integer weights on the vertices of the unit square so that all the sides have equal weights. Denoting by $P_{00}, P_{01}, P_{10}, P_{11}$ the vertices (see figure) and by $p_{00}, p_{01}, p_{10}, p_{11}$ their corresponding weights, we are led to the following Diophantine system:

$$\mathcal{S}_2\colon \quad \left\| \begin{aligned} p_{00} + p_{01} - p_{10} - p_{11} &= 0 \\ p_{00} - p_{01} + p_{10} - p_{11} &= 0 \end{aligned} \right.,$$



The general solution to this problem may be expressed as the formal series

$$F_2(y_{00}, y_{01}, y_{10}, y_{11}) = \sum_{p\in\mathcal{S}_2} y_{00}^{p_{00}} y_{01}^{p_{01}} y_{10}^{p_{10}} y_{11}^{p_{11}} = \frac{1}{(1-y_{00}y_{11})(1-y_{01}y_{10})}.$$

In particular, making the substitution $y_{00} = y_{01} = y_{10} = y_{11} = q$ we derive that the enumerator of solutions by total weight is given by the generating function

$$G_2(q) = \sum_{d \geqslant 0} m_d(2) q^{2d} = \frac{1}{(1-q^2)^2},$$

with $m_d(2)$ giving the number of solutions of total weight $2d$.

This problem generalizes to arbitrary dimensions. That is we seek to enumerate the distinct ways of placing weights on the vertices of the unit k-dimensional hypercube so that all hyperfaces have the same weight. Denoting by $p_{\epsilon_1\epsilon_2\cdots\epsilon_k}$ the weight we place on the vertex of coordinates $(\epsilon_1, \epsilon_2, \ldots, \epsilon_k)$ we obtain a Diophantine system $\mathcal{S}_k$ of k equations in the 2^k variables $\{p_{\epsilon_1\epsilon_2\cdots\epsilon_k}\}_{\epsilon_i=0,1}$.

The relevance of all this to the previous problem is a consequence of the following identity.

Theorem 1.2. *Denoting by $m_d(k)$ the number of solutions of the system $\mathcal{S}_k$ of total weight $2d$ and setting*

$$G_k(q) = \sum_{d \geqslant 0} m_d(k) q^{2d}, \tag{7}$$

we have

$$G_k(q) = \sum_{d \geqslant 0} q^{2d} \langle h_{d,d} \odot h_{d,d} \odot \cdots \odot h_{d,d}, S_{2d} \rangle,$$

where, $h_{d,d}$ denotes the homogeneous basis element indexed by the two part partition (d, d), and in each term, the Kronecker product has k factors.

For this reason, we will refer to the task of constructing the series $G_k(q)$ as the *Hdd Problem*.

Theorem 1.2 shows that the algorithmic machinery of Diophantine analysis may be used in the construction of generating functions of Kronecker coefficients as well as Hilbert series of ring of invariants. More precisely we are referring here to the *constant term methods* of MacMahon partition analysis which have been recently translated into computer software by Andrews et al. [1] and Xin [10].

To see what this leads to, we start by noting that using MacMahon's approach the solutions of $\mathcal{S}_2$ may be obtained by the following identity

$$F_2(y_{00}, y_{01}, y_{10}, y_{11}) = \sum_{p_{00} \geqslant 0} \sum_{p_{01} \geqslant 0} \sum_{p_{10} \geqslant 0} \sum_{p_{11} \geqslant 0} y_{00}^{p_{00}} y_{01}^{p_{01}} y_{10}^{p_{10}} y_{11}^{p_{11}} a_1^{p_{00}+p_{01}-p_{10}-p_{11}} a_2^{p_{00}-p_{01}+p_{10}-p_{11}} \Big|_{a_1^0 a_2^0},$$

where the symbol "$|_{a_1^0 a_2^0}$" denotes the operator of taking the constant term in a_1, a_2. This identity may also be written in the form

$$F_2(y_{00}, y_{01}, y_{10}, y_{11}) = \frac{1}{(1 - y_{00}a_1a_2)(1 - y_{01}a_1/a_2)(1 - y_{10}a_2/a_1)(1 - y_{11}/a_1a_2)} \Bigg|_{a_1^0 a_2^0}.$$

In particular the enumerator of the solutions of $\mathcal{S}_2$ by total weight may be computed from the identity

$$G_2(q) = \frac{1}{(1 - qa_1a_2)(1 - qa_1/a_2)(1 - qa_2/a_1)(1 - q/a_1a_2)} \Bigg|_{a_1^0 a_2^0}.$$

More generally we have

$$G_k(q) = \frac{1}{\prod_{S \subseteq [1,k]} (1 - q \prod_{i \in S} a_i / \prod_{j \notin S} a_j)} \Bigg|_{a_1^0 a_2^0 \cdots a_k^0}, \tag{8}$$

where we use (and will often use) $[m,n]$ to denote the set $\{m, m+1, \ldots, n\}$. Now, standard methods of Invariant Theory yield that we also have

$$W_k(q) = \frac{\prod_{i=1}^{k}(1-a_i^2)}{\prod_{S\subseteq[1,k]}(1-q\prod_{i\in S}a_i/\prod_{j\notin S}a_j)}\bigg|_{a_1^0a_2^0\cdots a_k^0}. \tag{9}$$

A comparison of (8) and (9) strongly suggests that a close study of the combinatorics of Diophantine systems such as $\mathcal{S}_k$ should yield a more revealing path to the construction of such Hilbert series. This idea turned out to be fruitful, as we shall see, in that it permitted the solution of a variety of similar problems (see [3,4]). In particular, we were eventually able to obtain that

$$G_5(\sqrt{q}) = \frac{N_5}{(1-q)^9(1-q^2)^8(1-q^3)^6(1-q^4)^3(1-q^5)}, \tag{10}$$

with

$$\begin{aligned}
N_5 = {} & q^{44} + 7q^{43} + 220q^{42} + 2606q^{41} + 24229q^{40} + 169840q^{39} + 951944q^{38} \\
& + 4391259q^{37} + 17128360q^{36} + 57582491q^{35} + 169556652q^{34} + 442817680q^{33} \\
& + 1036416952q^{32} + 2192191607q^{31} + 4219669696q^{30} + 7433573145q^{29} + 12041305271q^{28} \\
& + 18003453305q^{27} + 24921751416q^{26} + 32017113319q^{25} + 38243274851q^{24} + 42524815013q^{23} \\
& + 44052440432q^{22} + 42524815013q^{21} + 38243274851q^{20} + 32017113319q^{19} + 24921751416q^{18} \\
& + 18003453305q^{17} + 12041305271q^{16} + 7433573145q^{15} + 4219669696q^{14} + 2192191607q^{13} \\
& + 1036416952q^{12} + 442817680q^{11} + 169556652q^{10} + 57582491q^{9} + 17128360q^{8} + 4391259q^{7} \\
& + 951944q^{6} + 169840q^{5} + 24229q^{4} + 2606q^{3} + 220q^{2} + 7q + 1.
\end{aligned}$$

Surprisingly, the presence of the numerator factor in (9) absent in (8) does not increase the complexity of the result, as we see by comparing (10) with the Luque–Thibon result

$$W_5(\sqrt{q}) = \frac{P_5}{(1-q^2)^4(1-q^3)(1-q^4)^6(1-q^5)(1-q^6)^5},$$

with

$$\begin{aligned}
P_5 = {} & q^{54} + q^{52} + 16q^{50} + 9q^{49} + 98q^{48} + 154q^{47} + 465q^{46} + 915q^{45} + 2042q^{44} + 3794q^{43} + 7263q^{42} \\
& + 12688q^{41} + 21198q^{40} + 34323q^{39} + 52205q^{38} + 77068q^{37} + 108458q^{36} + 147423q^{35} \\
& + 191794q^{34} + 241863q^{33} + 292689q^{32} + 342207q^{31} + 386980q^{30} + 421057q^{29} + 443990q^{28} \\
& + 451398q^{27} + 443990q^{26} + 421057q^{25} + 386980q^{24} + 342207q^{23} + 292689q^{22} + 241863q^{21} \\
& + 191794q^{20} + 147423q^{19} + 108458q^{18} + 77068q^{17} + 52205q^{16} + 34323q^{15} + 21198q^{14} \\
& + 12688q^{13} + 7263q^{12} + 3794q^{11} + 2042q^{10} + 915q^{9} + 465q^{8} + 154q^{7} + 98q^{6} + 9q^{5} \\
& + 16q^{4} + q^{2} + 1.
\end{aligned}$$

It should be apparent from the size of the numerators of $W_5(q)$ and $G_5(q)$ that the problem of computing these rational functions explodes beyond $k = 4$. In fact it develops that all available computer packages (including *Omega* and *Latte*) fail to directly compute the constant terms in (8) for $k = 5$. This notwithstanding, we were eventually able to get the partial fraction algorithm of Xin [10] to deliver us $G_5(q)$.

This paper covers the variety of techniques we developed in our efforts to compute these remarkable rational functions. Our efforts in obtaining $W_6(q)$ and $G_6(q)$ are still in progress, so far they only resulted in reducing the computer time required to obtain $W_5(q)$ and $G_5(q)$. Using combinatorial ideas, group actions, in conjunction with the partial fraction algorithm of Xin, we developed three essentially distinct algorithms for computing these rational functions as well as other closely related families. Our most successful algorithm reduces the computation time for $W_5(q)$ down to about five minutes. The crucial feature of this algorithm is an inductive process for successively computing the series $G_k(q)$ and $W_k(q)$, based on a surprising role of divided differences.

This paper is the shortened version of [2]. We organize the contents in 5 sections. Section 1 is this introduction. In Section 2 we relate these Hilbert series to constant terms and derive a collection of identities to be used in later sections. In Section 3 we develop the combinatorial model that reduces the computation of our Kronecker products to solutions of Diophantine systems. In Section 4 we develop the divided difference algorithm for the computation of the complete generating functions yielding $W_k(q)$ and $G_k(q)$. In Section 5, after an illustration of what can be done with bare hands we expand the combinatorial ideas acquired from this experimentation into our three algorithms that yielded $G_5(q)$ and our fastest computation of $W_5(q)$.

The readers are referred to the papers of Luque and Thibon [5,6] and Wallach [8,9] for an understanding of how these Hilbert series are related to problem arising in the study of quantum computing.

2. Hilbert series of invariants as constant terms

Let us recall that given two matrices $A = [a_{ij}]_{i,j=1}^m$ and $B = [b_{ij}]_{i,j=1}^n$ we use the notation $A \otimes B$ to denote the $nm \times nm$ block matrix $A \otimes B = [a_{ij}B]_{i,j=1}^m$. For instance, if $m = n = 2$, then

$$A \otimes B = \begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22} \\ a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{11} & a_{22}b_{12} \\ a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22} \end{bmatrix}.$$

Here and in the following, we define $T_A P(x)$ to be the action of an $m \times m$ matrix $A = [a_{ij}]_{i,j=1}^m$ on a polynomial $P(x) = P(x_1, x_2, \dots, x_m)$ in $\mathbf{R}_m := \mathbb{C}[x_1, x_2, \dots, x_m]$ by

$$T_A P(x_1, x_2, \dots, x_m) = P\left(\sum_{i=1}^m x_i a_{i1}, \sum_{i=1}^m x_i a_{i2}, \dots, \sum_{i=1}^m x_i a_{im}\right). \tag{11}$$

In matrix notation (viewing $x = (x_1, x_2, \dots, x_m)$ as a row vector) we may simply rewrite this as

$$T_A P(x) = P(xA).$$

Recall that if G is a group of $m \times m$ matrices we say that P is *G-invariant* if and only if

$$T_A P(x) = P(x) \quad \forall A \in G.$$

The subspace of $\mathbf{R}_m$ of G-invariant polynomials is usually denoted $\mathbf{R}_m^G$. Clearly, the action in (11) preserves homogeneity and degree. Thus we have the direct sum decomposition

$$\mathbf{R}_m^G = \mathcal{H}_o(\mathbf{R}_m^G) \oplus \mathcal{H}_1(\mathbf{R}_m^G) \oplus \mathcal{H}_2(\mathbf{R}_m^G) \oplus \cdots \oplus \mathcal{H}_d(\mathbf{R}_m^G) \oplus \cdots$$

where $\mathcal{H}_d(\mathbf{R}_m^G)$ denotes the subspace of G-invariants that are homogeneous of degree d. The *Hilbert series* of $\mathbf{R}_m^G$ is simply given by the formal power series

$$F_G(q) = \sum_{d\geqslant 0} q^d \dim\bigl(\mathcal{H}_d\bigl(\mathbf{R}_m^G\bigr)\bigr).$$

This is a well-defined formal power series since $\dim \mathcal{H}_d(\mathbf{R}_m^G) \leqslant \dim(\mathcal{H}_d(\mathbf{R}_m)) = \binom{d+m-1}{m-1}$.

When G is a finite group the Hilbert series $F_G(q)$ is immediately obtained from Molien's formula

$$F_G(q) = \frac{1}{|G|}\sum_{A\in G}\frac{1}{\det(I-qA)}.$$

For an infinite group G which possesses a unit invariant measure ω this identity becomes

$$F_G(q) = \int\limits_{A\in G}\frac{1}{\det(I-qA)}\,d\omega. \tag{12}$$

For the present developments we need to specialize all this to the case $G = SL[2]^{\otimes k}$, that is the group of $2^k \times 2^k$ matrices obtained by tensoring a k-tuple of elements of $SL[2]$. More precisely

$$SL[2]^{\otimes k} = \bigl\{A_1 \otimes A_2 \otimes \cdots \otimes A_k\colon\ A_i \in SL[2]\ \forall i = 1,2,\ldots,k\bigr\}. \tag{13}$$

Our first task in this section is to derive the identity in (9). That is

Theorem 2.1. *Setting for $k \geqslant 1$,*

$$W_k(q) = F_{SL[2]^{\otimes k}}(q) = \sum_{d\geqslant 0} q^d \dim\bigl(\mathcal{H}_d\bigl(\mathbf{R}_{2^k}^{SL[2]^{\otimes k}}\bigr)\bigr), \tag{14}$$

we have

$$W_k(q) = \frac{\prod_{i=1}^k (1-a_i^2)}{\prod_{S\subseteq[1,k]}(1-q\prod_{i\in S}a_i/\prod_{j\notin S}a_j)}\bigg|_{a_1^0a_2^0\cdots a_k^0}. \tag{15}$$

We need the following result.

Proposition 2.2. *If $Q(a_1,a_2,\ldots,a_k)$ is a Laurent polynomial in $\mathbb{C}[a_1,a_2,\ldots,a_k;1/a_1,1/a_2,\ldots,1/a_k]$ then*

$$\left(\frac{1}{2\pi}\right)^k \int\limits_{-\pi}^{\pi}\cdots\int\limits_{-\pi}^{\pi} Q\left(e^{i\theta_1},e^{i\theta_2},\ldots,e^{i\theta_k}\right)d\theta_1\,d\theta_2\cdots d\theta_k = Q(a_1,a_2,\ldots,a_k)\big|_{a_1^0a_2^0\cdots a_k^0}. \tag{16}$$

Proof. By multilinearity, it suffices to consider $Q(a_1,a_2,\ldots,a_k) = a_1^{r_1}a_2^{r_2}\cdots a_k^{r_k}$, in which case (16) obviously holds. $\square$

Proof of Theorem 2.1. To keep our exposition within reasonable limits we will need to assume here some well-known facts (see [9] for proofs). Since $SL[2]$ has no finite measure the first step is to note that a polynomial $P(x) \in \mathbb{C}[x_1, x_2, \ldots, x_{2^k}]$ is $SL[2]^{\otimes k}$-invariant if and only if it is $SU[2]^{\otimes k}$-invariant, where $SU[2] := SU(2,\mathbb{C})$ and as in (13)

$$SU[2]^{\otimes k} = \{A_1 \otimes A_2 \otimes \cdots \otimes A_k\colon\ A_i \in SU[2]\ \forall i = 1, 2, \ldots, k\}.$$

In particular we derive that $F_{SL[2]^{\otimes k}}(q) = F_{SU[2]^{\otimes k}}(q)$. This fact allows us to compute $F_{SL[2]^{\otimes k}}(q)$ using Molien's identity (12). Note however that if

$$A = A_1 \otimes A_2 \otimes \cdots \otimes A_k$$

and A_i has eigenvalues $t_i, 1/t_i$ then (using plethistic notation) we have

$$\frac{1}{\det(I - qA)} = \sum_{m\geqslant 0} q^m h_m\big[(t_1 + 1/t_1)(t_2 + 1/t_2)\cdots(t_k + 1/t_k)\big].$$

Denoting by $d\omega_i$ the invariant measure of the ith copy of $SU[2]$ we see that (12) reduces to

$$F_{SU[2]^{\otimes k}}(q) = \sum_{m\geqslant 0} q^m \int\limits_{SU[2]} \cdots \int\limits_{SU[2]} h_m\big[(t_1 + 1/t_1)\cdots(t_k + 1/t_k)\big]\, d\omega_1 \cdots d\omega_k. \tag{17}$$

Now it is well know that if an integrand $f(A)$ of $SU[2]$ is invariant under conjugation then

$$\int\limits_{SU[2]} f(A)\, d\omega = \frac{1}{\pi} \int\limits_{-\pi}^{\pi} f\left(\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}\right) \sin^2\theta\, d\theta.$$

This identity converts the right-hand side of (17) to

$$\sum_{m\geqslant 0} q^m \frac{1}{\pi^k} \int\limits_{-\pi}^{\pi} \cdots \int\limits_{-\pi}^{\pi} h_m\big[\big(e^{i\theta_1} + e^{-i\theta_1}\big)\cdots\big(e^{i\theta_k} + e^{-i\theta_k}\big)\big] \sin^2\theta_1 \cdots \sin^2\theta_k\, d\theta_1 \cdots d\theta_k. \tag{18}$$

The substitution

$$\sin^2\theta_j = \frac{1 - \frac{e^{2i\theta_j} + e^{-2i\theta_j}}{2}}{2}$$

reduces the coefficient of q^m to

$$\frac{1}{(2\pi)^k} \int\limits_{-\pi}^{\pi} \cdots \int\limits_{-\pi}^{\pi} h_m\big[\big(e^{i\theta_1} + e^{-i\theta_1}\big)\cdots\big(e^{i\theta_k} + e^{-i\theta_k}\big)\big] \prod_{i=1}^{k}\left(1 - \frac{e^{2i\theta_j} + e^{-2i\theta_j}}{2}\right) d\theta_1 \cdots d\theta_k. \tag{19}$$

However the factor $h_m[(e^{i\theta_1} + e^{-i\theta_1})\cdots(e^{i\theta_k} + e^{-i\theta_k})]$ is invariant under any of the interchanges $e^{i\theta_j} \leftrightarrow e^{-i\theta_j}$. Thus the integral in (19) may be simplified to

$$\frac{1}{(2\pi)^k} \int\limits_{-\pi}^{\pi} \cdots \int\limits_{-\pi}^{\pi} h_m\big[\big(e^{i\theta_1} + e^{-i\theta_1}\big)\cdots\big(e^{i\theta_k} + e^{-i\theta_k}\big)\big] \prod_{i=1}^{k}\big(1 - e^{2i\theta_j}\big)\, d\theta_1 \cdots d\theta_k.$$

Proposition 2.2 then yields that this integral may be computed as the constant term

$$h_m\big[(a_1+1/a_1)(a_2+1/a_2)\cdots(a_k+1/a_k)\big]\prod_{i=1}^{k}\big(1-a_i^2\big)\big|_{a_1^0a_2^0\cdots a_k^0}.$$

Using this in (18) we derive that

$$\begin{aligned}
F_{SU[2]^{\otimes k}}(q) &= \sum_{m\geqslant 0} q^m h_m\big[(a_1+1/a_1)(a_2+1/a_2)\cdots(a_k+1/a_k)\big]\prod_{i=1}^{k}\big(1-a_i^2\big)\big|_{a_1^0a_2^0\cdots a_k^0}\\
&= \sum_{m\geqslant 0} q^m h_m\Bigg[\sum_{S\subseteq[1,k]}\frac{\prod_{i\in S}a_i}{\prod_{j\notin S}a_j}\Bigg]\prod_{i=1}^{k}\big(1-a_i^2\big)\big|_{a_1^0a_2^0\cdots a_k^0}\\
&= \Bigg(\prod_{S\subseteq[1,k]}\frac{1}{1-q\frac{\prod_{i\in S}a_i}{\prod_{j\notin S}a_j}}\Bigg)\prod_{i=1}^{k}\big(1-a_i^2\big)\big|_{a_1^0a_2^0\cdots a_k^0}.
\end{aligned}$$

This completes the proof of Theorem 2.1. □

Note that if we restrict our action of $SU[2]^{\otimes k}$ to the subgroup of matrices

$$T_2^{\otimes k} = \left\{\begin{bmatrix} t_1 & 0\\ 0 & \bar t_1\end{bmatrix}\otimes\begin{bmatrix} t_2 & 0\\ 0 & \bar t_2\end{bmatrix}\otimes\cdots\otimes\begin{bmatrix} t_k & 0\\ 0 & \bar t_k\end{bmatrix}:\ t_r=e^{i\theta_r}\right\}$$

then a similar use of Molien's formula yields the following result.

Theorem 2.3. *The Hilbert series of the ring of invariants* $\mathbf{R}_{2^k}^{T_2^{\otimes k}}$ *is given by the constant term*

$$F_{T_2^{\otimes k}}(q) = \frac{1}{\prod_{S\subseteq[1,k]}(1-q\prod_{i\in S}a_i/\prod_{j\notin S}a_j)}\Bigg|_{a_1^0a_2^0\cdots a_k^0}. \tag{20}$$

Proof. The integrand $1/\det(1-qA)$ is the same as in the previous proof and only the Haar measure changes. In this case we must take $dw = d\theta_1 d\theta_2\cdots d\theta_k/(2\pi)^k$ in (12), and Molien's formula gives

$$F_{T_2^{\otimes k}}(q) = \frac{1}{(2\pi)^k}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\frac{1}{\prod_{S\subseteq[1,k]}(1-q\prod_{i\in S}t_i/\prod_{j\notin S}t_j)}\,d\theta_1 d\theta_2\cdots d\theta_k.$$

Thus (20) follows from Proposition 2.2. □

Remark 2.4. There is another path leading to the same result that is worth mentioning here since it gives a direct way of connecting Invariants to Diophantine systems. For notational simplicity we will deal with the case $k=3$. Note that the element

$$\begin{bmatrix} t_1 & 0\\ 0 & \bar t_1\end{bmatrix}\otimes\begin{bmatrix} t_2 & 0\\ 0 & \bar t_2\end{bmatrix}\otimes\begin{bmatrix} t_3 & 0\\ 0 & \bar t_3\end{bmatrix}\in T_2^{\otimes 3}$$

is none other than the 8×8 diagonal matrix

$$A(t_1,t_2,t_3) = \operatorname{diag}(t_1t_2t_3,\ t_1t_2/t_3,\ t_1t_3/t_2,\ t_1/t_2t_3,\ t_2t_3/t_1,\ t_2/t_1t_3,\ t_3/t_1t_2,\ 1/t_1t_2t_3).$$

This gives that for any monomial $x^p = x_1^{p_1} x_2^{p_2} \cdots x_8^{p_8}$ we have

$$A(t_1, t_2, t_3)x^p = t_1^{p_1+p_2+p_3+p_4-p_5-p_6-p_7-p_8} t_2^{p_1+p_2-p_3-p_4+p_5+p_6-p_7-p_8} t_3^{p_1-p_2+p_3-p_4+p_5-p_6+p_7-p_8} \times x^p.$$

Thus all the monomials are eigenvectors and a polynomial $P(x_1, x_2, \ldots, x_8)$ will be invariant if and only if all its monomials are eigenvectors of eigenvalue 1. It then follows that the Hilbert series $F_{T_2^{\otimes 3}}(q)$ of $\mathbb{C}[x_1, x_2, \ldots, x_8]^{T_2^{\otimes 3}}$ is obtained by q-counting these monomials by total degree. That is q-counting by the statistic $p_1 + p_2 + p_3 + p_4 + p_5 + p_6 + p_7 + p_8$ the solutions of the Diophantine system

$$\mathcal{S}_3 = \left\| \begin{array}{l} p_1 + p_2 + p_3 + p_4 - p_5 - p_6 - p_7 - p_8 = 0 \\ p_1 + p_2 - p_3 - p_4 + p_5 + p_6 - p_7 - p_8 = 0 \\ p_1 - p_2 + p_3 - p_4 + p_5 - p_6 + p_7 - p_8 = 0 \end{array} \right. \tag{21}$$

and MacMahon partition analysis gives

$$\begin{aligned} F_{T_2^{\otimes 3}}(q) = {} & \frac{1}{1 - qa_1a_2a_3} \frac{1}{1 - qa_1a_2/a_3} \frac{1}{1 - qa_1a_3/a_2} \frac{1}{1 - qa_1/a_2a_3} \frac{1}{1 - qa_2a_3/a_1} \\ & \times \frac{1}{1 - qa_2/a_1a_3} \frac{1}{1 - qa_3/a_1a_2} \frac{1}{1 - qa/a_1a_2a_3} \bigg|_{a_1^0 a_2^0 a_3^0}. \end{aligned}$$

This gives another proof of the case $k = 3$ of (20). It is also clear that the same argument can be used for all $k > 3$ as well.

Remark 2.5. Full information about the solutions of our systems is given by the complete generating function

$$F_k(x_1, x_2, \ldots, x_{2^k}) = \sum_{p \in \mathcal{S}_k} x_1^{p_1} x_2^{p_2} \cdots x_{2^k}^{p_{2^k}}. \tag{22}$$

Using the notation adopted for $\mathcal{S}_3$ in (21), our system $\mathcal{S}_k$ may be written in vector form

$$p_1 V_1 + p_2 V_2 + \cdots + p_{2^k} V_{2^k} = 0,$$

where $V_1, V_2, \ldots, V_{2^k}$ are the k-vectors $(\pm 1, \pm 1, \ldots, \pm 1)$ yielding the vertices of the hypercube of semiside 1 centered at the origin. In this notation, MacMahon partition analysis gives that the rational function in (22) is obtained by taking the constant term

$$F_k(x_1, x_2, \ldots, x_{2^k}) = \prod_{i=1}^{2^k} \frac{1}{1 - x_i A_i} \bigg|_{a_1^0 a_2^0 \cdots a_k^0}$$

with the A_i Laurent monomials in $a_1, a_2, \ldots, a_k$ which may be written in the form

$$A_i = \prod_{i=1}^{k} a_i^{1-2\epsilon_i}$$

where $\epsilon_1 \epsilon_2 \cdots \epsilon_k$ are the binary digits of $i - 1$.

In the same vein the companion rational function $W(x_1, x_2, \dots, x_{2^k})$ associated to the Sdd problem is obtained by taking the constant term

$$W_k(x_1, x_2, \dots, x_{2^k}) = \prod_{j=1}^{k}\left(1 - a_j^2\right)\prod_{i=1}^{2^k}\frac{1}{1 - x_i A_i}\bigg|_{a_1^0 a_2^0 \cdots a_k^0}.$$

Of course we have

$$G_k(q) = F_k(x_1, x_2, \dots, x_{2^k})|_{x_i = q} \quad \text{and} \quad W_k(q) = W_k(x_1, x_2, \dots, x_{2^k})|_{x_i = q}.$$

In Section 4 we will show that, at least in principle, these rational functions could be constructed by a succession of elementary steps interspersed by single constant term extractions.

3. Diophantine systems, Constant terms and Kronecker products

We have seen, by MacMahon partition analysis, that the generating function $G_k(q)$ defined in (7), which counts solutions of the Diophantine system $\mathcal{S}_k$, is given by the constant term identity in (8):

$$G_k(q) = \frac{1}{\prod_{S\subseteq[1,k]}(1 - q\prod_{i\in S} a_i/\prod_{j\notin S} a_j)}\bigg|_{a_1^0 a_2^0 \cdots a_k^0}. \tag{23}$$

In the last section we proved (in Theorem 2.1) that the Hilbert series $W_k(q)$ of invariants in (14) is given by the constant term

$$W_k(q) = \frac{\prod_{i=1}^{k}(1 - a_i^2)}{\prod_{S\subseteq[1,k]}(1 - q\prod_{i\in S} a_i/\prod_{j\notin S} a_j)}\bigg|_{a_1^0 a_2^0 \cdots a_k^0}. \tag{24}$$

A comparison of (23) and (24) clearly suggests that these two results must be connected. This connection has a beautiful combinatorial underpinning which leads to another interpretation of these remarkable constant terms. The idea is best explained in the simplest case $k = 2$. Then (24) reduces to

$$W_2(q) = \frac{1 - a_1^2 - a_2^2 + a_1^2 a_2^2}{(1 - qa_1a_2)(1 - qa_1/a_2)(1 - qa_2/a_1)(1 - q/a_1a_2)}\bigg|_{a_1^0 a_2^0}.$$

Expanding the inner rational function as product of four formal power series in q we get

$$\begin{aligned}
W_2(q) = {} & \sum_{p_{00}\geqslant 0}\sum_{p_{01}\geqslant 0}\sum_{p_{10}\geqslant 0}\sum_{p_{11}\geqslant 0} q^{p_{00}+p_{01}+p_{10}+p_{11}} a_1^{p_{00}+p_{01}-p_{10}-p_{11}} a_2^{p_{00}-p_{01}+p_{10}-p_{11}}\big|_{a_1^0 a_2^0} \\
& - \sum_{p_{00}\geqslant 0}\sum_{p_{01}\geqslant 0}\sum_{p_{10}\geqslant 0}\sum_{p_{11}\geqslant 0} q^{p_{00}+p_{01}+p_{10}+p_{11}} a_1^{p_{00}+p_{01}-p_{10}-p_{11}+2} a_2^{p_{00}-p_{01}+p_{10}-p_{11}}\big|_{a_1^0 a_2^0} \\
& - \sum_{p_{00}\geqslant 0}\sum_{p_{01}\geqslant 0}\sum_{p_{10}\geqslant 0}\sum_{p_{11}\geqslant 0} q^{p_{00}+p_{01}+p_{10}+p_{11}} a_1^{p_{00}+p_{01}-p_{10}-p_{11}} a_2^{p_{00}-p_{01}+p_{10}-p_{11}+2}\big|_{a_1^0 a_2^0} \\
& + \sum_{p_{00}\geqslant 0}\sum_{p_{01}\geqslant 0}\sum_{p_{10}\geqslant 0}\sum_{p_{11}\geqslant 0} q^{p_{00}+p_{01}+p_{10}+p_{11}} a_1^{p_{00}+p_{01}-p_{10}-p_{11}+2} a_2^{p_{00}-p_{01}+p_{10}-p_{11}+2}\big|_{a_1^0 a_2^0}.
\end{aligned} \tag{25}$$

Now by MacMahon partition analysis, the ith term counts solutions of the Diophantine system

$$\mathcal{S}_2^i = \left\| \begin{array}{l} p_{00} + p_{01} - p_{10} - p_{11} = c_i \\ p_{00} - p_{01} + p_{10} - p_{11} = d_i \end{array} \right., \tag{26}$$

where (c_i, d_i) equals $(0,0), (-2,0), (0,-2), (-2,-2)$ for $i = 1,2,3,4$, respectively. Note that the first term of (25) is none other than (23) for $k = 2$.

Applying the same decomposition in the general case we see that the series $W_k(q)$ may be viewed as the end product of an inclusion exclusion process applied to a family of Diophantine systems. To derive some further consequences of this fact, it is more convenient to use another combinatorial model for these systems. In this alternate model our family of objects consists of the collection $\mathcal{F}_d$ of d-subsets of the $2d$-element set

$$\Omega_{2d} = \{1, 2, 3, \ldots, 2d\}.$$

For a given $A = \{1 \leqslant i_1 < i_2 < \cdots < i_d \leqslant 2d\} \in \mathcal{F}_d$ and σ in the symmetric group S_{2d} we set

$$\sigma A = \{\sigma_{i_1}, \sigma_{i_2}, \ldots, \sigma_{i_d}\}.$$

This clearly defines an action of S_{2d} on $\mathcal{F}_d$ as well as on the k-fold cartesian product

$$\mathcal{F}_d^k = \mathcal{F}_d \times \mathcal{F}_d \times \mathcal{F}_d \times \cdots \times \mathcal{F}_d.$$

Theorem 3.1. *The number $m_d(k)$ of solutions of the Diophantine system $\mathcal{S}_k$ is equal to the number of orbits in the action of S_{2d} on $\mathcal{F}_d^k$.*

Proof. It will be sufficient to see this for $k = 2$. Then leaving d generic we can visualize an element of $\mathcal{F}_d \times \mathcal{F}_d$ by the Ven diagram of Fig. 1. There we have depicted the pair (A_1, A_2) as it lies in Ω_{2d}. Using these two sets we can decompose Ω_{2d} into 4 parts labeled by $A_{00}, A_{01}, A_{10}, A_{11}$. More precisely "$A_{00}$" labels the set $A_1 \cap A_2$, "A_{01}" labels the set $A_1 \cap {}^c A_2$, "A_{10}" labels the set ${}^c A_1 \cap A_2$ and "A_{11}" labels the set ${}^c A_1 \cap {}^c A_2$. Here we use "${}^c A_i$" to denote the complement of A_i in Ω_{2d}. This given, if we let $p_{00}, p_{01}, p_{10}, p_{11}$ denote the respective cardinalities of these sets, the condition that the pair (A_1, A_2) belongs to $\mathcal{F}_d \times \mathcal{F}_d$ yields that we must have

$$p_{00} + p_{01} + p_{10} + p_{11} = 2d,$$

$$p_{00} + p_{01} = |A_1| = d,$$

$$p_{00} + p_{10} = |A_2| = d.$$

Note that this system of equations is equivalent to the system

$$p_{00} + p_{01} + p_{10} + p_{11} = 2d,$$

$$p_{00} + p_{01} - p_{10} - p_{11} = 0,$$

$$p_{00} - p_{01} + p_{10} - p_{11} = 0.$$

It is easily seen that for any solution $(p_{00}, p_{01}, p_{10}, p_{11})$ of this system, we can immediately construct a pair of subsets $(A_1, A_2) \in \mathcal{F}_d \times \mathcal{F}_d$ by simply filling the sets $A_{00}, A_{01}, A_{10}, A_{11}$ in the diagram of Fig. 1 with $p_{00}, p_{01}, p_{10}, p_{11}$ respective elements from the set Ω_{2d}. Moreover, any two such fillings can be seen to be images of each other under suitable permutations of S_{2d}. In other words by this construction we obtain a bijection between the orbits of $\mathcal{F}_d \times \mathcal{F}_d$ under $\mathcal{S}_{2d}$ and the solutions of

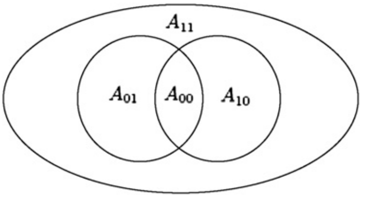


Fig. 1. The Ven diagram for $\mathcal{F}_d^2$.

the system $\mathcal{S}_2$ we have previously encountered. This proves the theorem for $k=2$. The general case follows by an entirely analogous argument. □

Now we are ready to prove Theorem 1.2 and then Theorem 1.1.

Proof of Theorem 1.2. We are to show that

$$m_k(d) = \langle h_{d,d} \odot h_{d,d} \odot \cdots \odot h_{d,d}, s_{2d} \rangle. \tag{27}$$

It is well known that a transitive action of a group G on a set Ω is equivalent to the action of G on the left G-cosets of the stabilizer of any element of Ω. In our case, pick the subset $[1,d]$ of Ω_{2d}. Then the stabilizer is the Young subgroup $S_{[1,d]} \times S_{[d+1,2d]}$ of S_{2d} and thus the Frobenius characteristic of this action is the homogeneous basis element $h_{d,d} = h_d h_d$. It follows then that the Frobenius characteristic of the action of S_{2d} on the k-tuples $(A_1, A_2, \ldots, A_k)$ of d-subsets of Ω_{2d} is given by the k-fold Kronecker product $h_{d,d} \odot h_{d,d} \odot \cdots \odot h_{d,d}$. Therefore the scalar product

$$\langle h_{d,d} \odot h_{d,d} \odot \cdots \odot h_{d,d}, s_{2d} \rangle$$

yields the multiplicity of the trivial under this action. But it is well known, and easy to see that this multiplicity is also equal to the number of orbits under this action. Thus (27) follows by Theorem 3.1. □

Proof of Theorem 1.1. Again we will only need to do it for $k=2$. To this end note that by Theorem 1.2 the number of solutions of the system $\mathcal{S}_2^1$ in (26) is given by the scalar product

$$\langle h_{d,d} \odot h_{d,d}, s_{2d} \rangle. \tag{28}$$

In the same vein we see that the number of solutions to the system $\mathcal{S}_2^2$ in (26) may be viewed as the number of orbits in the action of S_{2d} on the pairs of subsets (A_1, A_2) of Ω_{2d} where $|A_2| = |{}^c A_2|$ and $|A_1| = |{}^c A_1| + 2$. We have seen that the Frobenius characteristic of the action of S_{2d} on subsets of cardinality d is $h_{d,d}$. On the other hand the action of S_{2d} on sets of cardinality $d+1$ is equivalent to the action of S_{2d} on left cosets of $S_{[1,d+1]} \times S_{[d+2,2d]}$ yielding that the Frobenius characteristic for this action is $h_{d+1}h_{d-1}$. Thus the Frobenius characteristic of the action of S_{2d} on such pairs must be the Kronecker product

$$h_{d+1}h_{d-1} \odot h_d h_d.$$

It then follows that the number of solutions of the system $\mathcal{S}_2^2$ is given by the scalar product

$$\langle h_{d+1}h_{d-1} \odot h_d h_d, s_{2d} \rangle. \tag{29}$$

The same reasoning gives that the number of solutions of the systems $\mathcal{S}_2^3$ and $\mathcal{S}_2^4$ in (26) are given by the scalar products

$$\langle h_d h_d \odot h_{d+1} h_{d-1}, s_{2d} \rangle \quad \text{and} \quad \langle h_{d+1} h_{d-1} \odot h_{d+1} h_{d-1}, s_{2d} \rangle. \tag{30}$$

It follows then that the coefficient of q^{2d} in the alternating sum of formal power series in (25) is none other than the following alternating sum of the scalar products in (28), (29) and (30):

$$\begin{aligned} W_2(q)\big|_{q^{2d}} &= \langle h_d h_d \odot h_d h_d, s_{2d} \rangle - \langle h_{d+1} h_{d-1} \odot h_d h_d, s_{2d} \rangle \\ &\quad - \langle h_d h_d \odot h_{d+1} h_{d-1}, s_{2d} \rangle + \langle h_{d+1} h_{d-1} \odot h_{d+1} h_{d-1}, s_{2d} \rangle \\ &= \big\langle (h_d h_d - h_{d+1} h_{d-1}) \odot (h_d h_d - h_{d+1} h_{d-1}), s_{2d} \big\rangle = \langle s_{d,d} \odot s_{d,d}, s_{2d} \rangle. \end{aligned}$$

Summing over d gives

$$W_2(q) = \sum_{d \geqslant 0} q^{2d} \langle s_{d,d} \odot s_{d,d}, s_{2d} \rangle.$$

An entirely analogous argument proves the general identity in (5). $\square$

4. Enter divided difference operators

There is a truly remarkable approach to the solutions of a variety of constant term problems which exhibit the same types of symmetries of the Hdd and Sdd problems. We will introduce the approach in some simple cases first. We define the *double* of the Diophantine system

$$\mathcal{S}_2 = \left\| \begin{array}{l} p_1 + p_2 - p_3 - p_4 = 0 \\ p_1 - p_2 + p_3 - p_4 = 0 \end{array} \right.$$

to be the system

$$\mathcal{SS}_2 = \left\| \begin{array}{l} p_1 + p_2 - p_3 - p_4 + p_5 + p_6 - p_7 - p_8 = 0 \\ p_1 - p_2 + p_3 - p_4 + p_5 - p_6 + p_7 - p_8 = 0 \end{array} \right. .$$

As we can easily see we have simply repeated twice each linear form and appropriately increased the indices of the variables. Now suppose that we are in possession of the complete generating function of $\mathcal{S}_2$, that is

$$F_{\mathcal{S}_2}(x_1, x_2, x_3, x_4) = \sum_{p \in \mathcal{S}_2} x_1^{p_1} x_2^{p_2} x_3^{p_3} x_4^{p_4}.$$

We claim that the complete generating function of $\mathcal{SS}_2$ is simply given by

$$F_{\mathcal{SS}_2}(x_1, x_2, \ldots, x_8) = \delta_{1,5} \delta_{2,6} \delta_{3,7} \delta_{4,8} F_{\mathcal{S}_2}(x_1, x_2, x_3, x_4), \tag{31}$$

where for any pair of indices (i, j) we let $\delta_{i,j}$ denote the divided difference operator defined for any function $f(x)$ by

$$\delta_{i,j} f(x) = \frac{f(x) - f(x)|_{x_i = x_j,\, x_j = x_i}}{x_i - x_j}.$$

Proof of (31). By MacMahon partition analysis we have

$$F_{\mathcal{S}_2}(x_1,x_2,x_3,x_4) = \frac{1}{(1-x_1a_1a_2)}\frac{1}{(1-x_2a_1/a_2)}\frac{1}{(1-x_3a_2/a_1)}\frac{1}{(1-x_4/a_1a_2)}\bigg|_{a_1^0a_2^0}. \tag{32}$$

Now note that since

$$\delta_{1,5}\frac{1}{(1-x_1a_1a_2)} = \left(\frac{1}{(1-x_1a_1a_2)} - \frac{1}{(1-x_5a_1a_2)}\right)\frac{1}{x_1-x_5} = \frac{a_1a_2}{(1-x_1a_1a_2)(1-x_5a_1a_2)},$$

we obtain similarly

$$\begin{aligned}
\delta_{2,6}\frac{1}{(1-x_2a_1/a_2)} &= \frac{a_1/a_2}{(1-x_2a_1/a_2)(1-x_6a_1/a_2)},\\
\delta_{3,7}\frac{1}{(1-x_3a_2/a_1)} &= \frac{a_2/a_1}{(1-x_3a_2/a_1)(1-x_7a_2/a_1)},\\
\delta_{4,8}\frac{1}{(1-x_4/a_1a_2)} &= \frac{1/a_1a_2}{(1-x_4/a_1a_2)(1-x_8/a_1a_2)}.
\end{aligned}$$

Thus applying the operator $\delta_{1,5}\delta_{2,6}\delta_{3,7}\delta_{4,8}$ to both sides of (32) gives

$$\begin{aligned}
\delta_{1,5}\delta_{2,6}\delta_{3,7}\delta_{4,8}F_{\mathcal{S}_2}(x_1,x_2,x_3,x_4) &= \frac{1}{(1-x_1a_1a_2)(1-x_2a_1/a_2)(1-x_3a_2/a_1)(1-x_4/a_1a_2)}\\
&\quad\times \frac{1}{(1-x_5a_1a_2)(1-x_6a_1/a_2)(1-x_7a_2/a_1)(1-x_8/a_1a_2)}\bigg|_{a_1^0a_2^0}.
\end{aligned} \tag{33}$$

Now we can easily recognize that (33) is precisely the constant term that MacMahon partition analysis would yield for the system $\mathcal{SS}_2$. This proves (31). □

Note that to obtain the equality in (33) we have used the simple fact that the divided difference operator and the constant term operator do commute. This is the fundamental property which is at the root of the present algorithm. This example should make it evident to have the following more general result (with *double* modified).

Theorem 4.1. *If $F_{\mathcal{S}}(x_1,x_2,\ldots,x_n)$ is the complete generating function of the Diophantine system*

$$\mathcal{S} = \left\| \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n}\\ \vdots & \vdots & \cdots & \vdots\\ b_{r1} & b_{r2} & \cdots & b_{rn}\end{bmatrix}\begin{bmatrix} p_1\\ \vdots\\ p_n\end{bmatrix} = \begin{bmatrix} c_1\\ \vdots\\ c_r\end{bmatrix}\right.,$$

then the complete generating function of the doubling *of $\mathcal{S}$ defined by*

$$\mathcal{SS} = \left\| \left[\begin{array}{cccc|cccc} b_{11} & b_{12} & \cdots & b_{1n} & b_{11} & b_{12} & \cdots & b_{1n}\\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots\\ b_{r1} & b_{r2} & \cdots & b_{rn} & b_{r1} & b_{r2} & \cdots & b_{rn}\end{array}\right]\begin{bmatrix} p_1\\ \vdots\\ p_{2n}\end{bmatrix} = \begin{bmatrix} c_1-b_{11}-b_{12}-\cdots-b_{1n}\\ \vdots\\ c_r-b_{r1}-b_{r2}-\cdots-b_{rn}\end{bmatrix}\right.$$

is given by the rational function

$$F_{\mathcal{SS}}(x_1, x_2, \ldots, x_{2n}) = \delta_{1,n+1}\delta_{2,n+2}\cdots\delta_{n,2n} F_{\mathcal{S}}(x_1, x_2, \ldots, x_n).$$

This result combined with the next simple observation yields a powerful algorithm for computing a variety of complete generating functions.

Theorem 4.2. *Let $F_{\mathcal{S}}(x_1, x_2, \ldots, x_n)$ be the complete generating function of a Diophantine system $\mathcal{S}$. Then the complete generating function $F_{\mathcal{SE}}(x_1, x_2, \ldots, x_n)$ of the system $\mathcal{SE}$ obtained by adding the equation*

$$\mathcal{E} = \| r_1 p_1 + r_2 p_2 + \cdots + r_n p_n = s$$

to $\mathcal{S}$ is obtained by taking the constant term

$$F_{\mathcal{SE}}(x_1, x_2, \ldots, x_n) = a^{-s} F_{\mathcal{S}}\big(a^{r_1}x_1, a^{r_2}x_2, \ldots, a^{r_n}x_n\big)\Big|_{a^0}.$$

Proof. By assumption

$$F_{\mathcal{S}}(x_1, x_2, \ldots, x_n) = \sum_{p \in \mathcal{S}} x_1^{p_1} x_2^{p_2} \cdots x_n^{p_n}.$$

Now we have

$$\begin{aligned} a^{-s} F_{\mathcal{S}}\big(a^{r_1}x_1, a^{r_2}x_2, \ldots, a^{r_n}x_n\big)\Big|_{a^0} &= \sum_{p \in \mathcal{S}} x_1^{p_1} x_2^{p_2} \cdots x_n^{p_n} a^{r_1 p_1 + r_2 p_2 + \cdots + r_n p_n - s}\Big|_{a^0} \\ &= \sum_{p \in \mathcal{SE}} x_1^{p_1} x_2^{p_2} \cdots x_n^{p_n} \\ &= F_{\mathcal{SE}}(x_1, x_2, \ldots, x_n). \qquad \square \end{aligned}$$

These two results provide us with algorithms for (at least in principle) computing all the Hdd series $G_k(q)$ as well as the Sdd series $W_k(q)$.

Algorithm 4.3 *(Hdd Case).*

$\mathbf{b_1}$) Initially compute the complete generating function for the Hdd problem for $k = 1$. That is, compute the constant term

$$F_1(x_1, x_2) = \frac{1}{(1 - x_1 a)(1 - x_2/a)}\bigg|_{a^0}.$$

$\mathbf{a_k}$) With $F_{k-1}(x_1, \ldots, x_{2^{k-1}})$ from step $\mathbf{b_{k-1}}$), compute by divided difference

$$FF_{k-1}(x_1, \ldots, x_{2^k}) = \delta_{1,1+2^{k-1}} \cdots \delta_{2^{k-1},2^k} F_{k-1}(x_1, \ldots, x_{2^{k-1}}).$$

$\mathbf{b_k}$) With $FF_{k-1}(x_1, \ldots, x_{2^{k-1}})$ from step $\mathbf{a_k}$), compute the complete generating function for the Sdd problem for k by the following constant term:

$$F_k(x_1, x_2, \ldots, x_{2^k}) = FF_{k-1}(ax_1, ax_2, \ldots, ax_{2^{k-1}}, x_{2^{k-1}+1}/a, \ldots, x_{2^k}/a)|_{a^0}.$$

Algorithm 4.4 *(Sdd Case).*

$\mathbf{b_1}$) Initially compute the complete generating function for the Sdd problem for $k=1$. That is, compute the constant term

$$W_1(x_1,x_2) = \frac{1-a^2}{(1-x_1a)(1-x_2/a)}\bigg|_{a^0}.$$

$\mathbf{a_k}$) With $W_{k-1}(x_1,\dots,x_{2^{k-1}})$ from step $\mathbf{b_{k-1}}$), compute by divided difference

$$WW_{k-1}(x_1,\dots,x_{2^k}) = \delta_{1,1+2^{k-1}}\cdots\delta_{2^{k-1},2^k}W_{k-1}(x_1,\dots,x_{2^{k-1}}).$$

$\mathbf{b_k}$) With $WW_{k-1}(x_1,\dots,x_{2^{k-1}})$ from step $\mathbf{a_k}$), compute the complete generating function for the Sdd problem for k by the following constant term:

$$W_k(x_1,x_2,\dots,x_{2^k}) = WW_{k-1}(ax_1,ax_2,\dots,ax_{2^{k-1}},x_{2^{k-1}+1}/a,\dots,x_{2^k}/a)\big(1-a^2\big)\big|_{a^0}.$$

Since the two algorithms are similar, we only explain here in detail the Sdd case. For the Hdd case, see [2]. Note that the sequence of steps in Algorithm 4.4 can be terminated by replacing step $\mathbf{b_k}$) by

$\mathbf{b'_k}$) To obtain the generating function $W_k(q)$ compute the constant term

$$W_k(q) = WW_{k-1}(aq,aq,\dots,aq,q/a,\dots,q/a)\big(1-a^2\big)\big|_{a^0}.$$

Only steps $\mathbf{b_1}$) and $\mathbf{a_2}$) can be carried out by hand. Though steps 3 and 4 are routine they are too messy to do by hand. But step 5 again needs further tricks to be carried out by computer. Step 6 appears beyond reach at the moment.

It will be instructive to see what some of these steps give.

$\mathbf{b_1}$)

$$W_1(x_1,x_2) = \frac{1-x_2^2}{1-x_1x_2}.$$

$\mathbf{a_2}$)

$$WW_1(x_1,\dots,x_4) = \frac{1-x_2^2-x_2x_4-x_4^2+x_1x_2^2x_4+x_2^2x_3x_4-x_1x_2x_3x_4+x_1x_2x_4^2+x_2x_3x_4^2-x_1x_2^2x_3x_4^2}{(1-x_1x_2)(1-x_3x_2)(1-x_1x_4)(1-x_3x_4)}.$$

$\mathbf{b_2}$)

$$W_2(x_1,x_2,x_3,x_4) = \frac{1-x_2x_4-x_3x_4+x_4^2}{(1-x_1x_4)(1-x_2x_3)}.$$

This gives

$$W_2(q) = \frac{1}{1-q^2}.$$

a₃)

$$WW_2(x_1,\dots,x_8)=\frac{\text{(large numerator)}}{(1-x_1x_4)(1-x_1x_8)(1-x_2x_3)(1-x_2x_7)(1-x_3x_6)(1-x_4x_5)(1-x_5x_8)(1-x_6x_7)}.$$

b₃)

$$W_3(x_1,\dots,x_8)=\frac{\text{(large numerator)}}{(1-x_1x_8)(1-x_2x_7)(1-x_3x_6)(1-x_4x_5)(1-x_1x_4x_6x_7)(1-x_2x_3x_5x_8)}.$$

b′₃) Notwithstanding the complexity of the previous results it turns out that to obtain $W_3(q)$ we need only compute the constant term

$$W_3(q)=\frac{1}{(1-q^2)}\times\frac{1-a^2}{(1-q^2a^2)(1-q^2/a^2)}\bigg|_{a^0}. \tag{34}$$

To this end we start by determining the coefficients A and B in the partial fraction decomposition

$$\frac{(1-a^2)a^2}{(1-q^2a^2)(a^2-q^2)}=\frac{1}{q^2}+\frac{A}{1-q^2a^2}+\frac{B}{a^2-q^2}$$

obtaining

$$A=\frac{(1-a^2)a^2}{(a^2-q^2)}\bigg|_{a^2=1/q^2}=\frac{(1-1/q^2)/q^2}{(1/q^2-q^2)}=-\frac{1}{q^2(1+q^2)},$$
$$B=\frac{(1-a^2)a^2}{(1-q^2a^2)}\bigg|_{a^2=q^2}=\frac{(1-q^2)q^2}{(1-q^4)}=\frac{q^2}{(1+q^2)}$$

(the exact value of B is not needed) and we can write

$$\frac{1-a^2}{(1-q^2a^2)(1-q^2/a^2)}=\frac{1}{q^2}-\frac{1}{q^2(1+q^2)}\times\frac{1}{(1-a^2q^2)}+\frac{1}{(1+q^2)}\times\frac{q^2/a^2}{1-q^2/a^2}.$$

Thus taking constant terms gives

$$\frac{1-a^2}{(1-q^2a^2)(1-q^2/a^2)}\bigg|_{a^0}=\frac{1}{q^2}-\frac{1}{q^2(1+q^2)}+0=\frac{1}{1+q^2}.$$

Using this in (34) we finally obtain

$$W_3(q)=\frac{1}{1-q^4}.$$

a₄)

$$WW_4(x_1,x_2,\dots,x_{16})=\text{(too large for typesetting)}$$

$\mathbf{b}_4'$) Notwithstanding the complexity of the previous result it turns out that to obtain $W_4(q)$ we need only compute the constant term

$$W_4(q) = \frac{(1+q^4)(1+q^6)}{(1-q^2)(1-q^4)^2} \times \frac{1-a^2}{(1-a^2q^4)(1-q^4/a^2)(1-a^4q^4)(1-q^4/a^4)}\bigg|_{a^0}.$$

To illustrate the power and flexibility of the partial fraction algorithm we will carry this out by hand. The reader is referred to [3] for a brief tutorial on the use of this algorithm. In the next few lines we will strictly adhere to the notation and terminology given in [3].

To begin we note that we need only calculate the constant term

$$C(x) = \frac{1-a}{(1-ax)(1-x/a)(1-a^2x)(1-x/a^2)}\bigg|_{a^0}, \tag{35}$$

since we can write

$$W_4(q) = \frac{(1+q^4)(1+q^6)}{(1-q^2)(1-q^4)^2} \times C\big(q^4\big). \tag{36}$$

Now we have

$$\frac{1}{(1-a^2x)(1-x/a^2)} = \frac{a^2}{(1-a^2x)(a^2-x)} = \frac{1}{1-x^2}\frac{1}{1-a^2x} + \frac{1}{1-x^2}\frac{x/a^2}{1-x/a^2}.$$

Thus (35) may be rewritten in the form

$$C(x) = \frac{1}{1-x^2}\left(\frac{(1-a)}{(1-ax)(1-x/a)}\frac{1}{1-a^2x}\bigg|_{a^0} + \frac{(1-a)}{(1-ax)(1-x/a)}\frac{x/a^2}{1-x/a^2}\bigg|_{a^0}\right). \tag{37}$$

Note that in the first constant term we have only one dually contributing term and on the second we have only one contributing term. This gives

$$\frac{(1-a)}{(1-ax)(1-x/a)}\frac{1}{1-a^2x}\bigg|_{a^0} = \frac{(1-a)}{(1-ax)}\frac{1}{1-a^2x}\bigg|_{a=x} = \frac{(1-x)}{(1-x^2)}\frac{1}{1-x^3}, \tag{38}$$

$$\frac{(1-a)}{(1-ax)(1-x/a)}\frac{x/a^2}{1-x/a^2}\bigg|_{a^0} = \frac{(1-a)}{(1-x/a)}\frac{x/a^2}{1-x/a^2}\bigg|_{a=1/x} = \frac{-(1-x)}{(1-x^2)}\frac{x^2}{1-x^3}. \tag{39}$$

Using (38) and (39) in (37) we get

$$C(x) = \frac{1}{1-x^2}\left(\frac{(1-x)}{(1-x^2)}\frac{1}{1-x^3} - \frac{(1-x)}{(1-x^2)}\frac{x^2}{1-x^3}\right) = \frac{1-x}{(1-x^2)(1-x^3)}.$$

Together with (36), we get

$$W_4(q) = \frac{(1+q^4)(1+q^6)}{(1-q^2)(1-q^4)^2} \times \frac{1-q^4}{(1-q^8)(1-q^{12})} = \frac{1}{(1-q^2)(1-q^4)^2(1-q^6)}.$$

We will see in Section 5 what needs to be done to carry out step $\mathbf{b}_5'$) on the computer.

The identities for $W_2(q), W_3(q), W_4(q)$ in (6) have also been derived in [3] by symmetric function methods from the relation (5). In fact, all three results in (6) are immediate consequences of the following deeper symmetric function identity. (For a proof see [3, Section 2].)

Theorem 4.5.

$$s_{d,d} \odot s_{d,d} = \sum_{\lambda \vdash 2d} s_\lambda \chi(\lambda \in EO_4)$$

where EO_4 denotes the set of partitions of length 4 whose parts are $\geqslant 0$ and all even or all odd.

5. Solving the Hdd problem for $k = 5$

This section is divided into three parts. In the first subsection we start with our computer findings and end by giving a combinatorial decomposition that works nicely to obtain $F_3(x)$. In the second subsection, this decomposition is described algebraically and, together with group actions, turned into manipulatory gyrations that will be used to extract $G_5(q)$ and $W_5(q)$ out of our computers. In the final subsection, by combining the idea of decomposition and the method of divided difference in Section 4, we give our best way that reduce the computation time for $G_5(q)$ and $W_5(q)$ down to a few minutes.

5.1. A combinatorial decomposition for $F_3(x)$

Our initial efforts at solving the Hdd an Sdd problems were entirely carried out by computer experimentation. After obtaining quite easily the series $G_2(q)$, $G_3(q)$, $G_4(q)$ and $W_2(q)$, $W_3(q)$, $W_4(q)$, all the computer packages available to us failed to directly deliver $G_5(q)$ and $W_5(q)$.

The computer data obtained for the Hdd problem for $k = 2, 3$ were combinatorially so revealing that we have been left with a strong impression that this problem should have a very beautiful combinatorial general solution. Only time will tell if this will ever be the case. To stimulate further research we will begin by reviewing our initial computer and manual combinatorial findings.

Recall that we denoted by $\mathcal{F}_d$ the collection of all d-subsets of the $2d$ element set Ω_{2d}. We also showed (in Theorem 3.1) that the coefficient $m_d(k)$ in the series $G_k(q) = \sum_{d \leqslant 0} q^{2d} m_d(k)$ counts the number of orbits under the action of the symmetric group $\mathcal{S}_{2d}$ on the k-fold cartesian product $\mathcal{F}_d \times \mathcal{F}_d \times \cdots \times \mathcal{F}_d$. Denoting by $(A_1, A_2, \ldots, A_k)$ a generic element of this cartesian product, then each orbit is uniquely determined by the 2^k cardinalities

$$p_{\epsilon_1,\epsilon_2,\ldots,\epsilon_k} = \left|A_1^{\epsilon_1} \cap A_2^{\epsilon_2} \cap \cdots \cap A_k^{\epsilon_k}\right|$$

where for each $1 \leqslant i \leqslant k$ we set

$$A_i^{\epsilon_i} = \begin{cases} A_i & \text{if } \epsilon_i = 0, \\ {}^c A_i & \text{if } \epsilon_i = 1. \end{cases} \qquad \left(\text{here } {}^c A_i = \Omega_{2d}/A_i\right).$$

It is also convenient to set $A_{\epsilon_1,\epsilon_2,\ldots,\epsilon_k} = A_1^{\epsilon_1} \cap A_2^{\epsilon_2} \cap \cdots \cap A_k^{\epsilon_k}$. This given we have seen that the condition $(A_1, A_2, \ldots, A_k) \in \mathcal{F}_d^k$ is equivalent to the Diophantine system

$$\mathcal{S}_k = \left\| \begin{array}{l} \sum_{\epsilon_1=0}^1 \sum_{\epsilon_2=0}^1 \cdots \sum_{\epsilon_k=0}^1 (1-2\epsilon_1) p_{\epsilon_1,\epsilon_2,\ldots,\epsilon_k} = 0 \\ \sum_{\epsilon_1=0}^1 \sum_{\epsilon_2=0}^1 \cdots \sum_{\epsilon_k=0}^1 (1-2\epsilon_2) p_{\epsilon_1,\epsilon_2,\ldots,\epsilon_k} = 0 \\ \vdots \\ \sum_{\epsilon_1=0}^1 \sum_{\epsilon_2=0}^1 \cdots \sum_{\epsilon_k=0}^1 (1-2\epsilon_k) p_{\epsilon_1,\epsilon_2,\ldots,\epsilon_k} = 0 \end{array} \right.$$

together with the condition $|\Omega_{2d}| = 2d$, that is $\sum_{\epsilon_1=0}^1 \sum_{\epsilon_2=0}^1 \cdots \sum_{\epsilon_k=0}^1 p_{\epsilon_1,\epsilon_2,\ldots,\epsilon_k} = 2d$.

There are several algorithms available to solve such a system. See for instance [7, Chapter 4.6]. The algorithm we used for our computer experimentations is the MacMahon algorithm which has been recently implemented in MATHEMATICA by Andrews, Paule and Riese and in MAPLE by Xin using the partial fraction method of computing constant terms.

The former can be downloaded from the web site http://www.risc.uni-linz.ac.at/research/combinat/software/Omega/ and the latter from the web site http://www.combinatorics.net.cn/homepage/xin/maple/ell2.rar. For computer implementation we found it more convenient to use the alternate notation adopted in Remark 2.5. That is

$$\mathcal{S}_k = \| p_1 V_1 + p_2 V_2 + \cdots + p_{2^k} V_{2^k} = 0. \tag{40}$$

These algorithms may yield quite a bit more than the number of solutions of such a system. For instance, in our case the "Omega package" of Andrews, Paule and Riese should, in principle, yield the formal power series

$$F_k(x_1, x_2, \ldots, x_{2^k}) = \sum_{(p_1, p_2, \ldots, p_{2^k}) \in \mathcal{S}_k} x_1^{p_1} x_2^{p_2} \cdots x_{2^k}^{p_{2^k}}.$$

It follows from the general theory of Diophantine systems that $F_k(x_1, x_2, \ldots, x_{2^k})$ is always the Taylor series of a rational function.

Now for $\mathcal{S}_2$ and $\mathcal{S}_3$ the Omega package gives

$$F_2(x_1, x_2, x_3, x_4) = \frac{1}{(1 - x_1 x_4)(1 - x_2 x_3)}, \tag{41}$$

$$F_3(x_1, x_2, \ldots, x_8) = \frac{1 - x_2 x_3 x_5 x_8 x_1 x_4 x_6 x_7}{(1 - x_1 x_8)(1 - x_2 x_7)(1 - x_3 x_6)(1 - x_4 x_5)(1 - x_2 x_3 x_5 x_8)(1 - x_1 x_4 x_6 x_7)}. \tag{42}$$

But this is as far as this package went in our computers. However we could go further by giving up full information about the solutions and only ask for the series

$$G_k(q) = F_k(x_1, x_2, \ldots, x_{2^k})|_{x_i = q},$$

which can be computed from its constant term representation in (8). For example, the program *Latte* by De Loera, Hemmecke, Tauzer, Yoshida, which is available at http://www.math.ucdavis.edu/~latte/ computed the $G_4(q)$ series in approximately 30 seconds. However, this is as far as *Latte* went on our machines. We should also mention that all the series $G_k(q)$ and $W_k(q)$ for $k \leqslant 4$ can be obtained in only a few seconds, from the software of Xin by computing the corresponding constant terms in (8) and (9).

To get our computers to deliver $G_5(q)$ and $W_5(q)$ in a matter of minutes a divide and conquer strategy had to be adopted. More precisely, these rational functions were obtained by decomposing the constant terms (8) and (9) as sums of constant terms. This decomposition had its origin from an effort to find a human proof of the identities in (41) and (42). More importantly, the surprising simplicity of (41) and (42) required a combinatorial explanation. Our findings there provided the combinatorial tools that were used in our early computations of $G_5(q)$ and $W_5(q)$. This given, before describing our work on these series, we will show how to obtain (41) and (42) entirely by hand.

Let us start by sketching the idea for $k = 2$. Beginning with

$$\mathcal{S}_2 = \left\| \begin{array}{l} p_1 + p_2 - p_3 - p_4 = 0 \\ p_1 - p_2 + p_3 - p_4 = 0 \end{array} \right.$$

we immediately notice that $(1, 0, 0, 1)$ and $(0, 1, 1, 0)$ are solutions. Set $a = \min(p_1, p_4)$ and $b = \min(p_2, p_3)$. It is clear that the following difference must also be a solution.

$$(q_1, q_2, q_3, q_4) = (p_1, p_2, p_3, p_4) - (a, b, b, a) = (p_1 - a, p_2 - b, p_3 - b, p_4 - a).$$

Now $q_1 q_4 = 0$ and $q_2 q_3 = 0$. This gives us four possibilities for (q_1, q_2, q_3, q_4):

$$(0,0,x,y), \quad (0,x,0,y), \quad (x,0,y,0), \quad (x,y,0,0),$$

for some nonnegative integers x, y. It is easy to show that we must always have $x = 0$, $y = 0$ and therefore the general solution of $\mathcal{S}_2$ is of the form (a,b,b,a). See [2] for details. We thus reobtain the full generating function (41) of solutions of $\mathcal{S}_2$:

$$F_2(x_1,x_2,x_3,x_4) = \sum_{a\geqslant 0}\sum_{b\geqslant 0} x_1^a x_2^b x_3^b x_4^a = \frac{1}{(1-x_1x_4)(1-x_2x_3)}.$$

It turns out that we can deal with $\mathcal{S}_3$ in a similar manner. Again we begin by noticing the four *symmetric* solutions

$$(1,0,0,0,0,0,0,1), \quad (0,1,0,0,0,0,1,0), \quad (0,0,1,0,0,1,0,0), \quad (0,0,0,1,1,0,0,0).$$

Next we set

$$a = \min(p_1,p_8), \quad b = \min(p_2,p_7), \quad c = \min(p_3,p_6), \quad d = \min(p_4,p_5),$$

and by subtraction we get a solution

$$(q_1,q_2,q_3,q_4,q_5,q_6,q_7,q_8) = (p_1,p_2,p_3,p_4,p_5,p_6,p_7,p_8) - (a,b,c,d,d,c,b,a) \tag{43}$$

with the property $q_i q_{9-i} = 0$ for $1 \leqslant i \leqslant 4$. It will be good here and after to call the set

$$\{i \in [1,n]:\ p_i \geqslant 1\}$$

the *support* of the composition $(p_1,p_2,\ldots,p_n)$. This given, we derive that the resulting composition in (43) will necessarily have its support contained in at least one of the following 16 patterns:

$$\begin{array}{llll}
(0,0,0,0,*,*,*,*), & (0,0,0,*,0,*,*,*), & (0,0,*,0,*,0,*,*), & (0,0,*,*,0,0,*,*),\\
(0,*,0,0,*,*,0,*), & (0,*,0,*,0,*,0,*), & (0,*,*,0,*,0,0,*), & (0,*,*,*,0,0,0,*),\\
(*,0,0,0,*,*,*,0), & (*,0,0,*,0,*,*,0), & (*,0,*,0,*,0,*,0), & (*,0,*,*,0,0,*,0),\\
(*,*,0,0,*,*,0,0), & (*,*,0,*,0,*,0,0), & (*,*,*,0,*,0,0,0), & (*,*,*,*,0,0,0,0).
\end{array} \tag{44}$$

Unlike the case $k = 2$ not all of these patterns force a trivial solution. To find out which it is helpful to resort to a Venn diagram imagery. To this end recall that a solution of $\mathcal{S}_3$ gives the cardinalities of the 8 regions of the Venn diagram of three d-subsets A_1, A_2, A_3 of Ω_{2d} (see Fig. 2).

In Fig. 3, each pattern is represented by a Venn diagram where in each region $A_1^{\epsilon_1} \cap A_2^{\epsilon_2} \cap A_3^{\epsilon_3}$ that corresponds to a $*$ in the pattern we placed a black dot. That means that only the regions with a dot may have $\geqslant 0$ cardinality. The miracle is that all but the two patterns $(0,*,*,0,*,0,0,*)$ and $(*,0,0,*,0,*,*,0)$ can be quickly excluded by a reasoning that only uses the positions of the dots in the Venn diagram. In fact, in each of the excluded cases, we show that it is impossible to replace the dots by $\geqslant 0$ integers in such a manner that the three sets A_1, A_2, A_3 and their complements ${}^cA_1$, ${}^cA_2$, ${}^cA_3$ end up having the same cardinality (except for all empty sets).

The reasoning is so cute that we are compelled to present it here in full. In what follows the jth diagram in the ith row will be referred to as "D_{ij}":

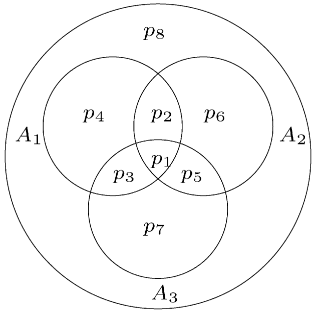


Fig. 2. The Ven diagram for $\mathcal{S}_3$.

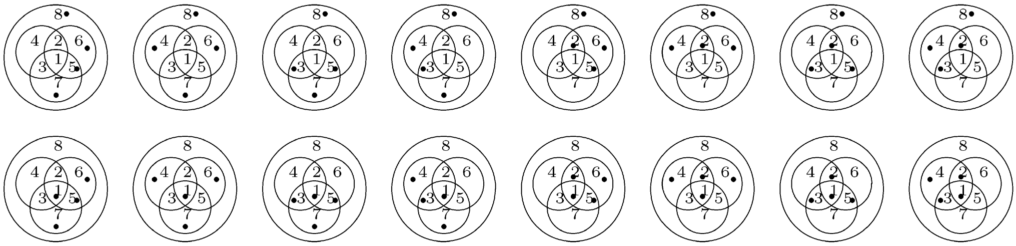


Fig. 3. The 16 support patterns for $\mathcal{S}_3$.

(1) D_{11}, D_{14}, D_{16}, D_{23}, D_{25}, and D_{28} can be immediately excluded because one of A_1, A_2, A_3, A_1^c, A_2^c or A_3^c would be empty.
(2) In D_{15} the dot next to 8 should give the cardinality of A_2^c (say d) and then the dot next to the 2 should also give d. But that forces the dots next to 5 and 6 to be 0, leaving A_3 empty, a contradiction. The same reasoning applies to D_{12}, D_{13}, D_{18}, D_{21}, D_{24}, D_{26}, and D_{27}.

That leaves only the two diagrams D_{17} and D_{22} which clearly correspond to the two above mentioned patterns. Now we see that for D_{22} we must have the equalities $p_1 + p_4 = p_1 + p_6 = p_1 + p_7 = p_6 + p_7$. This forces $p_1 = p_4 = p_6 = p_7$. In summary this pattern can only support the composition $(u, 0, 0, u, 0, u, u, 0)$. The same reasoning yields that the diagram D_{17} can only support the composition $(0, v, v, 0, v, 0, 0, v)$. It follows that the general solution of $\mathcal{S}_3$ must be of the form $(a, b, c, d, d, c, b, a) + (u, v, v, u, v, u, u, v)$.

Now recall that after the subtraction of a symmetric solution we are left with an *asymmetric* solution. Thus to avoid over counting we must impose the condition $uv = 0$. This leaves only three possibilities $u = v = 0$, $u > 0$, $v = 0$ or $u = 0$, $v > 0$. Thus

$$F_3(x) = \sum_{a\geqslant 0}\sum_{b\geqslant 0}\sum_{c\geqslant 0}\sum_{d\geqslant 0}(x_1x_8)^a(x_2x_7)^b(x_3x_6)^c(x_4x_5)^d\left(1 + \sum_{u\geqslant 1}(x_1x_4x_6x_7)^u + \sum_{v\geqslant 1}(x_2x_3x_5x_8)^v\right)$$
$$= \frac{1}{(1-x_1x_8)(1-x_2x_7)(1-x_3x_6)(1-x_4x_5)}\left(1 + \frac{x_1x_4x_6x_7}{1-x_1x_4x_6x_7} + \frac{x_2x_3x_5x_8}{1-x_2x_3x_5x_8}\right),$$

which is only another way of writing (42).

5.2. Algebraic decompositions and group actions

It is easy to see that the decomposition of a solution into a sum of a symmetric plus an asymmetric solution can be carried out for general k. In fact, note that if $0 \leqslant i \leqslant 2^k - 1$ has binary digits $\epsilon_1\epsilon_2\cdots\epsilon_k$

then the binary digits of 2^k-1-i are $\bar\epsilon_1\bar\epsilon_2\cdots\bar\epsilon_k$ (with $\bar\epsilon=1-\epsilon$). Thus we see from (40) that in each equation p_i and p_{2^k+1-i} appear with opposite signs. This shows that for each $k\geqslant 2$ the system $\mathcal{S}_k$ has 2^{k-1} symmetric solutions, which may be symbolically represented by the monomials $x_ix_{i'}$ for $i=1,\dots,2^{k-1}$, where we use (and will often use) i' to denote 2^k+1-i when k is fixed.

Proceeding as we did for $\mathcal{S}_2$ and $\mathcal{S}_3$ we arrive at a unique decomposition of each solution of $\mathcal{S}_k$ into

$$(p_1,p_2,\dots,p_{2^k})=(u_1,u_2,\dots,u_2,u_1)+(q_1,q_2,\dots,q_{2^k})$$

with the first summand symmetric and the second asymmetric, that is $u_i=u_{i'}$ and $q_iq_{i'}=0$ for $1\leqslant i\leqslant 2^{k-1}$, and thereby obtain a factorization of $F_k(x)$ in the form

$$F_k(x)=\left(\prod_{i=1}^{2^{k-1}}\frac{1}{1-x_ix_{i'}}\right)F_k^A(x) \tag{45}$$

with $F_k^A(x)$ denoting the complete generating function of the asymmetric solutions.

This given it is tempting to try to apply, in the general case, the same process we used for $k=3$ and obtain the rational function $F_k^A(x)$ by selecting the patterns that do contain the support of an asymmetric solution. Note that the total number of asymmetric patterns to be examined is $2^{2^{k-1}}$ which is already 256 for $k=4$. For $k=5$ the number grows to 65,536 and doing this by hand is out of the question. Moreover, it is easy to see, by going through a few cases, that even for $k=4$ the geometry of the Venn Diagrams is so intricate that the only way that we can find out if a given pattern contains the support of a solution is to solve the corresponding reduced system.

Nevertheless, using some inherent symmetries of the problem, the complexity of the task can be substantially reduced to permit the construction of $G_5(q)$ by computer. To describe how this was done we need some notation. We will start with the complete generating function of the system $\mathcal{S}_k$ as given in Remark 2.5, that is

$$F_k(x_1,x_2,\dots,x_{2^k})=\prod_{i=1}^{2^k}\frac{1}{1-x_iA_i}\bigg|_{a_1^0a_2^0\cdots a_k^0},$$

where $A_i=\prod_{i=1}^k a_i^{1-2\epsilon_i}$, with $\epsilon_1\epsilon_2\cdots\epsilon_k$ being the binary digits of $i-1$. Note that since (as we previously observed) the binary digits of 2^k-1-i are $\bar\epsilon_1\bar\epsilon_2\cdots\bar\epsilon_k$, we have $A_{i'}=1/A_i$. It then follows that

$$\frac{1-x_ix_{i'}}{(1-x_iA_i)(1-x_{i'}A_{i'})}=\left(\frac{1}{1-x_iA_i}+\frac{x_{i'}/A_i}{1-x_{i'}/A_i}\right).$$

Thus combining the factors containing A_i and $A_{i'}$ we may rewrite (45) in the form

$$F_k(x_1,x_2,\dots,x_{2^k})=\prod_{i=1}^{2^{k-1}}\frac{1}{1-x_ix_{i'}}\prod_{i=1}^{2^{k-1}}\left(\frac{1}{(1-x_iA_i)}+\frac{x_{i'}/A_i}{(1-x_{i'}/A_i)}\right)\bigg|_{a_1^0a_2^0\cdots a_k^0}. \tag{46}$$

Comparing with (45) we derive that the complete generating function of the asymmetric solutions is given by the following sum:

$$F_k^A(x)=\sum_{S\subseteq[1,2^{k-1}]}F_S(x), \tag{47}$$

where

$$F_S(x) = \left(\prod_{i\notin S} \frac{1}{(1-x_iA_i)}\right) \times \left(\prod_{i\in S} \frac{x_{i'}/A_i}{(1-x_{i'}/A_i)}\right)\bigg|_{a_1^0a_2^0\cdots a_k^0}. \tag{48}$$

In this way we have described our decomposition algebraically. Using notation as of (40), we can see that $F_S(x)$ is none other than the complete generating function of the reduced system

$$\sum_{i\notin S} p_iV_i + \sum_{i\in S} p_{i'}V_{i'} = 0$$

with the added condition that $p_{i'} \geqslant 1$ for all $i \in S$.

Note that for $k=3$ the summands in (47) correspond precisely to the 16 patterns in (44) with the added condition that the "$*$" in position $i \geqslant 5$ should represent $p_i \geqslant 1$ in the corresponding solution vector. This extra condition is precisely what is needed to eliminate overcounting.

Perhaps all this is best understood with an example. For instance for $k=3$ the patterns

$$(*,0,0,*,0,*,*,0) \quad \text{and} \quad (0,*,*,0,*,0,0,*)$$

were the only ones that supported an asymmetric solution. They represent the two reduced systems

$$\mathcal{S}_{\{14\}} = \left\| \begin{array}{l} p_1+p_4-p_6-p_7=0 \\ p_1-p_4+p_6-p_7=0 \\ p_1-p_4-p_6+p_7=0 \end{array} \right., \qquad \mathcal{S}_{\{23\}} = \left\| \begin{array}{l} p_2+p_3-p_5-p_8=0 \\ p_2-p_3+p_5-p_8=0 \\ -p_2+p_3+p_5-p_8=0 \end{array} \right.$$

and correspond to the following two summands of (47) for $k=3$,

$$F_{\{1,4\}}(x) = \frac{1}{1-x_1a_1a_2a_3}\,\frac{1}{1-x_4a_1/a_2a_3}\,\frac{x_6a_2/a_1a_3}{1-x_6a_2/a_1a_3}\,\frac{x_7a_3/a_1a_2}{1-x_7a_3/a_1a_2}\bigg|_{a_1^0a_2^0a_3^0} = \frac{x_1x_4x_6x_7}{1-x_1x_4x_6x_7}, \tag{49}$$

$$F_{\{2,3\}}(x) = \frac{1}{1-x_2a_1a_2/a_3}\,\frac{1}{1-x_3a_1a_3/a_2}\,\frac{x_5a_2a_3/a_1}{1-x_5a_2a_3/a_1}\,\frac{x_8/a_1a_2a_3}{1-x_8/a_1a_2a_3}\bigg|_{a_1^0a_2^0a_3^0} = \frac{x_2x_3x_5x_8}{1-x_2x_3x_5x_8}. \tag{50}$$

A close look at these two expressions should reveal the key ingredient that needs to be added to our algorithms that will permit reaching $k=5$ in the Hdd and Sdd problems. Indeed we see that $F_{\{1,4\}}(x)$ goes onto $F_{\{2,3\}}(x)$ if we act on the vector $(x_1, x_2, \ldots, x_8)$ by the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 4 & 1 & 2 & 7 & 8 & 5 & 6 \end{pmatrix} \tag{51}$$

and on the triple (a_1, a_2, a_3) by the operation $a_2 \to a_2^{-1}$. In fact, σ is none other than an image of the map $(\epsilon_1, \epsilon_2, \epsilon_3) \to (\epsilon_1, \bar{\epsilon}_2, \epsilon_3)$ on the binary digits of $0, 1, \ldots, 7$, as we can easily see when we replace each i in (51) by the binary digits of $i-1$

$$\sigma = \begin{pmatrix} 000 & 001 & 010 & 011 & 100 & 101 & 110 & 111 \\ 010 & 011 & 000 & 001 & 110 & 111 & 100 & 101 \end{pmatrix}.$$

What goes on is quite simple. Recall that solutions p of our system $\mathcal{S}_k$ can also be viewed as assignments of weights to the vertices of the k-hypercube giving all hyperfaces equal weight. Then clearly any rotation or reflection of the hypercube will carry this assignment onto an assignment with the same property. Thus the Hyperoctahedral group $\mathcal{B}_k$ will act on all the constructs we used to solve $\mathcal{S}_k$.

To make precise the action of $\mathcal{B}_k$ on $[1, 2^k]$ we need some conventions.

(1) We will view the elements of $\mathcal{B}_k$ as pairs (α, η) with a permutation $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k) \in S_k$ and a binary vector $\eta = (\eta_1, \eta_2, \dots, \eta_k)$.

(2) Next, for any binary vector $\epsilon = (\epsilon_1, \epsilon_2, \dots, \epsilon_k)$ let us set

$$(\alpha, \eta)\epsilon = (\epsilon_{\alpha_1} + \eta_1, \epsilon_{\alpha_2} + \eta_2, \dots, \epsilon_{\alpha_k} + \eta_k) \tag{52}$$

with "*mod 2*" addition.

(3) This given, to each element $g = (\alpha, \eta) \in \mathcal{B}_k$ there corresponds a permutation $\sigma(g)$ by setting

$$\sigma(g) = \begin{pmatrix} 1 & 2 & \cdots & 2^k \\ \sigma_1 & \sigma_2 & \cdots & \sigma_{2^k} \end{pmatrix},$$

where $\sigma_i = j$ if and only if the k-vector $\epsilon = (\epsilon_1, \epsilon_2, \dots, \epsilon_k)$ giving the binary digits of $i - 1$ is sent by (52) onto the k-vector giving the binary digits of $j - 1$. In particular we will set

$$g(x_1, x_2, \dots, x_{2^k}) = (x_{\sigma_1}, x_{\sigma_2}, \dots, x_{\sigma_{2^k}}). \tag{53}$$

(4) In the same vein we will make $\mathcal{B}_k$ act on the k-tuple $(a_1, a_2, \dots, a_k)$ by setting, again for $g = (\alpha, \eta)$,

$$g(a_1, a_2, \dots, a_k) = \left(a_{\alpha_1}^{1-2\eta_1}, a_{\alpha_2}^{1-2\eta_2}, \dots, a_{\alpha_k}^{1-2\eta_k}\right). \tag{54}$$

With these conventions we can easily derive that $g x_i A_i = x_{\sigma_i} A_{\sigma_i}$. Thus

$$g \prod_{i=1}^{2^k} \frac{1}{1 - x_i A_i}\bigg|_{a_1^0 a_2^0 \cdots a_k^0} = \prod_{i=1}^{2^k} \frac{1}{1 - x_{\sigma_i} A_{\sigma_i}}\bigg|_{a_1^0 a_2^0 \cdots a_k^0} = \prod_{i=1}^{2^k} \frac{1}{1 - x_i A_i}\bigg|_{a_1^0 a_2^0 \cdots a_k^0},$$

from which we again derive the $\mathcal{B}_k$ invariance of the complete generating function $F_k(x_1, x_2, \dots, x_{2^k})$.

If we let $\mathcal{B}_{k-1}$ not only act on the indices $1, 2, \dots, 2^{k-1}$, but also on $1', 2', \dots, 2^{k-1'}$ by $\sigma_{i'} = \sigma_i'$. Then $\mathcal{B}_{k-1}$ permutes the summands in (47) as well as the factors in the product $\prod_{i=1}^{2^{k-1}} \frac{1}{1 - x_i x_{i'}}$. Note further that if we only want the q-series $G_k(q)$ we can reduce (47) to

$$G_k^A(q) = \sum_{S \subseteq [1, 2^{k-1}]} G_S(q), \tag{55}$$

where $G_S(q) = F_S(x)|_{x_i = q}$. But if for some $g \in \mathcal{B}_{k-1}$ we have $F_{S_1}(x_{\sigma_1}, x_{\sigma_2}, \dots, x_{\sigma_{2^k}}) = F_{S_2}(x)$, then replacing each x_i by q converts this to the equality $G_{S_1}(q) = G_{S_2}(q)$. That means that we need only compute the constant terms in (55) for orbit representatives, then replace (55) by a sum over orbit representatives multiplied by orbit sizes. More precisely we get

$$G_k^A(q) = \sum_i m_i G_{S_i}(q), \tag{56}$$

where the sum ranges over all orbits and m_i denotes the cardinality of the orbit of the representative $F_{S_i}(x)$. In the computer implementation we obtain orbit representatives as well as orbit sizes, by acting with $\mathcal{B}_{k-1}$ on the monomials $M_S = \prod_{i \in S} x_i$.

For detailed work along this line for $k = 3, 4, 5$, we refer the reader to [2]. Here, for brevity, we only report that we can produce $G_5(q)$ in about 15 minutes using this decomposition.

The decomposition in (55) is only $\mathcal{B}_{k-1}$ invariant, and it is natural from the geometry of the hypercube labelings, to ask of a $\mathcal{B}_k$ invariant decomposition. To obtain such a decomposition of $F_k(x)$ we will pair off the factors containing A_i and $A_{i'}$ by means of the more symmetric identity

$$\frac{1-x_i x_{2^k+1-i}}{(1-x_i A_i)(1-x_{2^k+1-i}A_{2^k+1-i})} = \left(1+\frac{x_i A_i}{1-x_i A_i}+\frac{x_{i'}A_{i'}}{1-x_{i'}A_{i'}}\right)$$

and derive that

$$F_k(x) = \sum_{S\cup T\subseteq[1,2^{k-1}]} F_{S,T}(x),$$

where S and T are disjoint and

$$F_{S,T}(x) = \left(\prod_{i\in S}^{k}\frac{x_i A_i}{1-x_i A_i}\right)\left(\prod_{i\in T}\frac{x_i'/A_i}{1-x_i'/A_i}\right).$$

Note that every pair (S,T) should be identified with the set $S\cup\{i'\colon\ i\in T\}\subseteq[1,2^k]$ when applying the action of $\mathcal{B}_k$.

Example 5.1. For $k=3$ we have $3^4=81$ summands with 9 orbits but only 2 orbits do contribute to F_3^A. The two orbits corresponds to the monomials 1 and $x_1x_4x_6x_7$ with respective orbit sizes 1 and 2. The orbit representative that corresponds to 1 is simply the case $F_{\phi,\phi}=1|_{a_1^0a_2^0a_3^0a_4^0}=1$ and that corresponds to $x_1x_4x_6x_7$ is

$$F_{\{1,4\},\{2,3\}}(x) = \frac{x_1A_1}{1-x_1A_1}\frac{x_4A_4}{1-x_4A_4}\frac{x_6A_6}{1-x_6A_6}\frac{x_7A_7}{1-x_7A_7}\bigg|_{a_1^0a_2^0a_3^0a_4^0} = \frac{x_1x_4x_6x_7}{1-x_1x_4x_6x_7}.$$

Example 5.2. For $k=4$ we have $3^8=6561$ summands with 62 orbits but only 10 orbits do contribute to F_4^A. We obtain the following complete generating functions for the 10 orbit representatives:

$$(1)\quad 1$$

$$(24)\quad \frac{x_1x_{15}x_4x_{14}}{1-x_1x_{15}x_4x_{14}}$$

$$(16)\quad \frac{x_{16}x_7(x_9)^2x_6x_4}{1-x_{16}x_7{x_9}^2x_6x_4}$$

$$(96)\quad \frac{x_{15}x_3x_7(x_{12})^2(x_9)^2(x_6)^3}{(1-x_{12}x_7x_9x_6)(1-x_{15}x_3x_{12}x_9{x_6}^2)}$$

$$(96)\quad \frac{x_{16}x_{14}x_5x_7(x_{11})^2(x_2)^2}{(1-x_2x_7x_{11}x_{14})(1-x_{16}x_5x_2x_{11})}$$

$$(192)\quad \frac{x_9x_{10}x_1(x_4)^4(x_{15})^3(x_5)^2(x_{14})^2}{(1-x_1x_{15}x_4x_{14})(1-x_{15}x_5x_{10}x_4)(1-x_{15}x_5x_9{x_4}^2x_{14})}$$

$$(64)\quad \frac{x_6x_{16}x_4(x_3)^2(x_5)^2(x_{15})^2(x_{10})^3}{(1-x_{15}x_5x_{10}x_4)(1-x_{16}x_5x_3x_{10})(1-x_{15}x_3x_{10}x_6)}$$

$$(64)\quad \frac{x_3x_7x_4(x_6)^5x_1(x_9)^3(x_{12})^3(x_{15})^3}{(1-x_1x_{15}x_{12}x_6)(1-x_{12}x_7x_9x_6)(1-x_{15}x_9x_6x_4)(1-x_{15}x_3x_{12}x_9{x_6}^2)}$$

$$(32) \quad \frac{(x_{13})^3(x_{12})^3 x_1 x_3 x_2 x_6 x_7 x_8 (1 - x_1 x_2 x_3 x_8 x_{12}{}^3 x_7 x_{13}{}^3 x_6)}{(1 - x_1 x_8 x_{12} x_{13})(1 - x_2 x_{12} x_7 x_{13})(1 - x_3 x_{12} x_{13} x_6)(1 - x_1 x_{12}{}^2 x_7 x_{13} x_6)(1 - x_2 x_3 x_8 x_{12} x_{13}{}^2)}$$

$$(8) \quad \frac{x_4 x_5 x_3 x_6 x_9 x_{10} x_{15} x_{16} (1 - 2x_{15} x_{16} x_5 x_3 x_{10} x_9 x_6 x_4 + x_{15}{}^2 x_{16}{}^2 x_5{}^2 x_3{}^2 x_{10}{}^2 x_9{}^2 x_6{}^2 x_4{}^2)}{(1 - x_{16} x_3 x_9 x_6)(1 - x_{16} x_5 x_9 x_4)(1 - x_{15} x_9 x_6 x_4)(1 - x_{15} x_5 x_{10} x_4)(1 - x_{16} x_5 x_3 x_{10})(1 - x_{15} x_3 x_{10} x_6)}.$$

Here the numbers in parentheses give the respective orbit sizes.

Replacing all the x_i by q and summing as in (56), we obtain

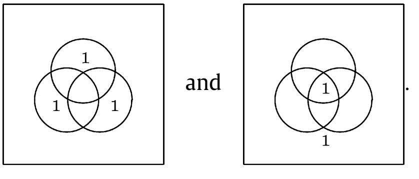
$$G_4(q) = \frac{1 + q^2 + 21q^4 + 36q^6 + 74q^8 + 86q^{10} + 74q^{12} + 36q^{14} + 21q^{16} + q^{18} + q^{20}}{(1-q^2)^7(1-q^4)^4(1-q^6)}.$$

We should mention that the partial fraction algorithm delivers this rational function in less than a second by directly computing the constant term in (8) for $k = 4$. We computed the above representatives because it contains more information and can be used for an alternate path to $G_5(q)$.

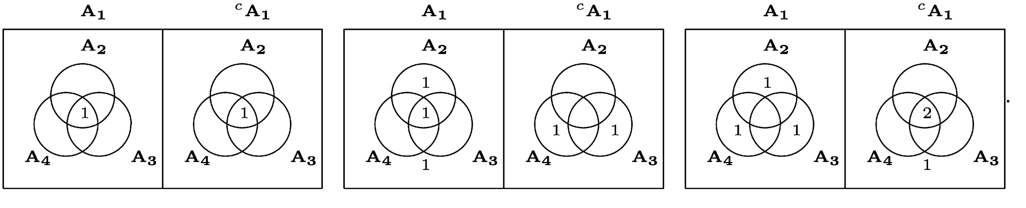
Computing the orbit representatives for $k = 5$ requires the construction of the $2^5 \times 5! = 3840$ elements of $\mathcal{B}_5$ and examining their action on the $3^{16} = 43046721$ symmetric supports. This took a few hours on our computers. We found in this manner that the 43046721 summands in (55) break up into 15418 orbits and of these 6341 contribute to the sum. Most of the orbits have denominators of less than 16 factors. It also took about 15 minutes to persuade MAPLE to deliver $G_5(q)$ in the form displayed in the introduction.

It turns out that the same orbit reduction idea can also be used to compute $W_5(q)$, but with a little more efforts. Detailed work can be found in [2].

Remark 5.3. It is interesting to point out that computing complete generating functions for orbit representatives of summands in (47) yielded as a byproduct orbit representatives of the extreme rays of our Diophantine cone for $k = 4$ and $k = 5$. Note that for $k = 3$ the representatives can be directly derived from our hand computation: there are only two and the corresponding Venn Diagrams are



Here the regions without numbers are empty. The number 1 indicates that the region has only one element. For $k = 4$ we found that there are only three orbits, containing 24, 8 and 16 elements respectively. The corresponding diagrams are depicted below:



Note, for $k = 4$ each Venn diagram is depicted as a pair of Venn diagrams of $k = 3$. The first member of the pair renders the Venn diagram of $A_1 \cap A_2$, $A_1 \cap A_3$, $A_1 \cap A_4$ and the second member renders the Venn diagram of ${}^cA_1 \cap A_2$, ${}^cA_1 \cap A_3$, ${}^cA_1 \cap A_4$.

For $k = 5$ we found that there are 2712 extreme rays which break up into 9 orbits. We give in Fig. 4 a set of representatives depicted as assignments of weights to the vertices of the 5-dimensional hypercube. We imagine that the vertices of this hypercube are indexed by the binary digits of $0, 1, 2, \ldots, 31$

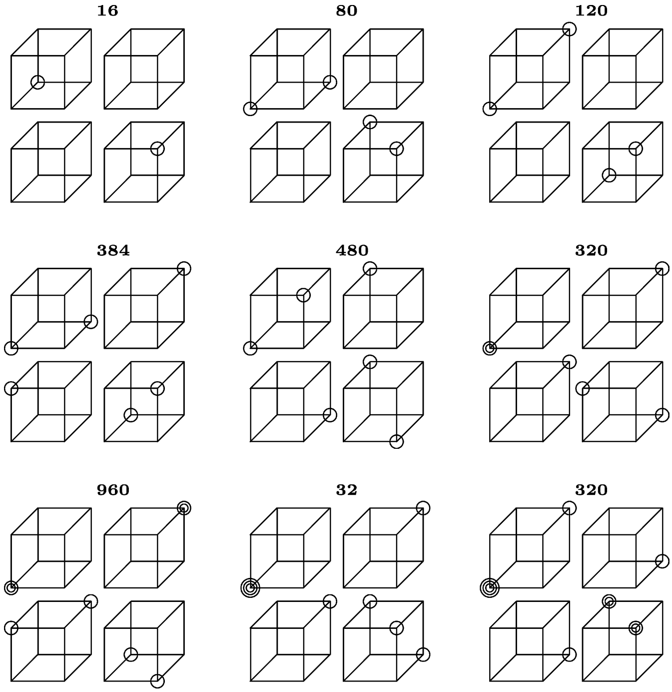


Fig. 4. Representatives of extreme rays for $k = 5$.

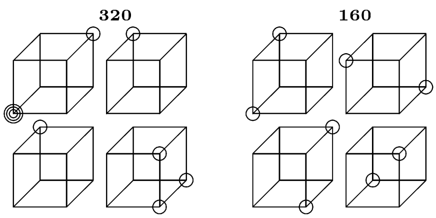


Fig. 5. Representatives for minimal solutions but not extreme rays.

with 00000 the vertex at the origin and 11111 the opposite vertex of the origin. In Fig. 4 each hypercube is represented by two rows of two cubes. The cubes in the first row, from left to right, have the vertices labeled with the binary digits of 1 to 16 (minus 1) and the cubes in the second row have the vertices labeled with the binary digits of 17 to 32 (minus 1). The vertices here have possible weights $0, 1, 2, 3$ and, correspondingly, are surrounded by $0, 1, 2, 3$ concentric circles. The integer on the top of each diagram gives the size of the corresponding orbit.

Each of the corresponding solutions of our system $\mathcal{S}_5$ is *minimal*, that is, it cannot be decomposed into a nontrivial sum of solutions. But we found that there are also 480 minimal solutions that do not come from extreme rays. The latter break up into two orbits, with representatives depicted in Fig. 5.

5.3. *Our fastest way for $G_5(q)$ and $W_5(q)$*

With the notations in the previous subsection and Section 4 handy, we can describe our best way to obtain $G_5(q)$ and $W_5(q)$.

Let us explain the idea for $k = 5$. In Example 5.2 we have obtained for $F_4^A(x)$ 10 orbit representatives with corresponding orbit sizes. Denote them by $R_\ell(x)$ the representatives and m_ℓ the orbit sizes for $\ell = 1, \dots, 10$. From this we can give explicit formula of $F_4^A(x)$ and hence of $F_4(x)$ with the help of $\mathcal{B}_4$ action as follows:

$$F_4(x) = \frac{F_4^A(x)}{\prod_{i=1}^8 (1 - x_i x_{17-i})} = \sum_{\ell=1}^{10} \frac{m_\ell}{|\mathcal{B}_4|} \sum_{g \in \mathcal{B}_4} g \frac{R_\ell}{\prod_{i=1}^8 (1 - x_i x_{17-i})}. \tag{57}$$

Applying Algorithm 4.3 $\mathbf{a_5}$), $\mathbf{b_5}$) to (57), we can obtain $F_5(x)$ by multilinearity:

$$F_5(x) = \sum_{\ell=1}^{10} \frac{m_\ell}{|\mathcal{B}_4|} \sum_{g \in \mathcal{B}_4} \left(g \delta_{1,17} \cdots \delta_{16,32} \frac{R_\ell}{\prod_{i=1}^8 (1 - x_i x_{17-i})} \right) \Bigg|_{x_j = x_j a, x_{16+j} = x_{16+j}/a}^{j=1,2,\dots,16} \Bigg|_{a^0}, \tag{58}$$

where we have used the straightforwardly checked fact: for any rational function $R(x_1, \dots, x_{16})$ and $g \in \mathcal{B}_k$, it holds that

$$\delta_{1,17} \cdots \delta_{16,32} g R(x_1, \dots, x_{16}) = g \delta_{1,17} \cdots \delta_{16,32} R(x_1, \dots, x_{16}),$$

where g is extended to permute also indices $16 + j$ by $g(16 + j) = 16 + g(j)$ for $j = 1, \dots, 16$.

Substituting $x_j = q$ for all j into (58) gives

$$G_5(q) = \sum_{\ell=1}^{10} m_\ell \left(\delta_{1,17} \cdots \delta_{16,32} \frac{R_\ell}{\prod_{i=1}^8 (1 - x_i x_{17-i})} \right) \Bigg|_{x_j = x_j a, x_{16+j} = x_{16+j}/a}^{j=1,2,\dots,16} \Bigg|_{a^0}. \tag{59}$$

That is to say, we only need representatives of $F_{k-1}(x)$ together with orbit sizes to compute $F_k(x)$, and this clearly extends for general k. Using (59), we can persuade Maple to deliver $G_5(q)$ as in (10) in about 12 minutes.

The idea for $W_5(q)$ is similar but much more complicated. In order to use a similar decomposition as for $F_4(x)$, we construct a $\mathcal{B}_4$-invariant constant term to replace the constant term in (15). But the new problem is that we do not have simple orbit representatives as in Example 5.2. We use (again) group action to give new representatives (together with orbit sizes) of reasonable sizes. Now we can apply a similar formula as (59), but (inspired by our computation of $W_3(q)$ and $W_4(q)$) this time we combine the summands before taking "$|_{a^0}$." Using this approach Maple can deliver $W_5(q)$ in only about 5 minutes in total which is the shortest time we have been able to compute this series. See [2] for detailed work.

Finally we mention that the extended version [2] of this paper also includes our first way to obtain $G_5(q)$ and $W_5(q)$ by another trick of divide and conquer.

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