Iterative sequence for asymptotically demicontractive maps in Banach spaces

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Abstract

Let $E$ be a real Banach space and $T : E \to E$ an asymptotically demicontractive and uniformly $L$-Lipschitzian map with $F(T) := \{x \in E : Tx = x\} \neq \emptyset$. We prove necessary and sufficient conditions for the strong convergence of the Mann iterative sequence to a fixed point of $T$.

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1. Introduction

Let $E$ be a real normed linear space, $E^*$ its dual and let $\langle \cdot, \cdot \rangle$ denote the generalized duality pairing between $E$ and $E^*$. Let $J : E \to 2^{E^*}$ be the normalized duality mapping defined for each $x \in E$ by

$$J(x) = \{f^* \in E^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2\}.$$  

It is well known that if $E$ is smooth then $J$ is single-valued. In the sequel we shall denote the single-valued normalized duality map by $j$.  

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The map $T$ is called *asymptotically nonexpansive* with sequence \( \{k_n\} \subset [1, \infty) \) if \( \lim_{n \to \infty} k_n = 1 \) and \( \forall x, y \in E \),
\[
\|T^n x - T^n y\| \leq k_n \|x - y\|, \quad \forall n \in \mathbb{N},
\]
and is called *asymptotically pseudocontractive* with sequence \( \{k_n\} \) if \( \lim_{n \to \infty} k_n = 1 \) and for all \( x, y \in E \) there exists \( j(x - y) \in J(x - y) \) such that
\[
\{T^n x - T^n y, j(x - y)\} \subseteq k_n \|x - y\|^2, \quad \forall n \in \mathbb{N}.
\]
\( T \) is said to be *uniformly L-Lipschitzian* if \( \exists L > 0 \), a constant, such that \( \forall x, y \in E \) and \( \forall n \in \mathbb{N} \),
\[
\|T^n x - T^n y\| \leq L \|x - y\|. \tag{1.3}
\]
Let \( F(T) := \{x \in E : x = Tx\} \not= \emptyset \) denote the set of fixed points of \( T \). If Eqs. (1.1) and (1.2) hold \( \forall x \in E \) and \( \forall y = x^* \in F(T) \), then the map \( T \) is said to be, respectively, *asymptotically quasi-nonexpansive*, and *asymptotically hemicontractive*.

Let \( E = H \) (the Hilbert space). A map \( T : E \to E \) is said to be *\( k \)-strictly asymptotically pseudocontractive* (see, e.g., [3,5]) if there exists a sequence \( \{a_n\}_{n=0}^\infty \) with \( \lim_{n \to \infty} a_n = 1 \) such that
\[
\|T^n x - T^n y\|^2 \leq a_n^2 \|x - y\|^2 + k \|(I - T^n)x - (I - T^n)y\|^2, \tag{1.4}
\]
for some \( k \in [0, 1) \) and for all \( x, y \in E \) and \( n \in \mathbb{N} \). \( T : E \to E \) is called *asymptotically demicontractive* (see, e.g., [3,5]) if there exists a sequence \( \{a_n\}_{n=0}^\infty \) with \( \lim_{n \to \infty} a_n = 1 \) such that
\[
\|T^n x - x^*\|^2 \leq a_n^2 \|x - x^*\|^2 + k \|x - T^n x\|^2, \tag{1.5}
\]
for some \( k \in [0, 1) \) and for all \( x \in E \). \( x^* \in F(T) \) and \( n \in \mathbb{N} \). The class of \( k \)-strictly asymptotically pseudocontractive and asymptotically demicontractive maps were introduced by Liu [3]. If \( k = 0 \) in (1.5) then \( T : E \to E \) is asymptotically quasi-nonexpansive. Let \( E \) be an arbitrary Banach space. \( T : E \to E \) is \( k \)-strictly asymptotically pseudocontractive (see, e.g., [4,5]) if \( \forall x, y \in E \) there exists \( j(x - y) \in J(x - y) \) and a constant \( k \in [0, 1) \) such that
\[
\Re\{1 - T^n\} x - (I - T^n)y, j(x - y)\} \geq \frac{1}{2} (1 - k) \| (I - T^n)x - (I - T^n)y \|^2 - \frac{1}{2} (a_n^2 - 1) \|x - y\|^2. \tag{1.6}
\]
Furthermore, it follows from (1.6) that \( T : E \to E \) is asymptotically demicontractive if \( F(T) \not= \emptyset \) and for all \( x \in E \) and \( x^* \in F(T) \) there exists \( j(x - x^*) \in J(x - x^*) \) and a constant \( k \in [0, 1) \) such that
\[
\Re\{1 - T^n\} x, j(x - x^*) \geq \frac{1}{2} (1 - k) \| x - T^n x \|^2 - \frac{1}{2} (a_n^2 - 1) \|x - x^*\|^2. \tag{1.7}
\]
In 1973, Petryshyn and Williamson [6] proved a necessary and sufficient condition for the Picard and the Mann iterative schemes to converge strongly to fixed points of quasi-nonexpansive mappings in Hilbert spaces. Liu [1,2] extended the above results and obtained some necessary and sufficient conditions for an Ishikawa-type iterative scheme with errors to converge to fixed points of asymptotically quasi-nonexpansive maps.
It is our purpose in this paper to prove necessary and sufficient conditions for the strong convergence of the Mann iteration process to a fixed point of an asymptotically demicontractive map in (real) Banach spaces. Our theorems thus improve and extend the results of Liu [1,2], Osilike [5] and several others.

2. Preliminaries

In the sequel we shall make use of the following lemma.

Lemma 2.1 (Tan and Xu [7, Lemma 1, p. 303]). Let \( \{\beta_n\}_{n \geq 0} \) and \( \{b_n\}_{n \geq 0} \) be sequences of nonnegative real numbers satisfying the inequality

\[
\beta_{n+1} \leq \beta_n + b_n, \quad n \geq 0.
\]

If \( \sum_{n \geq 0} b_n < \infty \) then \( \lim_{n \to \infty} \beta_n \) exists.

Remark 2.2. Let \( E \) be a real normed linear space. Then \( \forall x, y \in E \) and for \( j(x - y) \in J(x - y) \) the following inequality holds:

\[
\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y) \rangle.
\] (2.1)

3. Main results

Lemma 3.1. Let \( E \) be a real normed linear space and \( T : E \to E \) a uniformly \( L \)-Lipschitzian asymptotically demicontractive map with sequence \( \{a_n^2\} \subset [1, \infty) \) and \( F(T) \neq \emptyset \). Let \( \{c_n\}_{n \geq 0} \subset [0, 1] \) be a real sequence such that \( \sum_{n \geq 0} c_n^2 < \infty \) and \( \sum_{n \geq 0} c_n(a_n^2 - 1) < \infty \). Let \( \{x_n\}_{n \geq 0} \) be the sequence generated from an arbitrary \( x_0 \in E \) by

\[
x_{n+1} = (1 - c_n)x_n + c_n T^n x_n, \quad n \geq 0.
\] (3.1)

Then \( \forall x^* \in F(T) \) and \( \forall n, m \in \mathbb{N} \),

(a) there exists \( M > 0 \) such that \( \|x_n - x^*\| \leq M \),

(b) \( \lim_{n \to \infty} \|x_n - x^*\| \) exists,

(c) \( \|x_{n+1} - x^*\| \leq (1 + c_n^2)\|x_n - x^*\| + \mu_n \) for some \( \{\mu_n\} \) with \( \sum_{n \geq 0} \mu_n < \infty \),

(d) \( \|x_{n+m} - x^*\| \leq D\|x_n - x^*\| + D \sum_{i=n}^{n+m-1} \mu_i^2 \), where \( D = e^{\sum_{i=n}^{n+m-1} c_i} \).

Proof of (a) and (b). Using (1.7), (2.1) and (3.1) we get that

\[
\|x_{n+1} - x^*\|^2 \leq (1 - c_n)^2 \|x_n - x^*\|^2 + 2c_n\|T^n x_n - x^*, j(x_{n+1} - x^*)\|
\]

\[
= (1 - c_n)^2 \|x_n - x^*\|^2 - 2c_n\|x_{n+1} - T^n x_{n+1}, j(x_{n+1} - x^*)\|
\]

\[
+ 2c_n\|x_{n+1} - x^*, j(x_{n+1} - x^*)\| + 2c_n\|T^n x_n - T^n x_{n+1}, j(x_{n+1})\|
\]

\[
\leq (1 - c_n)^2 \|x_n - x^*\|^2 + 2c_n(k_n - 1)\|x_{n+1} - x^*\|^2
\]

\[
+ 2c_n\|x_{n+1} - x^*\|^2 + 2c_n\|T^n x_n - T^n x_{n+1}\|\|x_{n+1} - x^*\|.
\]
Moreover,
\[ \|T^n x_n - T^n x_{n+1}\| \leq L \|x_n - x_{n+1}\| \leq c_n L (1 + L) \|x_n - x^*\| \]
and
\[ \|x_{n+1} - x^*\| \leq \|x_{n+1} - x_n\| + \|x_n - x^*\| \leq [c_n (1 + L) + 1] \|x_n - x^*\|. \]

And hence,
\[ \|x_{n+1} - x^*\|^2 \leq (1 - c_n)^2 \|x_n - x^*\|^2 + c_n (a_n^2 - 1) [(c_n (1 + L) + 1)]^2 \|x_n - x^*\|^2 + 2 c_n [c_n (1 + L) + 1] \|x_n - x^*\|^2 + 2 c_n^2 L (1 + L) \|x_n - x^*\|^2 \]
\[ = (1 + c_n^2) \|x_n - x^*\|^2 + [c_n (a_n^2 - 1) [(c_n (1 + L) + 1)]^2 + 2 c_n [c_n (1 + L) + 1] + 2 c_n^2 \|x_n - x^*\|^2 \]
\[ = (1 + \gamma_n) \|x_n - x^*\|^2, \tag{3.2} \]

where \( \gamma_n = c_n (a_n^2 - 1) [(c_n (1 + L) + 1)]^2 + 2 c_n^2 L (1 + L) \|x_n - x^*\|^2 + 2 (1 + L) \|x_n - x^*\|^2 \). Observe that \( \sum_{n=0}^{\infty} \gamma_n < \infty \). From (3.2) we get
\[ \|x_{n+1} - x^*\|^2 \leq \prod_{i=0}^{n} (1 + \gamma_i) \|x_1 - x^*\|^2 \leq e^{\sum_{i=0}^{\infty} \gamma_i} \|x_0 - x^*\|^2. \]

So that \( \|x_n - x^*\| \leq M \) for some \( M > 0 \). If we set \( \beta_n = \|x_n - x^*\|^2 \) and \( b_n = \gamma_n M^2 \) then, by Lemma 2.1, \( \lim_{n \to \infty} \|x_n - x^*\| \) exists.

**Proof of (c).** From (3.2) we get
\[ \|x_{n+1} - x^*\|^2 \leq (1 + c_n^2 + \lambda_n) \|x_n - x^*\|^2, \]
where \( \lambda_n = \gamma_n - c_n^2 \). Moreover,
\[ \|x_{n+1} - x^*\| \leq (1 + c_n^2 + \lambda_n)^{1/2} \|x_n - x^*\| \]
\[ \leq (1 + c_n^2) \|x_n - x^*\| + \lambda_n M = (1 + c_n^2) \|x_n - x^*\| + \mu_n, \]
where \( \mu_n = \lambda_n M = (\gamma_n - c_n^2) M \). Observe that \( \sum_{n=0}^{\infty} \mu_n < \infty \).

**Proof of (d).** From (c) and \( \forall n, m \in \mathbb{N} \) we get
\[ \|x_{n+m} - x^*\| \leq (1 + c_n^2 + \mu_{n+m-1}) \|x_n - x^*\| + \mu_{n+m-1} \]
\[ \leq (1 + c_n^2 + \mu_{n+m-1}) (1 + c_{n+m-2}^2) \|x_{n+m-2} - x^*\| \]
\[ + (1 + c_{n+m-1}^2) \mu_{n+m-2} + \mu_{n+m-1} \]
\[ : \]
Let $E$ be a real Banach space and $T : E \to E$ be a uniformly $L$-Lipschitzian asymptotically demicontractive map with sequence $\{a_n\}$ and $F(T) \neq \emptyset$. Let $\{c_n\}_{n \geq 0} \subset [0, 1]$ be a real sequence such that $\sum_{n \geq 0} c_n^2 < \infty$ and $\sum_{n \geq 0} c_n(a_n^2 - 1) < \infty$. Let $\{x_n\}_{n \geq 0}$ be the sequence generated from an arbitrary $x_0 \in E$ by (3.1). Then $\{x_n\}_{n \geq 0}$ converges strongly to a fixed point of $T$ if and only if $\liminf_{n \to \infty} d(x_n, F(T)) = 0$.

**Proof.** From (c) of Lemma 3.1 we obtain

$$d(x_{n+1}, F(T)) \leq (1 + c_n^2)d(x_n, F(T)) + \delta_n.$$ 

Since $\liminf_{n \to \infty} d(x_n, F(T)) = 0$ we have from (b) of Lemma 3.1 that

$$\lim_{n \to \infty} d(x_n, F(T)) = 0.$$

It now suffices to show that $\{x_n\}_{n \geq 0}$ is Cauchy. For this, let $\varepsilon > 0$ be given. Since $\lim_{n \to \infty} d(x_n, F(T)) = 0$ and $\sum_{i=0}^{\infty} \delta_i < \infty$, there exists a positive integer $N_1$ such that $\forall n \geq N_1$,

$$d(x_n, F(T)) \leq \frac{\varepsilon}{3D} \quad \text{and} \quad \sum_{i=0}^{\infty} \delta_i \leq \frac{\varepsilon}{6D}.$$

In particular there exists $x^* \in F(T)$ such that $d(x_{N_1}, x^*) \leq \varepsilon/(3D)$. Now from Lemma 3.1(c) we have that, $\forall n \geq N_1$, that

$$\|x_{n+m} - x_n\| \leq \|x_{N_1+m} - x_{N_1}\| + \|x_{N_1} - x^*\| + D \sum_{i=N_1}^{N_1+m-1} \delta_i + D\|x_{N_1} - x^*\| + D \sum_{i=N_1}^{N_1+m-1} \delta_i \leq \varepsilon.$$

Hence $\lim_{n \to \infty} x_n$ exists (since $E$ is complete). Suppose that $\lim_{n \to \infty} x_n = x^*$. We now show that $x^* \in F(T)$. But given any $\hat{\varepsilon} > 0$ there exists a positive integer $N_2 \geq N_1$ such that $\forall n \geq N_2$,

$$\|x_n - x^*\| \leq \frac{\hat{\varepsilon}}{2(1 + L)} \quad \text{and} \quad d(x_n, F(T)) \leq \frac{\hat{\varepsilon}}{2(1 + 3L)}.$$

Thus, there exists $y^* \in F(T)$ such that

$$\|x_{N_2} - y^*\| = d(x_{N_2}, y^*) \leq \frac{\hat{\varepsilon}}{2(1 + 3L)}.$$
We then have the following estimates:

\[
\|Tx^* - x^*\| \leq \|Tx_N - y^*\| + 2\|x_N - y^*\| + \|x_N - x^*\| + \|x_N - x^*\|
\]

\[
\leq L\|x_N - y^*\| + 2L\|x_N - y^*\| + \|x_N - y^*\| + \|x_N - x^*\|
\]

\[
\leq L\|x_N - x^*\| + L\|x_N - y^*\| + (1 + 2L)\|x_N - y^*\| + \|x_N - x^*\|
\]

\[
= (1 + L)\|x_N - x^*\| + (1 + 3L)\|x_N - y^*\| \leq \bar{\epsilon}.
\]

Since \(\bar{\epsilon} > 0\) is arbitrary we have that \(Tx^* = x^*\). This completes the proof. \(\square\)

**Theorem 3.3.** Let \(E\) be a real Banach space and \(T : E \rightarrow E\) be a uniformly \(L\)-Lipschitzian asymptotically demicontractive map with sequence \(\{a_n\}\) and \(F(T) \neq \emptyset\). Let \(\{c_n\}_{n \geq 0} \subset [0, 1]\) be a real sequence such that \(\sum_{n \geq 0} c_n^2 < \infty\) and \(\sum_{n \geq 0} c_n (a_n^2 - 1) < \infty\). Let \(\{x_n\}_{n \geq 0}\) be the sequence generated from an arbitrary \(x_0 \in E\) by (3.1). Then \(\{x_n\}_{n \geq 0}\) converges strongly to \(x^* \in F(T)\) if and only if there exists an infinite subsequence of \(\{x_n\}_{n \geq 0}\) which converges strongly to \(x^* \in F(T)\).

**Proof.** Let \(x^* \in F(T)\) and \(\{x_{n_j}\}_{j \geq 0}\) a subsequence of \(\{x_n\}_{n \geq 0}\) such that

\[
\lim_{j \to \infty} \|x_{n_j} - x^*\| = 0.
\]

Since, by Lemma 3.1(b), \(\lim_{n \to \infty} \|x_n - x^*\|\) exists then \(\lim_{n \to \infty} \|x_n - x^*\| = 0.\) \(\square\)

**Remark 3.4.** The extension of our theorems to Ishikawa-type iteration process and to iteration processes with errors is now routine.

**References**


**Further reading**
