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Group actions on central simple algebras: A geometric approach

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Abstract

We study actions of linear algebraic groups on central simple algebras using algebro-geometric techniques. Suppose an algebraic group G acts on a central simple algebra A of degree n. We are interested in questions of the following type: (a) Do the G-fixed elements form a central simple subalgebra of A of degree n? (b) Does A have a G-invariant maximal subfield? (c) Does A have a splitting field with a G-action, extending the G-action on the center of A?

Somewhat surprisingly, we find that under mild assumptions on A and the actions, one can answer these questions by using techniques from birational invariant theory (i.e., the study of group actions on algebraic varieties, up to equivariant birational isomorphisms). In fact, group actions on central simple algebras turn out to be related to some of the central problems in birational invariant theory, such as the existence of sections, stabilizers in general position, affine models, etc. In this paper we explain these connections and explore them to give partial answers to questions (a)–(c).

Keywords: Linear algebraic group; Group action; Central simple algebra; Division algebra

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1. Introduction

In this paper we study actions of linear algebraic groups G on central simple algebras A in characteristic zero. As usual, we will denote the center of A by $\mathrm{Z}(A)$ and the subalgebra of G-fixed elements of A by

$$A^G = \{a \in A \mid g(a) = a \ \forall g \in G\}.$$

We will be interested in questions such as the following:

(1.1) (a) Is A^G a central simple algebra of the same degree as A?

(b) Does A have a G-invariant maximal subfield?

(c) Can the G-action on $\mathrm{Z}(A)$ be extended to a splitting field L, and if so, what is the minimal possible value of $\mathrm{trdeg}_{\mathrm{Z}(A)} L$?

Actions of finite groups on central simple algebras have been extensively studied in the 1970s and 80s in the context of group actions on noncommutative rings; for an overview see [M]. More recently, torus actions were considered in [RV_1,RV_2], and actions of solvable groups in [V_3], all by purely algebraic methods (cf. also [V_1,V_2]). Inner actions of compact groups were studied in [Sa]. The purpose of this paper is to introduce a geometric approach to the subject by relating it to “birational invariant theory,” i.e., to the study of group actions on algebraic varieties, up to birational isomorphism. In particular, we will see that the questions posed in (1.1) are related to some of the central problems in birational invariant theory, such as existence of affine models, quotients, stabilizers in general positions, sections, etc. (For an overview of birational invariant theory, see [PV, Chapters 1, 2, 7] and [P, Part 1].) To make the algebro-geometric techniques applicable, we always assume that the centers of our simple algebras are finitely generated field extensions of a fixed algebraically closed base field k of characteristic zero. All algebraic groups are assumed to be linear and defined over k.

Let G be an algebraic group and A be a finite-dimensional central simple algebra. Of course, we are primarily interested in studying G-actions on A which respect the structure of G as an algebraic (and not just an abstract) group. The following definition is natural in the geometric context.

It is well known that a finitely generated field extension of k can be interpreted as the field of rational functions $k(X)$ on some irreducible variety X, where X is unique up to birational isomorphism. Similarly, a central simple algebra A of degree n is isomorphic (as a k-algebra) to the algebra $k_n(X)$ of PGL_n-equivariant rational functions $X \dashrightarrow \mathrm{M}_n(k)$, where X is an irreducible variety with a generically free PGL_n-action. Here X is unique up to birational isomorphism of PGL_n-varieties. For details, see [RV_4, Theorem 7.8 and Section 8].

We will say that a G-action on a central simple algebra $A = k_n(X)$ is *geometric*, if it is induced by a regular G-action on X, via

$$(gf)(x) = f\bigl(g^{-1}x\bigr) \tag{1.2}$$

for $x \in X$ in general position. One can check that all rational functions $gf : X \dashrightarrow \mathrm{M}_n(k)$ lie in $k_n(X)$ (i.e., are PGL_n-equivariant) if and only if the actions of G and PGL_n on X commute. So a regular G-action on X induces a G-action on $A = k_n(X)$ precisely if X is a $G \times \mathrm{PGL}_n$-variety. To sum up:

1.3. Definition. An action of an algebraic group G on a central simple algebra A of degree n is said to be *geometric* if there is an irreducible $G \times \mathrm{PGL}_n$-variety X such that A is G-equivariantly isomorphic to $k_n(X)$. We will call X *the associated variety* for this action.

The second part of the definition makes sense since the associated variety X is unique up to birational isomorphism (as a $G \times \mathrm{PGL}_n$-variety); see Corollary 3.2. Note that the PGL_n-action on X is necessarily generically free, since $A \simeq k_n(X)$ is a central simple algebra of degree n; see Lemma 2.8. Conversely, any $G \times \mathrm{PGL}_n$-variety X, which is PGL_n-generically free, is the associated variety for the geometric action of G on the central simple algebra $A = k_n(X)$ given by (1.2).

From an algebraic point of view it is natural to consider another class of actions, introduced in [V_3, Section 2] (and in the special case of torus actions in [RV_2, Section 5]). We shall call such actions *algebraic*; for a precise definition, see Section 5. The relationship between algebraic and geometric actions is discussed in Sections 5 and 8. In particular, every algebraic action on a central simple algebra is geometric; see Theorem 5.3.

We are now ready to address the questions posed in (1.1), in the context of geometric actions.

1.4. Theorem. *Consider a geometric action of an algebraic group G on a central simple algebra A of degree n, with associated $G \times \mathrm{PGL}_n$-variety X.*

(a) *The fixed algebra A^G is a central simple algebra of degree n if and only if for $x \in X$ in general position,*

$$\mathrm{Stab}_{G\times \mathrm{PGL}_n}(x) \subseteq G \times \{1\}.$$

(b) *The fixed algebra A^G contains an element with n distinct eigenvalues if and only if for every $x \in X$ in general position there exists a torus T_x of PGL_n such that*

$$\mathrm{Stab}_{G\times \mathrm{PGL}_n}(x) \subseteq G \times T_x.$$

We now turn to question (b) in (1.1).

1.5. Theorem. *Consider a geometric action of an algebraic group G on a central simple algebra A of degree n, with associated $G \times \mathrm{PGL}_n$-variety X.*

(a) *A has a G-invariant maximal étale subalgebra if and only if there exists a $G \times \mathrm{PGL}_n$-equivariant rational map $X \dashrightarrow \mathrm{PGL}_n/N$, where N is the normalizer of a maximal torus in PGL_n and G acts trivially on the homogeneous space PGL_n/N.*
(b) *If A has a G-invariant maximal étale subalgebra, then for every $x \in X$ in general position there exists a maximal torus T_x of PGL_n such that*

$$\mathrm{Stab}_{G\times \mathrm{PGL}_n}(x) \subseteq G \times \mathrm{N}(T_x).$$

Here $\mathrm{N}(T_x)$ denotes the normalizer of T_x in PGL_n.
(c) *If the orbit Gx has codimension $< n^2 - n$ in X for $x \in X$ in general position, then A has no G-invariant maximal étale subalgebras.*

Here by an *étale subalgebra* of A we mean a subalgebra of A which is an étale algebra over $\mathrm{Z}(A)$; cf. 2.9. If A is a division algebra, the maximal étale subalgebras are just the maximal subfields.

The converse to Theorem 1.5(b) is false in general; see Proposition 15.3. Note that the points of the homogeneous space PGL_n/N parameterize the maximal tori in PGL_n (see the beginning of Section 9). The converse to part (b) is thus true if and only if the tori T_x can be chosen so that $x \dashrightarrow T_x$ is a rational map. We also remark that Theorem 1.4(b) gives a necessary and sufficient condition for A to have a G-invariant maximal étale algebra of the form $\mathrm{Z}(A)[a]$, where $a \in A^G$; see Corollary 6.3.

Our final result addresses question (c) in (1.1). We begin with the following definition.

1.6. Definition. Suppose a group G acts on a central simple algebra A of degree n. We will say that A is *G-split*, if A is G-equivariantly isomorphic to $\mathrm{M}_n(\mathrm{Z}(A)) = \mathrm{M}_n(k) \otimes_k \mathrm{Z}(A)$, where G acts via the second factor. We will say that a G-equivariant field extension $L/\mathrm{Z}(A)$ is a *G-splitting field* for A if $A \otimes_{\mathrm{Z}(A)} L$ is G-split.

Note that if G acts trivially on A, then a G-splitting field is just a splitting field for A in the usual sense. Note also that a G-action on a split central simple algebra (i.e., a matrix algebra over a field) need not be G-split (cf. Example 6.2).

1.7. Theorem. *Every geometric action of an algebraic group G on a central simple algebra A of degree n has a G-splitting field of the form $L = k(X_0)$, where X_0 is a G-variety and* $\operatorname{trdeg}_{Z(A)}(L) = n^2 - 1$. *Moreover, if G acts algebraically on A, then X_0 can, in addition, be chosen to be affine.*

In general, the value of $\operatorname{trdeg}_{Z(A)} L$ given in Theorem 1.7 is the smallest possible; see Proposition 13.1(b). If G is connected, we give a different construction of G-splitting fields in Section 11.

At the end of the paper we will present four examples illustrating our main results, Theorems 1.4, 1.5, and 1.7, and two appendices. Appendix A deals with inner actions on division algebras which need not be geometric, while Appendix B treats regular actions of algebraic groups (see Definition 5.1) on prime affine PI-algebras. Using Theorem 1.7, we show that such actions are "induced" by regular actions on commutative domains. Further results on geometric actions will appear in the paper [RV_5].

2. Preliminaries

2.1. Conventions. We work over a fixed algebraically closed base field k of characteristic zero. All algebras are k-algebras, and division algebras and central simple algebras are assumed to be finite-dimensional over their centers, which in turn are assumed to be finitely generated field extensions of k. All actions on algebras are by k-algebra automorphisms. Algebraic groups are always assumed to be linear algebraic groups over k, and G will always denote an algebraic group. Regular actions are meant to be regular over k; similarly for algebraic actions (see Definition 5.2). If K is a field, we shall denote the algebra of $n \times n$ matrices over K by $\mathrm{M}_n(K)$. If $K = k$, we will write M_n in place of $\mathrm{M}_n(k)$. We will sometimes view M_n as a k-algebra and sometimes as an algebraic variety, isomorphic to the affine space $\mathbb{A}^{n^2}$.

2.2. G-varieties. By a G-variety X we mean an algebraic variety with a regular action of G. By a morphism $X \to Y$ of G-varieties, we mean a G-equivariant morphism. The notions of isomorphism, rational map, birational isomorphism, etc. of G-varieties are defined in a similar manner. As usual, given a G-action on X, we denote the orbit of $x \in X$ by Gx and the stabilizer subgroup of x by $\mathrm{Stab}_G(x) \subseteq G$. Throughout this paper we use [PV] as a reference for standard notions from invariant theory, such as rational and categorical quotients, stabilizers in general position, sections, etc.

2.3. Definition. We shall say that a G-action on X is

(a) *faithful* if every $1 \neq g \in G$ acts nontrivially on X,
(b) *generically free* if $\mathrm{Stab}_G(x) = \{1\}$ for $x \in X$ in general position, and
(c) *stable* if the orbit Gx is closed in X for $x \in X$ in general position.

2.4. Lemma. *Suppose the group G is either* (a) *finite or* (b) *diagonalizable. Then every faithful irreducible G-variety X is generically free.*

Proof. (a) Since the G-action is faithful, $X^g = \{x \in X \mid gx = x\} \neq X$ for every $1 \neq g \in G$. Since each X^g is a closed subvariety of X, every point of the Zariski dense open subset $X - \bigcup_{1 \neq g \in G} X^g$ has a trivial stabilizer in G.

Part (b) is an immediate corollary of a theorem of Richardson [Ri, Theorem 9.3.1]; see also [PV, Theorem 7.1]. □

The following example shows that, contrary to the assertion in [PV, Proposition 7.2], Lemma 2.4 fails if we only assume that the connected component of G is a torus. We shall return to this example in Section 14.

2.5. Example. Consider the natural linear action of the orthogonal group $G = O_2$ on $\mathbb{A}^2$. This action is faithful but not generically free: $\mathrm{Stab}_G(v)$ has order 2 for $v \in k^2$ in general position. Indeed, for every non-isotropic vector v in k^2, there is a unique non-trivial element of O_2, leaving v invariant; this element is the orthogonal reflection in v. Note also O_2 is a semidirect product of a one-dimensional torus with $\mathbb{Z}/2\mathbb{Z}$.

2.6. Lemma *(Popov). Let G be a reductive group, X be an affine G-variety and V be a G-representation. Suppose the G-orbit of $x \in X$ is closed in X and* $\mathrm{Stab}(x) \subseteq \mathrm{Stab}(v)$ *for some $v \in V$. Then there exists a G-equivariant morphism $f : X \to V$ such that $f(x) = v$.*

Proof. In the case where $\mathrm{Stab}(x) = \{1\}$, this lemma is stated and proved in [P, Theorem 1.7.12]. The same argument goes through in our slightly more general setting. □

2.7. Algebras of rational maps. If X is a PGL_n-variety, we will denote by

$$\mathrm{RMaps}_{\mathrm{PGL}_n}(X, \mathrm{M}_n)$$

the k-algebra of PGL_n-equivariant rational maps $f : X \dashrightarrow \mathrm{M}_n$, with addition and multiplication induced from M_n.

2.8. Lemma. *Let Y be an irreducible* PGL_n*-variety. Then the following are equivalent*:

(a) *The* PGL_n*-action on Y is generically free.*
(b) $A = \mathrm{RMaps}_{\mathrm{PGL}_n}(Y, \mathrm{M}_n)$ *is a central simple algebra of degree n.*

If (a) *and* (b) *hold then the center of A is* $\mathrm{RMaps}_{\mathrm{PGL}_n}(Y, k) = k(Y)^{\mathrm{PGL}_n}$. *Here elements of k are identified with scalar matrices in* M_n.

Proof. (b) ⇒ (a): Note that the center of A contains $k(Y)^{\mathrm{PGL}_n}$. Choose $f_1, \ldots, f_{n^2} \in A$ which are linearly independent over $k(Y)^{\mathrm{PGL}_n}$. By [Re, Lemma 7.4], $f_1(y), \ldots, f_{n^2}(y)$ are k-linearly independent in M_n for $y \in Y$ in general position. Now consider the PGL_n-equivariant rational map

$$f = (f_1, \ldots, f_{n^2}) : Y \dashrightarrow (\mathrm{M}_n)^{n^2}.$$

For $y \in Y$ in general position, $\mathrm{Stab}(f(y)) = \{1\}$, so that also $\mathrm{Stab}(y) = \{1\}$. Hence Y is PGL_n-generically free.

The implication (a) $\Rightarrow$ (b) and the last assertion of the lemma are proved in [Re, Lemma 8.5] (see also [Re, Definition 7.3 and Lemma 9.1]). □

If the PGL_n-action on X is generically free, we will denote the central simple algebra $\mathrm{RMaps}_{\mathrm{PGL}_n}(X, \mathrm{M}_n)$ by $k_n(X)$.

2.9. Maximal étale subalgebras. Let A be a central simple algebra of degree n. By an étale subalgebra of A we mean a subalgebra of A which is an étale algebra over $\mathrm{Z}(A)$, i.e., a finite direct sum of (separable) field extensions of $\mathrm{Z}(A)$. Note that since we are working in characteristic zero, the term "étale" could be replaced by "commutative semisimple." We are interested in maximal étale subalgebras, i.e., étale subalgebras E of A satisfying the following equivalent conditions:

(a) $\dim_{\mathrm{Z}(A)} E = \deg(A)$,
(b) E is maximal among commutative subalgebras of A;

cf. [Ro_2, Exercise 7.1.1]. Using the double centralizer theorem, one easily verifies that every étale subalgebra of A is contained in a maximal étale subalgebra, see, e.g., [J, Theorem 4.10 and Exercise 4.6.12] and [Ro_2, Exercise 7.1.2]. Of course, if A is a division algebra, then maximal étale subalgebras are just maximal subfields.

We will repeatedly use the following characterization of maximal étale subalgebras, which follows easily from [B, §V.7.2, Proposition 3].

2.10. Lemma. *Let A be a central simple algebra of degree n with center K. Let $a \in A$. Then $K[a]$ is a maximal étale subalgebra of A if and only if the eigenvalues of a are distinct.*

3. The uniqueness of the associated variety

Recall that given a generically free PGL_n-variety X, we write $A = k_n(X)$ for the algebra of PGL_n-equivariant functions $a : X \dashrightarrow \mathrm{M}_n$. A PGL_n-equivariant dominant rational map $f : X' \dashrightarrow X$ induces an embedding $f^* : A \hookrightarrow A'$ of central simple algebras, where $A' = k_n(X')$ and $f^*(a) = a \circ f : X' \dashrightarrow \mathrm{M}_n$.

We now deduce a simple consequence of the functoriality of the maps $X \mapsto k_n(X)$ and $f \mapsto f^*$; see [RV_4, Theorem 1.2]. Recall that if X has a G-action, which commutes with the PGL_n-action, then (1.2) defines a G-action on $A = k_n(X)$, which we call geometric.

3.1. Lemma. *Let X and X' be $G \times \mathrm{PGL}_n$-varieties, which are PGL_n-generically free.*

(a) *If $f : X' \dashrightarrow X$ is a dominant rational map of $G \times \mathrm{PGL}_n$-varieties then the induced embedding $f^* : k_n(X) \hookrightarrow k_n(X')$ of central simple algebras is G-equivariant.*

(b) *Every G-equivariant embedding $j : k_n(X) \hookrightarrow k_n(X')$ induces a dominant rational $G \times \mathrm{PGL}_n$-equivariant map $j_* : X' \dashrightarrow X$.*

Proof. (a) By [RV₄, Theorem 1.2], since the diagram

$$\begin{array}{ccc} X' & \overset{f}{\dashrightarrow} & X \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g} \\ X' & \overset{f}{\dashrightarrow} & X \end{array}$$

commutes for every $g \in G$, so does the induced diagram

$$\begin{array}{ccc} k_n(X) & \xrightarrow{f^*} & k_n(X') \\ \downarrow{\scriptstyle g^{-1}} & & \downarrow{\scriptstyle g^{-1}} \\ k_n(X) & \xrightarrow{f^*} & k_n(X'). \end{array}$$

(b) Conversely, since the diagram

$$\begin{array}{ccc} k_n(X) & \xrightarrow{j} & k_n(X') \\ \downarrow{\scriptstyle g^{-1}} & & \downarrow{\scriptstyle g^{-1}} \\ k_n(X) & \xrightarrow{j} & k_n(X') \end{array}$$

commutes, so does the induced diagram

$$\begin{array}{ccc} X' & \overset{j_*}{\dashrightarrow} & X \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g} \\ X' & \overset{j_*}{\dashrightarrow} & X. \end{array} \quad \square$$

3.2. Corollary. *Given a geometric action of an algebraic group G on a central simple algebra A, the $G \times \mathrm{PGL}_n$-variety associated to this action is unique up to birational isomorphism.*

Proof. Suppose two $G \times \mathrm{PGL}_n$-varieties X and X' are both associated varieties for this action, i.e., $k_n(X)$ and $k_n(X')$ are both G-equivariantly isomorphic to A. In other words, there are mutually inverse G-equivariant algebra isomorphisms $i : k_n(X) \xrightarrow{\simeq} k_n(X')$ and $j : k_n(X') \xrightarrow{\simeq} k_n(X)$. Applying Lemma 3.1, i and j induce mutually inverse dominant $G \times$

PGL_n-equivariant rational map $i_* : X' \overset{\simeq}{\dashrightarrow} X$ and $j_* : X \overset{\simeq}{\dashrightarrow} X'$. We conclude that X and X' are birationally isomorphic $G \times \mathrm{PGL}_n$-varieties. □

3.3. Example. Let G be a subgroup of PGL_n, and consider the conjugation action of G on $A = \mathrm{M}_n(k)$. We claim that the associated $G \times \mathrm{PGL}_n$-variety for this action is $X = \mathrm{PGL}_n$, with G acting by translations on the right and PGL_n acting by translations on the left. More precisely, for $(g, h) \in G \times \mathrm{PGL}_n$ and $x \in X$, $(g, h) \cdot x = hxg^{-1}$. Consequently for $f \in k_n(X)$,

$$(g \cdot f)(x) = f\big((g, 1)^{-1} \cdot x\big) = f(xg),$$

see (1.2). Note that since X is a single PGL_n-orbit, every PGL_n-equivariant rational map $f : \mathrm{PGL}_n \dashrightarrow \mathrm{M}_n$ is necessarily regular. It is now easy to check that the k-algebra isomorphism

$$\phi\colon\ k_n(X) = \mathrm{RMaps}_{\mathrm{PGL}_n}(\mathrm{PGL}_n, \mathrm{M}_n) \overset{\simeq}{\longrightarrow} A = \mathrm{M}_n$$

given by $\phi(f) = f(1)$ is G-equivariant.

3.4. Example. Let $m \geqslant 2$, and consider the PGL_n-variety $X = (\mathrm{M}_n)^m$, where PGL_n acts by simultaneous conjugation, i.e., via

$$g \cdot (a_1, \ldots, a_m) = \big(ga_1g^{-1}, \ldots, ga_mg^{-1}\big).$$

Since $m \geqslant 2$, this action is generically free. The associated division algebra $k_n(X)$ is called the *universal division algebra* of m generic $n \times n$ matrices and is denoted by $UD(m, n)$. Identify the function field of X with $k(x_{ij}^{(h)})$, where for each $h = 1, \ldots, m$, $x_{ij}^{(h)}$ are the n^2 coordinate functions on copy number h of M_n, and identify the algebra of all rational maps $X \dashrightarrow \mathrm{M}_n$ with $\mathrm{M}_n(k(x_{ij}^{(h)}))$. Now we can think of $UD(m, n)$ as the division subalgebra of $\mathrm{M}_n(k(x_{ij}^{(h)}))$ generated by the *m generic $n \times n$ matrices* $X^{(h)} = (x_{ij}^{(h)})$, $h = 1, \ldots, m$. Here $X^{(h)}$ corresponds to projection $(\mathrm{M}_n)^m \to \mathrm{M}_n$ given by $(a_1, \ldots, a_m) \mapsto a_h$. For details of this construction, see [Pr$_2$, Section 2] or [lB, Theorem 5].

Now observe that the GL_m-action on $X = (\mathrm{M}_n)^m$ given by

$$g \cdot (a_1, \ldots, a_m) = \left(\sum_{j=1}^{m} g_{1j}a_j, \ldots, \sum_{j=1}^{m} g_{mj}a_j\right) \tag{3.5}$$

commutes with the above PGL_n-action. Here $g = (g_{ij}) \in \mathrm{GL}_m$, with $g_{ij} \in k$. Using formula (1.2), we see that this gives rise to a GL_m-action on $UD(m, n)$ such that for $g \in \mathrm{GL}_m$,

$$g \cdot X^{(h)} = \sum_{j=1}^{m} g'_{hj} X^{(j)}, \tag{3.6}$$

where $g^{-1} = (g'_{ij})$. In other words, this GL_m-action on $UD(m,n)$ is geometric, with associated $G \times \mathrm{PGL}_n$-variety $X = (\mathrm{M}_n)^m$. We will return to this important example later in this paper (in Example 5.5 and Sections 13 and 14), as well as in [RV$_5$].

3.7. Remark. The k-subalgebra of $UD(m,n)$ generated by $X^{(1)}, \dots, X^{(m)}$ is called the *generic matrix ring* generated by m generic $n \times n$ matrices; we denote it by $G_{m,n}$. Note that the action (3.6) of GL_m on $UD(m,n)$ restricts to an action on $G_{m,n}$. Consequently, the GL_m-action on $G_{m,n}$ is induced by the GL_m-action on $(\mathrm{M}_n)^m$ in the sense of formula (1.2).

4. Brauer–Severi varieties

Let A/K be a central simple algebra of degree n. Throughout much of this paper, we associate to A a PGL_n-variety X/k such that A is the algebra of PGL_n-equivariant rational maps $X \dashrightarrow \mathrm{M}_n(k)$. Another variety that can be naturally associated to A is the Brauer–Severi variety $\mathrm{BS}(A)$, defined over K. Any algebra automorphism $g : A \to A$, defined over the base field k, induces k-automorphisms of K and $\mathrm{BS}(A)$ such that the diagram

$$\begin{array}{ccc} \mathrm{BS}(A) & \xrightarrow{g_*} & \mathrm{BS}(A) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(K) & \xrightarrow{(g_{|K})_*} & \mathrm{Spec}(K), \end{array}$$

commutes; conversely, g can be uniquely recovered from this diagram. If a group G acts on A, it is natural to ask if $\mathrm{BS}(A)$ can be G-equivariantly represented by an algebraic variety over k. In this short section we will address this question, following a suggestion of the referee. Our main result, Proposition 4.1 below, will not be used in the sequel.

4.1. Proposition. *Consider a geometric action ϕ of an algebraic group G on a central simple algebra A/K of degree n. Then there exists a morphism $\sigma : S \to Y$ of irreducible G-varieties (of finite type over k) such that*

(a) *S is a Brauer–Severi variety over Y;*
(b) *$k(Y) = K$ and $\sigma^{-1}(\eta)$ is the Brauer–Severi variety of A, where η is the generic point of Y;*
(c) *the G-actions on S and Y induce the action ϕ on A.*

Proof. Let X be the $G \times \mathrm{PGL}_n$-variety associated to ϕ and H be the maximal parabolic subgroup of PGL_n consisting of matrices of the form

$$\begin{pmatrix} * & 0 & \dots & 0 \\ * & * & \dots & * \\ \vdots & \vdots & & \vdots \\ * & * & \dots & * \end{pmatrix}.$$

Consider the natural dominant rational map $\sigma : X/H \dashrightarrow X/\mathrm{PGL}_n$ given by the inclusion $k(X)^{\mathrm{PGL}_n} \hookrightarrow k(X)^H$. Recall that the rational quotient varieties X/H and X/PGL are a priori only defined up to birational isomorphism. However, we can choose models for these varieties such that the induced G-actions are regular; cf. [PV, Proposition 2.6 and Corollary 1.1]. For notational convenience, we will continue to denote these G-varieties by X/H and X/PGL_n. Note also that since the actions of G and PGL_n on X commute, the resulting map $\sigma : X/H \dashrightarrow X/\mathrm{PGL}_n$ is G-equivariant.

By [RV_4, Section 9], X/H is a Brauer–Severi variety over a dense open subset U of X/PGL_n, and is isomorphic to $\mathrm{BS}(A)$ over the generic point of X/PGL_n. Since σ is G-equivariant, X/H is a Brauer–Severi variety over $g(U)$, for every $g \in G$. Setting Y to be the union of the $g(U)$ inside X/PGL_n, as g ranges over G, and setting S to be the preimage of this set in X/H, we obtain a G-equivariant morphism $\sigma : S \to Y$ with the desired properties. □

5. Algebraic actions

5.1. Definition. We shall say that the action of an algebraic group G on a (not necessarily commutative) k-algebra R is *regular*,[3] if every finite-dimensional k-subspace of R is contained in a G-invariant finite-dimensional k-subspace V, such that the G-action on V induces a homomorphism $G \to \mathrm{GL}(V)$ of algebraic groups.

Every regular action of a connected algebraic group on a division algebra (or even a field) must be trivial (see, e.g., [V_2, A.1]), so this notion is too restrictive for our purposes. However, it naturally leads to the following definition, made in [V_3, Section 2]. (The special case where G is a torus had been considered earlier in [RV_2, Section 5].)

5.2. Definition. Let G be an algebraic group acting on a k-algebra A by k-algebra automorphisms. We call the action *algebraic*[4] (over k) if there is a G-invariant subalgebra R of A and a G-invariant multiplicatively closed subset S of R consisting of central non-zero divisors of R such that (1) G acts regularly on R, and (2) $A = RS^{-1}$.

Note that a regular action on A is algebraic (use $S = \{1\}$). We shall be primarily interested in the case where A is a central simple algebra; in this case R is an order in A (and in particular, R is prime). For basic properties of algebraic actions, see [V_3, Section 2].

The purpose of this section is to investigate the relationship between algebraic and geometric actions (cf. Definition 1.3).

5.3. Theorem. (a) *Algebraic actions on central simple algebras are geometric.*

[3] Such actions are usually called rational; we prefer the term regular, since the term "rational action" has a different meaning in the context of birational invariant theory.

[4] In [V_3], S is not required to be central; it is, however, proved there that S can always be chosen to be central if A is a central simple algebra.

(b) *Let G be an algebraic group acting geometrically on a central simple algebra A of degree n. Then the action of G on A is algebraic if and only if there is an associated $G \times \mathrm{PGL}_n$-variety X with the following two properties*: *X is affine, and the* PGL_n*-action on X is stable* (*cf. Definition* 2.3(c)).

We begin with a result which is a G-equivariant version of [RV_4, Theorem 6.4].

Let $U_{m,n}$ be the open subset of $(\mathrm{M}_n)^m$ consisting of those m-tuples $(a_1, \dots, a_m)$ of $n \times n$ matrices with the property that $a_1, \dots, a_m$ generate M_n as a k-algebra. Recall that an n-variety is a closed PGL_n-invariant subvariety of $U_{m,n}$. To every n-variety $Y \subset U_{m,n}$ we associate the ideal $\mathcal{I}(Y) \subset G_{m,n}$ given by

$$\mathcal{I}(Y) = \big\{ f \in G_{m,n} \mid f(a_1, \dots, a_m) = 0 \ \forall (a_1, \dots, a_m) \in Y \big\},$$

and the polynomial identity coordinate ring $k_n[Y] = G_{m,n}/\mathcal{I}(Y)$. Conversely, an ideal $I \in G_{m,n}$ cuts out the n-variety

$$\mathcal{Z}(I) = \big\{ (a_1, \dots, a_m) \in U_{m,n} \mid f(a_1, \dots, a_m) = 0 \ \forall f \in I \big\}.$$

For details of these constructions see [RV_4].

5.4. Proposition. *Let G be an algebraic group acting regularly on a finitely generated prime k-algebra R of PI-degree n. Then there is an n-variety Y with a regular G-action such that R is G-equivariantly isomorphic to $k_n[Y]$.*

Proof. We may assume that G acts faithfully on R. There is a finite-dimensional G-stable k-subspace W of R which generates R as a k-algebra. Set $m = \dim_k(W)$, and consider the generic matrix ring $G_{m,n}$ with its GL_m-action as in Remark 3.7. Denote by V the k-subspace of $G_{m,n}$ generated by the m generic $n \times n$ matrices. Let $\psi_0 : V \to W$ be a k-vector space isomorphism. Define a regular action of G on V by making ψ_0 G-equivariant. The action of G on V extends to a regular action on $G_{m,n}$. By the universal mapping property of $G_{m,n}$, ψ_0 extends to a G-equivariant surjective k-algebra homomorphism $\psi : G_{m,n} \to R$. Replacing G by an isomorphic subgroup of GL_m, we may assume that G acts on V as in (3.6). Then the action of G on $G_{m,n}$ is induced (as in (1.2)) from the action of G on $(\mathrm{M}_n)^m$ given by (3.5). Note that the actions of G and PGL_n on $(\mathrm{M}_n)^m$ commute.

Let I be the kernel of ψ, and let $Y = \mathcal{Z}(I) \subset (\mathrm{M}_n)^m$ be the irreducible n-variety associated to I, see [RV_4, Corollary 4.3]. Note that Y is G-stable for the action of G on $(\mathrm{M}_n)^m$. By [RV_4, Proposition 5.3], $\mathcal{I}(Y) = I$, so that R is G-equivariantly isomorphic to $k_n[Y] = G_{m,n}/\mathcal{I}(Y) = G_{m,n}/I$. $\square$

Proof of Theorem 5.3. (a) Let G be an algebraic group acting algebraically on a central simple algebra A of degree n. Let R be a G-stable finitely generated prime PI-algebra contained in A such that A is the total ring of fractions of R. By Proposition 5.4, there is an n-variety Y with a regular action of G such that R is G-equivariantly isomorphic to $k_n[Y]$. Then A is G-equivariantly isomorphic to the total ring of fractions of $k_n[Y]$, i.e., to $k_n(Y)$, see [RV$_4$, Proposition 7.3]. As the proof of Proposition 5.4 showed, $(\mathrm{M}_n)^m$ is a $G \times \mathrm{PGL}_n$-variety (where G acts via some subgroup of GL_m as in (3.5)), and Y is a G-stable subset of $(\mathrm{M}_n)^m$. Hence the closure X of Y in $(\mathrm{M}_n)^m$ is an affine $G \times \mathrm{PGL}_n$-variety. It is clear that $k_n(Y)$ and $k_n(X)$ are G-equivariantly isomorphic, and that the PGL_n-action on X is generically free and stable. So G acts geometrically on A, and the associated $G \times \mathrm{PGL}_n$-variety X has the two additional properties from part (b).

(b) If the action of G on A is algebraic then an associated $G \times \mathrm{PGL}_n$-variety X with the desired properties was constructed in the proof of part (a).

Conversely, assume that there is an associated $G \times \mathrm{PGL}_n$-variety X which is affine and on which the PGL_n-action is stable. We may assume that $A = k_n(X)$. So A is a central simple algebra with center $K = k(X)^{\mathrm{PGL}_n}$; cf. Lemma 2.8. Since X is affine, and since PGL_n-orbits in X in general position are closed, $k[X]^{\mathrm{PGL}_n}$ separates PGL_n-orbits in general position, so that $\mathrm{Q}(k[X]^{\mathrm{PGL}_n}) = k(X)^{\mathrm{PGL}_n} = K$; see [PV, Lemma 2.1]. (Here Q stands for the quotient field.) Denote by R the subalgebra of A consisting of the regular PGL_n-equivariant maps $X \to \mathrm{M}_n$. It is clearly G-invariant. Note that G acts regularly on $k[X]$. Consequently, G acts regularly on $\mathrm{M}_n(k[X])$, the set of regular maps $X \to \mathrm{M}_n$. Hence, G also acts regularly on its subalgebra R. It remains to show that R is a prime subalgebra of A, and that its total ring of fractions is equal to A.

Let $v \in (\mathrm{M}_n)^2$ be a pair of matrices generating M_n as k-algebra, and let $x \in X$ be such that its stabilizer in PGL_n is trivial and such that its PGL_n-orbit is closed. Then by Lemma 2.6, there is a PGL_n-equivariant regular map $X \to (\mathrm{M}_n)^2$ such that $f(x) = v$. Write $f = (f_1, f_2)$, where f_1 and f_2 are PGL_n-equivariant regular maps $X \to \mathrm{M}_n$, i.e., elements of R. Since $f_1(x)$ and $f_2(x)$ generate M_n, the central polynomial g_n [Ro$_1$, p. 26] does not vanish on R. Since g_n is t^2-normal, it vanishes on every proper K-subspace of A, see [Ro$_1$, 1.1.35]. Consequently $RK = A$, and R is prime and has PI-degree n. Clearly, R contains $k[X]^{\mathrm{PGL}_n}$. Since $R\,\mathrm{Q}(k[X]^{\mathrm{PGL}_n}) = RK = A$, $\mathrm{Q}(R) = A$. Hence, G acts algebraically on A. □

5.5. Example. It follows easily from Definition 5.2 that the action (3.6) of GL_m on $UD(m,n)$ is algebraic. So by Theorem 5.3(b), there is an associated $\mathrm{GL}_m \times \mathrm{PGL}_n$-variety X with the following two properties: X is affine, and the PGL_n-action on X is stable. Indeed, the natural associated variety $X = (\mathrm{M}_n)^m$ has these properties.

6. Proof of Theorem 1.4

We begin with the following simple observation:

6.1. Remark. Consider a geometric action of an algebraic group G on a central simple algebra A, with associated $G \times \mathrm{PGL}_n$-variety X. Elements of A are thus PGL_n-equivariant rational maps $a : X \dashrightarrow \mathrm{M}_n$. Such an element is G-fixed if and only if it factors through the rational quotient map $X \dashrightarrow X/G$. In other words, A^G is isomorphic to $\mathrm{RMaps}_{\mathrm{PGL}_n}(X/G, \mathrm{M}_n)$.

We are now ready to proceed with the proof of Theorem 1.4.

(a) We may assume that $A = k_n(X)$. Combining Remark 6.1 with Lemma 2.8, we see that A^G is a central simple algebra of degree n if and only if $Y = X/G$ is a generically free

PGL_n-variety. The latter condition is equivalent to $\mathrm{Stab}_{G\times\mathrm{PGL}_n}(x) \subseteq G \times \{1\}$ for $x \in X$ in general position.

(b) First suppose that there is an $a \in A^G$ with n distinct eigenvalues. Adding to a some constant in k, we may assume that the eigenvalues of a are distinct and non-zero. Hence for $x \in X$ in general position, the eigenvalues of $a(x) \in \mathrm{M}_n$ are also distinct and non-zero. The stabilizer of $a(x)$ in PGL_n is thus a maximal torus T_x of PGL_n. Let $(g, p) \in \mathrm{Stab}_{G\times\mathrm{PGL}_n}(x)$. Then $a(x) = g(a)(x) = a(g^{-1}(x)) = a(p(x)) = p\,a(x)p^{-1}$. Thus $p \in T_x$, so that $\mathrm{Stab}_{G\times\mathrm{PGL}_n}(x) \subseteq G \times T_x$.

We will now prove the converse. Assume $\mathrm{Stab}_{G\times\mathrm{PGL}_n}(x)$ is contained in $G \times T_x$ for some torus T_x of PGL_n (depending on x). Denote by Y the rational quotient PGL_n-variety X/G. To produce an $a \in A^G$ with distinct eigenvalues, it suffices to construct a PGL_n-equivariant rational map $a : Y \dashrightarrow \mathrm{M}_n$ whose image contains a matrix with distinct eigenvalues. By our assumption, $\mathrm{Stab}_{\mathrm{PGL}_n}(y)$ is contained in a torus $T_x \subset \mathrm{PGL}_n$ for $y \in Y$ in general position. Hence, $\mathrm{Stab}_{\mathrm{PGL}_n}(y)$ is diagonalizable (and, in particular, reductive). By [RV$_3$, Theorem 1.1], after replacing Y by a birationally equivalent PGL_n-variety, we may assume that Y is affine and the PGL_n-action on Y is stable.

We are now ready to construct a map $a : Y \dashrightarrow \mathrm{M}_n$ with the desired properties. Let $y \in Y$ be a point whose orbit is closed and whose stabilizer S is diagonalizable, and let $v \in \mathrm{M}_n$ be a matrix with distinct eigenvalues. Then $\mathrm{Stab}(v)$ is a maximal torus in PGL_n; after replacing v by a suitable conjugate, we may assume $S \subseteq \mathrm{Stab}(v)$. Now Lemma 2.6 asserts that there exists a PGL_n-equivariant morphism $a : Y \to \mathrm{M}_n$ such that $a(y) = v$. This completes the proof of Theorem 1.4.

6.2. Example. Let G be a subgroup of PGL_n, acting by conjugation on $A = \mathrm{M}_n(k)$. The associated variety for this action is $X = \mathrm{PGL}_n$, with $G \times \mathrm{PGL}_n$ acting on it by $(g, h) \cdot x = hxg^{-1}$; see Example 3.3. Since all of X is a single PGL_n-orbit, the stabilizer of any $x \in X$ is conjugate to the stabilizer of 1_{PGL_n}, which is easily seen to be $\{(g, g) \mid g \in G\}$. So in this setting, Theorem 1.4 reduces to the following familiar facts:

(a) $\mathrm{M}_n(k)^G = \mathrm{M}_n(k)$ if and only if $G = \{1\}$, and
(b) $\mathrm{M}_n(k)^G$ contains an element with n distinct eigenvalues if and only if G centralizes a maximal torus in GL_n, i.e., if and only if G is contained in maximal torus of PGL_n.

Using Lemma 2.10, we can rephrase Theorem 1.4(b) in a way that makes its relationship to Question 1.1(b) more transparent.

6.3. Corollary. *Consider a geometric action of an algebraic group G on a central simple algebra A of degree n, with associated $G \times \mathrm{PGL}_n$-variety X. The following conditions are equivalent.*

(a) *A has a maximal étale subalgebra E of the form $E = \mathrm{Z}(A)[a]$ for some $a \in A^G$.*
(b) *A^G contains a separable element of degree n over $\mathrm{Z}(A)$.*
(c) *For $x \in X$ in general position, $\mathrm{Stab}_{G\times\mathrm{PGL}_n}(x)$ is contained in $G \times T_x$, where T_x is a torus in PGL_n.*

Here by a separable element of A we mean an element whose minimal polynomial over $\mathrm{Z}(A)$ is separable, i.e., has no repeated roots.

6.4. Remark. It is necessary in Corollary 6.3(b) to require that a is separable over $\mathrm{Z}(A)$. Indeed, in Example 6.2 set $n = 2$, $A = \mathrm{M}_2(k)$ and $G = \{ \left(\begin{smallmatrix} 1 & g \\ 0 & 1 \end{smallmatrix}\right) \mid g \in k \}$. Then the fixed algebra A^G consists of all matrices of the form $\left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right)$ with $a, b \in k$. In particular, A^G contains elements of degree $n = 2$ over $\mathrm{Z}(A) = k$, but the minimal polynomial of any such element has repeated roots.

7. The G-action on the center of A

Throughout this section, we consider a geometric action of an algebraic group G on a central simple algebra A of degree n with associated $G \times \mathrm{PGL}_n$-variety X. It is sometimes possible to deduce information about the G-action on A from properties of the G-action on the center $\mathrm{Z}(A)$. In this section we find conditions on the G-action on $\mathrm{Z}(A)$ which allow us to answer question (a) in (1.1).

Recall that the field of rational functions on X/PGL_n is G-equivariantly isomorphic to the center $\mathrm{Z}(A)$ of A (see Lemma 2.8). Of course, a priori X/PGL_n is only defined up to birational isomorphism. From now on we will fix a particular model W equipped with a regular G-action and a G-equivariant rational quotient map for the PGL_n-action on X

$$\pi : X \dashrightarrow W.$$

It will not matter in the sequel which model W of X/PGL_n we use. Note that the G-variety W is just a birational model for the G-action on $\mathrm{Z}(A)$. In many (perhaps, most) cases, W is much easier to construct than X; for an example of this phenomenon, see Section 15.

We begin with a simple observation, relating stabilizers in X and W.

7.1. Lemma. *Let X be a $G \times \mathrm{PGL}_n$-variety which is PGL_n-generically free. Denote by $\pi : X \dashrightarrow X/\mathrm{PGL}_n$ the rational quotient map for the PGL_n-action. Then for $x \in X$ in general position, the projection $G \times \mathrm{PGL}_n \to G$ onto the first factor induces an isomorphism between $\mathrm{Stab}_{G \times \mathrm{PGL}_n}(x)$ and $\mathrm{Stab}_G(\pi(x))$.*

Proof. For $x \in X$ in general position, π is defined at x, the fiber over $\pi(x)$ is the orbit $\mathrm{PGL}_n x$, and $\mathrm{Stab}_{\mathrm{PGL}_n}(x)$ is trivial. For such x, the projection p restricts to a surjective map

$$\mathrm{Stab}_{G \times \mathrm{PGL}_n}(x) \to \mathrm{Stab}_G\big(\pi(x)\big)$$

whose kernel is $\mathrm{Stab}_{\mathrm{PGL}_n}(x) = \{1\}$, and the lemma follows. □

7.2. Proposition.

(a) *Suppose that for $w \in W$ in general position, the stabilizer $\mathrm{Stab}_G(w)$ does not admit a non-trivial homomorphism to PGL_n. Then A^G is a central simple algebra of degree $n = \deg(A)$.*

(b) *Suppose that for $w \in W$ in general position, $\mathrm{Stab}_G(w)$ is an abelian group consisting of semisimple elements and the n-torsion subgroup of $\mathrm{Stab}_G(w)/\mathrm{Stab}_G(w)^0$ is cyclic. Then there exists an $a \in A^G$ with n distinct eigenvalues.*

Note that the condition of part (a) is satisfied if the G-action on W is generically free.

Proof. (a) By Lemma 7.1, $\mathrm{Stab}_{G\times \mathrm{PGL}_n}(x) \subseteq G \times \{1\}$ for x in general position in X. The desired conclusion follows from Theorem 1.4(a).

(b) Let H be the projection of $\mathrm{Stab}_{G\times \mathrm{PGL}_n}(x)$ to PGL_n. By Lemma 7.1, H is an abelian group consisting of semisimple elements, and H/H^0 is a homomorphic image of $\mathrm{Stab}_G(w)/\mathrm{Stab}_G(w)^0$. Using the fundamental theorem of finite abelian groups, one checks that surjective homomorphisms of finite abelian groups preserve the property that the n-torsion subgroup is cyclic. By [St, Corollary 2.25(a)], H is contained in a maximal torus of PGL_n. (Note that the torsion primes for PGL_n are the primes dividing n; see [St, Corollaries 1.13 and 2.7].) The desired conclusion now follows from Theorem 1.4(b). □

We will now use Proposition 7.2, to study inner actions. Recall that an automorphism ϕ of a central simple algebra A is called *inner* if there exists an invertible element $a \in A$ such that $\phi(x) = axa^{-1}$ for every $x \in A$, and *outer* otherwise. By the Skolem–Noether theorem ϕ is inner if and only $\phi(x) = x$ for every $x \in \mathrm{Z}(A)$.

7.3. Corollary. *Let G be a finite group or a torus acting geometrically on a central simple algebra A of degree n. The elements of G that act by inner automorphisms form a normal subgroup of G; denote this subgroup by N.*

(a) *If $N = \{1\}$ (i.e., if G acts on A by outer automorphisms), then A^G is a central simple algebra of degree n.*
(b) *If N is a cyclic group, then there is an element $a \in A^G$ with n distinct eigenvalues.*

In the case where the group G is finite, part (a) is proved by algebraic means and under weaker hypotheses in [M, Theorem 2.7 and Corollary 2.10]. Note also that since every action of a finite group on a central simple algebra is algebraic (see Definition 5.2), our assumption that the action is geometric is only relevant if G is a torus. Moreover, if G is a torus then every geometric action is algebraic; see Corollary 8.4.

Proof. We may assume that the action is faithful. Indeed, if $K \subseteq G$ is the kernel of this action, we can replace G by G/K and N by N/K.

Now let W be an irreducible G-variety whose function field $k(W)$ is G-equivariantly isomorphic to $\mathrm{Z}(A)$ (over k); see the beginning of Section 7. Clearly an element of G acts trivially on W if and only if it acts on A by an inner automorphism. Now recall that if G is a finite group or a torus then the stabilizer in general position for the G-action on W is precisely the kernel N of this action; cf. Lemma 2.4.

The desired conclusions in parts (a) and (b) now follow from parts (a) and (b) of Proposition 7.2, respectively. □

8. Which geometric actions are algebraic?

Theorem 5.3(a) says that every algebraic action is geometric. It is easy to see that the converse is not true. For example, let Y be a generically free PGL_n-variety (e.g., we can take $Y = \mathrm{PGL}_n$ where PGL_n acts on Y by translations), and consider the $G \times \mathrm{PGL}_n$-variety

$X = (G/P) \times Y$, where G is a non-solvable connected algebraic group, and P is a proper parabolic subgroup. Here G acts by translations on the first factor, and PGL_n acts on the second factor. Since the PGL_n-action on X is generically free, $A = k_n(X)$ is a central simple algebra of degree n. On the other hand, since G/P is complete, it is easy to see that X is not birationally isomorphic to an affine $G \times \mathrm{PGL}_n$-variety; hence by Theorem 5.3(b), this action is not algebraic.

Nevertheless, we will now show that under fairly mild assumptions, the converse of Theorem 5.3(a) holds, i.e., every geometric action is, indeed, algebraic.

8.1. Lemma. *Let G be an algebraic group, and let X be an irreducible $G \times \mathrm{PGL}_n$-variety which is PGL_n-generically free. Assume that X has a stable affine model as a $G \times \mathrm{PGL}_n$-variety. Then the induced action of G on $k_n(X)$ is algebraic.*

Proof. We may assume without loss of generality that X itself is affine and stable as a $G \times \mathrm{PGL}_n$-variety. By Theorem 5.3(b) it suffices to show that X is stable as a PGL_n-variety, i.e., that PGL_n-orbits in general position in X are closed. Let $x \in X$ be a point in general position. Then the $G \times \mathrm{PGL}_n$-orbit $(G \times \mathrm{PGL}_n) \cdot x$ is closed in X and can be naturally identified with the homogeneous space $(G \times \mathrm{PGL}_n)/H$, where $H = \mathrm{Stab}_{G \times \mathrm{PGL}_n}(x)$. The PGL_n-orbit $(\mathrm{PGL}_n) \cdot x$ is then identified with the image Z of PGL_n in $(G \times \mathrm{PGL}_n)/H$. It thus remains to show that Z is closed in $(G \times \mathrm{PGL}_n)/H$. Indeed, Z is also the image of the product $\mathrm{PGL}_n H$, which is a closed subgroup of $G \times \mathrm{PGL}_n$ (because PGL_n is normal; see [H, Section 7.4]). Since $\mathrm{PGL}_n H$ is a closed subgroup of $G \times \mathrm{PGL}_n$ containing H, its image Z in $(G \times \mathrm{PGL}_n)/H$ is closed; see [H, Section 12.1]. □

8.2. Corollary. *Let G be an algebraic group, and let X be an irreducible $G \times \mathrm{PGL}_n$-variety which is $G \times \mathrm{PGL}_n$-generically free. Then the induced action of G on $k_n(X)$ is algebraic.*

Proof. By [RV_3, Theorem 1.2(i)] X has a stable affine birational model as a $G \times \mathrm{PGL}_n$-variety. The desired conclusion is now immediate from Lemma 8.1. □

The criterion for a geometric action to be algebraic given by Lemma 8.1 can be further simplified by considering the G-action on the center of A, as in Section 7.

8.3. Proposition. *Consider a geometric action of an algebraic group G on a central simple algebra A, and let W be a birational model for the G-action on $\mathrm{Z}(A)$. Then the G-action on A is algebraic, provided one of the following conditions holds*:

(a) *The G-action on W is generically free.*
(b) *The normalizer $H = \mathrm{N}_G(G_w)$ is reductive for $w \in W$ in general position.*
(c) *G is reductive and the stabilizer G_w is reductive for $w \in W$ in general position.*
(d) *G is reductive and W has a stable affine model as G-variety.*

Proof. Let X be an associated $G \times \mathrm{PGL}_n$-variety for the G-action on A. Recall that the PGL_n-action on X is generically free and W is the rational quotient X/PGL_n. In view of Lemma 8.1, it suffices to show that X has a stable affine model as a $G \times \mathrm{PGL}_n$-variety.

(a) Immediate from Corollary 8.2 and Lemma 7.1.

(b) Choose $x \in X$ in general position, and set $w = \pi(x) \in W$. Let $S_x = \mathrm{Stab}_{G\times \mathrm{PGL}_n}(x)$. We claim that $\mathrm{N}_{G\times\mathrm{PGL}_n}(S_x)$ is reductive for $x \in X$ in general position. The desired conclusion follows from this claim by [RV_3, Theorem 1.2(ii)].

The proof of the claim is based on two simple observations. First of all, if $H = \mathrm{N}_G(G_w)$ is reductive, then so is $S_x \simeq G_w$. Indeed, the unipotent radical $\mathrm{R}_u(G_w)$ is characteristic in G_w, hence, normal in H. Since H is reductive, this implies $\mathrm{R}_u(G_w) = \{1\}$, i.e., G_w is reductive, as claimed.

Secondly, by Lemma 7.1, the normalizer $\mathrm{N}_{G\times\mathrm{PGL}_n}(S_x)$ is a priori contained in $H \times \mathrm{PGL}_n$, i.e.,

$$\mathrm{N}_{G\times\mathrm{PGL}_n}(S_x) = \mathrm{N}_{H\times\mathrm{PGL}_n}(S_x).$$

Since both $H \times \mathrm{PGL}_n$ and S_x are reductive, the normalizer $\mathrm{N}_{H\times\mathrm{PGL}_n}(S_x)$ is reductive as well; see [LR, Lemma 1.1]. This concludes the proof of the claim and thus of part (b).

(c) If G and G_w are both reductive then using [LR, Lemma 1.1] once again we see that $\mathrm{N}_G(G_w)$ is also reductive. Part (c) now follows from part (b).

(d) After replacing W by a stable affine model, we see that for $w \in W$ in general position, the orbit $Gw \simeq G/G_w$ is affine, so that G_w is reductive by Matsushima's theorem, see [PV, Theorem 4.17]. Now use part (c). □

8.4. Corollary. *Let G be an algebraic group whose connected component is a torus. Then every geometric action of G on a central simple algebra is algebraic.*

Proof. In this case, every subgroup of G is reductive, so that part (c) of Proposition 8.3 applies. □

9. Proof of Theorem 1.5

9.1. The generic torus. Let T be a maximal torus in GL_n, and let N be the normalizer of the image of T in PGL_n. Since PGL_n permutes the maximal tori in GL_n transitively, one can think of PGL_n/N as the variety of maximal tori of GL_n (or equivalently, of PGL_n). We briefly recall how one can construct a PGL_n-equivariant rational map

$$\pi : \mathrm{M}_n \dashrightarrow \mathrm{PGL}_n/N$$

which sends a non-singular matrix $\alpha \in \mathrm{M}_n$ with distinct eigenvalues to the unique maximal torus in GL_n containing α. The map π is sometimes called the *generic torus* of GL_n; cf. [Vos, 4.1].

Denote by $\mathrm{Gr}(n, n^2)$ the Grassmannian of n-dimensional subspaces of M_n. The action of PGL_n on M_n induces a regular action of PGL_n on $\mathrm{Gr}(n, n^2)$. Define a rational, PGL_n-equivariant map $\pi_1 : \mathrm{M}_n \dashrightarrow \mathrm{Gr}(n, n^2)$ by sending a non-singular matrix α with distinct eigenvalues to $\mathrm{Span}(1, \alpha, \ldots, \alpha^{n-1})$. The unique maximal torus $T_{(\alpha)}$ of GL_n containing α is characterized by $\mathrm{Span}(T_{(\alpha)}) = \pi_1(\alpha)$. The image of π_1 consists thus of a single PGL_n-orbit O. Since the stabilizer of both T and $\mathrm{Span}(T)$ is N, $gN \mapsto g\,\mathrm{Span}(T)g^{-1}$ defines an isomorphism $\pi_2 : \mathrm{PGL}_n/N \to O$. Here T is the maximal torus in GL_n which we chose (and fixed) at the beginning of this section and N is the normalizer of the image of T in PGL_n. Now $\pi = \pi_2^{-1} \circ \pi_1$ is a PGL_n-equivariant rational map $\mathrm{M}_n \dashrightarrow \mathrm{PGL}_n/N$ such that for any α as above, $\pi(\alpha) = gN$ if and only if gTg^{-1} is the unique torus of GL_n containing α.

9.2. Proof of Theorem 1.5. (a) Suppose $A = k_n(X)$ has a G-invariant maximal étale subalgebra E. It follows easily from the primitive element theorem that there is an $a \in E$ so that $E = \mathrm{Z}(A)[a]$. Choose one such a. By Lemma 2.10, a has distinct eigenvalues. Adding some constant in k to a, we may assume that the eigenvalues of a are distinct and non-zero. Then for $x \in X$ in general position $a(x)$ is a matrix whose eigenvalues are distinct and non-zero. We now define a rational map $\varphi : X \dashrightarrow \mathrm{PGL}_n/N$ by $\varphi(x) = \pi(a(x))$. This map is PGL_n-equivariant by construction. Moreover, for every $g \in G$, $g(a) \in E$ commutes with a. Thus, for $x \in X$ in general position, $a(x)$ and $g^{-1}(a)(x) = a(g(x))$ lie in the same maximal torus, and consequently, $\varphi(x) = \varphi(g(x))$.

Conversely, suppose there exists a $G \times \mathrm{PGL}_n$-equivariant rational map $X \dashrightarrow \mathrm{PGL}_n/N$. After removing the indeterminacy locus from X, we may assume this map is regular. We may also assume that PGL_n acts freely on X, and that N is the normalizer of the diagonal torus in PGL_n. Let X_0 be the preimage of the coset $N \in \mathrm{PGL}_n/N$ in X. Note that X_0 is $G \times N$-invariant, that $X = \mathrm{PGL}_n \cdot X_0$, and that the N-action on X_0 is generically free. Moreover, X is birationally isomorphic as PGL_n-variety to $\mathrm{PGL}_n *_N X_0$, see [P, Theorem 1.7.5].

Let $\Delta \simeq \mathbb{A}^n$ be the variety of diagonal $n \times n$ matrices. By [Re, Proposition 7.1] there exists an N-equivariant rational map $a : X_0 \dashrightarrow \Delta$ whose image contains a matrix with distinct eigenvalues. (Note that here we use the fact that Δ is a vector space and N acts on it linearly.) This rational map then naturally extends to a PGL_n-equivariant rational map

$$X \simeq \mathrm{PGL}_n *_N X_0 \dashrightarrow \mathrm{PGL}_n *_N \Delta \simeq \mathrm{M}_n$$

induced by $(g, x_0) \mapsto (g, a(x_0))$. By abuse of notation, we denote this extended rational map by a as well.

We now view a as an element of $A = k_n(X)$. Since the image of a contains a matrix with distinct eigenvalues, Lemma 2.10 tells us that $E = \mathrm{Z}(A)[a]$ is a maximal étale subalgebra of A. It remains to show that E is G-invariant. To do this it suffices to prove that $g(a) \in E$ for every $g \in G$. Since $E = C_A(E)$, we only need to establish that $g(a)$ commutes with a, i.e., that the commutator $b = [a, g(a)]$ equals 0. Indeed, for any $x \in X_0$,

$$b(x) = \bigl[a(x), a\bigl(g^{-1}(x)\bigr)\bigr] = \bigl[a(x), a(y)\bigr],$$

where $y = g^{-1}(x) \in X_0$. By our construction a maps every element of X_0 to a diagonal matrix. In particular, $a(x)$ and $a(y)$ commute, and thus $b(x) = 0$ for every $x \in X_0$. Since b is a PGL_n-equivariant rational map $X \dashrightarrow \mathrm{M}_n$ and since $\mathrm{PGL}_n \cdot X_0 = X$, we conclude that $b = [a, g(a)]$ is identically zero on X, as claimed. This completes the proof of part (a).

(b) The action of $G \times \mathrm{PGL}_n$ on PGL_n/N has stabilizer of the form $G \times \mathrm{N}(S)$ at every point, where S is a maximal torus of PGL_n. Part (b) is now an immediate consequence of part (a).

(c) Assume that A has a G-invariant maximal étale subalgebra. Let $x \in X$ be a point in general position. We claim that

$$\dim(Gx \cap \mathrm{PGL}_n\, x) \leqslant n - 1. \tag{9.3}$$

Indeed, $Gx \cap \mathrm{PGL}_n\, x$ is easily seen to be the image of the morphism from $\mathrm{Stab}_{G\times \mathrm{PGL}_n}(x)$ to X given by $(g, p) \mapsto px$. Since $\mathrm{Stab}_{G\times \mathrm{PGL}_n}(x) \subseteq G \times \mathrm{N}(T_x)$ by part (b), we conclude that

$$\dim(Gx \cap \mathrm{PGL}_n\, x) \leqslant \dim \mathrm{N}(T_x) = n - 1$$

as claimed.

Consider the rational quotient map $\pi : X \dashrightarrow X/\mathrm{PGL}_n$. We may assume without loss of generality that π is defined at x. Now restrict π to the (well-defined) rational map $\pi_{G^0x} : G^0x \dashrightarrow X/\mathrm{PGL}_n$, where G^0 is the connected component of G. For $y \in G^0x$ in general position, the fiber over $\pi_{G^0x}(y)$ is $G^0x \cap \mathrm{PGL}_n\, y = G^0y \cap \mathrm{PGL}_n\, y$. By (9.3),

$$\begin{aligned}\dim(Gx) = \dim\bigl(G^0x\bigr) &\leqslant \dim(X/\mathrm{PGL}_n) + n - 1\\ &= \dim(X) - \dim(\mathrm{PGL}_n) + n - 1\\ &= \dim(X) - n^2 + n.\end{aligned}$$

So $\dim(X) - \dim(Gx) \geqslant n^2 - n$. This proves part (c). □

10. Proof of Theorem 1.7

We begin by spelling out what it means for an algebraic group action on a central simple algebra to be split in terms of the associated variety.

10.1. Lemma. *A geometric action of an algebraic group G on a central simple algebra A of degree n is G-split in the sense of Definition* 1.6 *if and only if its associated $G \times \mathrm{PGL}_n$-variety is birationally isomorphic to $X_0 \times \mathrm{PGL}_n$, for some G-variety X_0.*

Here G acts on the first factor and PGL_n acts on the second factor by translations.

Proof. Suppose $X = X_0 \times \mathrm{PGL}_n$. Then we have the following G-equivariant isomorphisms,

$$\mathrm{RMaps}_{\mathrm{PGL}_n}(X, \mathrm{M}_n) \simeq \mathrm{RMaps}(X_0, \mathrm{M}_n) \simeq \mathrm{M}_n(k) \otimes_k k(X_0),$$

where the first isomorphism is given by $f \mapsto f|_{X_0\times 1_{\mathrm{PGL}_n}}$ for every PGL_n-equivariant rational map $f : X \dashrightarrow \mathrm{M}_n$. In other words, the induced G-action on $A = k_n(X)$ is G-split in the sense of Definition 1.6.

Conversely, suppose a geometric G-action on A is G-split. Denote the associated $G \times \mathrm{PGL}_n$-variety by X. Let $X_0 = X/\mathrm{PGL}_n$ be the rational quotient of X by the PGL_n-action. Note that $k(X_0) = \mathrm{Z}(A)$. Then, as we saw above, $\mathrm{RMaps}_{\mathrm{PGL}_n}(X_0 \times \mathrm{PGL}_n, \mathrm{M}_n)$ is G-equivariantly isomorphic to $\mathrm{M}_n(k) \otimes_k k(X_0)$, which is G-equivariantly isomorphic to A (because A is G-split). By Corollary 3.2, we conclude that X is birationally isomorphic to $X_0 \times \mathrm{PGL}_n$. □

10.2. Corollary. *Consider a geometric action of an algebraic group G on a central simple algebra A, with associated $G \times \mathrm{PGL}_n$-variety X. Then for any G-variety X_0 the following are equivalent*:

(a) *$L = k(X_0)$ is a G-splitting field for A.*
(b) *There exists a dominant rational map $f : X_0 \times \mathrm{PGL}_n \dashrightarrow X$ which is $G \times \mathrm{PGL}_n$-equivariant.*

Here G acts on the first factor of $X_0 \times \mathrm{PGL}_n$ and PGL_n acts on the second factor by translations, as in Lemma 10.1.

Proof. (a) ⇒ (b): The G-action on $A' = A \otimes_{\mathrm{Z}(A)} L \simeq \mathrm{M}_n(k) \otimes_k L$ is geometric, with associated variety $X' = X_0 \times \mathrm{PGL}_n$; see Lemma 10.1. The embedding $j : A \hookrightarrow A'$ induces a $G \times \mathrm{PGL}_n$-equivariant dominant rational map $j_* : X' \dashrightarrow X$; see Lemma 3.1.

(b) ⇒ (a): Let $X' = X_0 \times \mathrm{PGL}_n$. By Lemma 3.1, f induces a G-equivariant embedding $f^* : A \hookrightarrow A'$ of central simple algebras, where $A' = k_n(X') \simeq \mathrm{M}_n(k) \otimes_k k(X_0)$; see Lemma 10.1. In other words, A' is G-equivariantly isomorphic to $A \otimes_{\mathrm{Z}(A)} k(X_0)$. □

10.3. Proof of Theorem 1.7. Let X be the associated $G \times \mathrm{PGL}_n$-variety for the G-action on A. Consider the dominant morphism $f : X \times \mathrm{PGL}_n \to X$ given by $(x, h) \mapsto hx$. If we let $(g, h) \in G \times \mathrm{PGL}_n$ act on $X \times \mathrm{PGL}_n$ by $(g, h) \cdot (x, h') = (gx, hh')$, as in Lemma 10.1 and Corollary 10.2, then we can easily check that f is $G \times \mathrm{PGL}_n$-equivariant. By Corollary 10.2, we conclude that $L = k(X)$ is a G-splitting field for A. Moreover,

$$\begin{aligned}\mathrm{trdeg}_{\mathrm{Z}(A)} L &= \mathrm{trdeg}_k(L) - \mathrm{trdeg}_k \mathrm{Z}(A) \\ &= \dim(X) - \dim(X/\mathrm{PGL}_n) = n^2 - 1,\end{aligned}$$

as claimed. Note that if G acts algebraically on A, we may assume that X is affine by Theorem 5.3(b). □

11. More on G-splitting fields

In this section we discuss G-splitting fields in the case where G is a connected group. Our main result is the following:

11.1. Proposition. *Consider a geometric action of a connected algebraic group G on a central simple algebra A of degree n. Then there exists an affine G-variety X_0 such that $L = k(X_0)$ is a G-splitting field of A and*

$$\operatorname{trdeg}_{Z(A)} L = \dim \operatorname{Stab}_{G\times \mathrm{PGL}_n}(x) = \dim \operatorname{Stab}_G(w), \tag{11.2}$$

where x and w are points in general position in the associated $G \times \mathrm{PGL}_n$-variety X and in the rational quotient $W = X/\mathrm{PGL}_n$, respectively. In particular,

$$\operatorname{trdeg}_{Z(A)} L \leqslant \dim(G).$$

Note that for $w \in W$ in general position we have

$$\begin{aligned} \dim \operatorname{Stab}_G(w) &= \dim(G) - \dim(Gw) \\ &= \dim(G) - \big(\dim(W) - \dim(W/G)\big) \end{aligned} \tag{11.3}$$

so that the integer $\dim \operatorname{Stab}_G(w)$ for $w \in W$ in general position, which appears in the statement of Proposition 11.1, is well defined. Similarly, the integer $\dim \operatorname{Stab}_{G\times \mathrm{PGL}_n}(x)$ for $x \in X$ in general position is also well defined. Since $\operatorname{trdeg}_{Z(A)^G} Z(A) = \dim(W) - \dim(W/G)$, (11.2) can be restated in algebraic terms as

$$\operatorname{trdeg}_{Z(A)} L = \dim(G) - \operatorname{trdeg}_{Z(A)^G} Z(A). \tag{11.2$'$}$$

In general, the value for $\operatorname{trdeg}_{Z(A)} L$ given in (11.2) and (11.2′) is the smallest possible, see Remark 11.8. Our proof of Proposition 11.1 will rely on the following lemma.

11.4. Lemma. *Let H be a connected algebraic group and let V be an irreducible H-variety. Then there exists an irreducible variety Y and an H-equivariant dominant morphism $Y \times H \to V$ such that* $\dim(Y) = \dim(V/H)$.

The action of H on $Y \times H$ is induced by the trivial action on Y and by the translation action on H.

Proof. See [P, (1.2)] or [PV, Proposition 2.7], where the term quasi-section is used to describe Y. □

11.5. Proof of Proposition 11.1. By Lemma 11.4 (with $H = G \times \mathrm{PGL}_n$) there is a $G \times \mathrm{PGL}_n$-equivariant dominant morphism $f : Y \times (G \times \mathrm{PGL}_n) \to X$, where

$$\dim(Y) = \dim\big(X/(G \times \mathrm{PGL}_n)\big) = \dim(W/G). \tag{11.6}$$

Note that since $G \times \mathrm{PGL}_n$ acts trivially on Y, we can take Y to be affine. Setting $X_0 = Y \times G$ (as a G-variety) and applying Corollary 10.2, we conclude that $L = k(X_0)$ is a G-splitting field for A. By our construction, $X_0 = Y \times G$ is affine. Since the second equality in (11.2) is an immediate consequence of Lemma 7.1, we only need to check that $\operatorname{trdeg}_{Z(A)} L = \dim \operatorname{Stab}_G(w)$ for $w \in W$ in general position. Indeed,

$$\begin{aligned}\operatorname{trdeg}_{Z(A)} L &= \operatorname{trdeg}_k(L) - \operatorname{trdeg}_k Z(A) = \dim(X_0) - \dim(X/\mathrm{PGL}_n) \\ &= \dim(Y) + \dim(G) - \dim(X/\mathrm{PGL}_n) \\ &= \dim(G) - \big(\dim(W) - \dim(W/G)\big) = \dim \operatorname{Stab}_G(w),\end{aligned}$$

where the two last equalities follow from (11.6) and (11.3), respectively. □

Specializing Proposition 11.1 to the case of torus actions, we recover a result which was proved in [V_3] for algebraic actions in arbitrary characteristic.

11.7. Corollary. *Suppose a torus T acts geometrically* (*or equivalently, algebraically*; *cf. Corollary* 8.4) *on a central simple algebra A. Let H be the kernel of the T-action on* $Z(A)$. *Then there exists a T-variety X_0 such that $L = k(X_0)$ is a T-splitting field for A and* $\operatorname{trdeg}_{Z(A)} L = \dim(H)$.

Proof. Let X be the associated $G \times \mathrm{PGL}_n$-variety and $W = X/\mathrm{PGL}_n$, as before. By Lemma 2.4, applied to the T-action on W, we have $H = \operatorname{Stab}_T(w)$ for $w \in W$ in general position. The corollary now follows from Proposition 11.1. □

11.8. Remark. If the T-action on A is faithful then the value of $\operatorname{trdeg}_{Z(A)} L$ given by Corollary 11.7 is the smallest possible. Indeed, since the T-action on both A and $L = k(X_0)$ is algebraic (cf. Corollary 8.4), [V_3, Theorem 2(b)] tells us that $\operatorname{trdeg}_{Z(A)} L \geqslant \dim(H)$ for every T-splitting field of the form $L = k(X_0)$, where X_0 is a T-variety.

11.9. Remark. Suppose a torus T acts geometrically (or equivalently, algebraically; cf. Corollary 8.4) on a division algebra D. Then [V_3, Theorem 2(c)] asserts that D has a T-splitting field L of the form $k(X_0)$ such that $[L\colon Z(D)] < \infty$.

We now give an alternative proof of this result based on Corollary 11.7. Let $T_0 \subset T$ be the kernel of the T-action on D. After replacing T by T/T_0, we may assume the action is faithful. Let H be the kernel of the T-action on $Z(A)$, i.e., the subgroup of T acting by inner automorphisms. By Corollary A.2, H is a finite group. By Corollary 11.7, there exists a T-splitting field $L = k(X_0)$ such that $\operatorname{trdeg}_{Z(D)} L = \dim(H) = 0$. Since L is finitely generated over k (and hence, over $Z(D)$), we conclude that $[L\colon Z(D)] < \infty$.

12. An example: Algebraic actions of unipotent groups

In this and the subsequent three sections we will present examples, illustrating Theorems 1.4, 1.5, and 1.7. We begin by applying Theorems 1.4 and 1.5 in the context of unipotent group actions on division algebras.

12.1. Proposition. *Let U be a unipotent group acting algebraically on a finite-dimensional division algebra D. Then D^U is a division algebra of the same degree as D.*

Proof. Say D has degree n, and let X be the associated $U \times \mathrm{PGL}_n$-variety. By Lemma 7.1, for $x \in X$ in general position, $\mathrm{Stab}_{U\times \mathrm{PGL}_n}(x)$ is a unipotent group (it is isomorphic to a subgroup of U). Consequently, the projection H_x of this group to PGL_n is unipotent.

On the other hand, by [V_3, Proposition 7], D has a U-invariant maximal subfield. In view of Theorem 1.5(b), this implies that H_x is a subgroup of the normalizer of a maximal torus in PGL_n; in particular, H_x has no non-trivial unipotent elements. This is only possible if $H_x = \{1\}$, i.e., if

$$\mathrm{Stab}_{U\times \mathrm{PGL}_n}(x) \subseteq U \times \{1\}.$$

The desired conclusion now follows from Theorem 1.4(a). □

12.2. Remark. The condition that D is a division algebra is essential here. Suppose $G = U$ is a non-trivial unipotent subgroup of PGL_n acting on $A = \mathrm{M}_n(k)$ by conjugation, as in Example 3.3. Since A is a finite-dimensional k-vector space, this action is easily seen to be algebraic. On the other hand, the fixed algebra A^U is not a central simple algebra of degree n; cf. Example 6.2(a) (see also Remark 6.4).

13. An example: The GL_m-action on $UD(m,n)$, $m \geqslant n^2$

We now return to the GL_m-action on the universal division algebra $A = UD(m,n)$, described in Example 3.4. In this section we will assume that $m \geqslant n^2$; in the next section we will set $m = n = 2$. The case where $m \leqslant n^2 - 1$ will be considered in [RV_5].

13.1. Proposition. *Let $A = UD(m,n)$, where $m \geqslant n^2$. Then*

(a) $A^{\mathrm{GL}_m} = k$;
(b) $\mathrm{trdeg}_{Z(A)} L \geqslant n^2 - 1$ *for every* GL_m*-splitting field L of A of the form $L = k(X_0)$, where X_0 is a* GL_m*-variety.*

Part (b) shows that the value of $\mathrm{trdeg}_{Z(A)} L$ given by Theorem 1.7 is optimal for this action.

Proof. The variety $(\mathrm{M}_n)^m$ is an associated $\mathrm{GL}_m \times \mathrm{PGL}_n$-variety for the GL_m-action on A; see Example 3.4. The key fact underlying the proof of both parts is that for $m \geqslant n^2$, $(\mathrm{M}_n)^m$ has a dense GL_m-orbit; denote this orbit by X. Since the actions of GL_m and PGL_n commute, X is PGL_n-stable, and therefore is also an associated $\mathrm{GL}_m \times \mathrm{PGL}_n$-variety for the GL_m-action on A.

(a) By Remark 6.1, $A^{\mathrm{GL}_m} = \mathrm{RMaps}_{\mathrm{PGL}_n}(X/\mathrm{GL}_m, \mathrm{M}_n)$. Since X is a single GL_m-orbit, the rational quotient X/GL_m is a point (with trivial PGL_n-action). Clearly, every PGL_n-equivariant rational map $f : \{pt\} \dashrightarrow \mathrm{M}_n$ is regular and has its image in the center of M_n. In other words,

$$A^{\mathrm{GL}_m} = \mathrm{RMaps}_{\mathrm{PGL}_n}(X/\mathrm{GL}_m, \mathrm{M}_n) = \mathrm{RMaps}\big(\{pt\}, k\big) = k,$$

as claimed.

(b) By Corollary 10.2 there exists a dominant rational map $f : X_0 \times \mathrm{PGL}_n = X' \dashrightarrow X$. Choose $x' \in X'$, so that f is defined at x' and set $x = f(x')$. Denote by S and S' the stabilizers in $\mathrm{GL}_m \times \mathrm{PGL}_n$ of x and x', respectively. Note that $S' \subseteq S \subseteq \mathrm{GL}_m \times \mathrm{PGL}_n$. Since GL_m acts transitively on X, the projection of S to PGL_n is all of PGL_n. On the other hand, we clearly have $S' \subseteq G \times \{1\}$. Consequently, $\dim(S) - \dim(S') \geqslant \dim(\mathrm{PGL}_n) = n^2 - 1$, and if O' is a $\mathrm{GL}_m \times \mathrm{PGL}_n$-orbit in general position in X', then $\dim(O') - \dim(X) \geqslant n^2 - 1$. We thus conclude that

$$\begin{aligned}\operatorname{trdeg}_{Z(A)} L &= \operatorname{trdeg}_k L - \operatorname{trdeg}_k Z(A)\\ &= \dim(X'/\mathrm{PGL}_n) - \dim(X/\mathrm{PGL}_n)\\ &= \dim(X') - \dim(X) \geqslant \dim(O') - \dim(X) \geqslant n^2 - 1,\end{aligned}$$

as claimed. □

13.2. Remark. One can show directly that the GL_m-splitting field L for $A = UD(m,n)$ given by Proposition 11.1 satisfies the inequality of Proposition 13.1(b) (assuming, of course, that $m \geqslant n^2$). Indeed, since G has a dense orbit in $W = X/\mathrm{PGL}_n$, for $w \in W$ in general position,

$$\dim \mathrm{Stab}_{\mathrm{GL}_m}(w) = \dim(\mathrm{GL}_m) - \dim(W) = m^2 - \dim(X/\mathrm{PGL}_n).$$

Since the associated variety $X = (\mathrm{M}_n)^m$ has dimension mn^2, this yields

$$\operatorname{trdeg}_{Z(A)} L = \dim \mathrm{Stab}_{\mathrm{GL}_m}(w) = m\big(m - n^2\big) + \big(n^2 - 1\big) \geqslant n^2 - 1,$$

as claimed.

14. An example: The GL_2-action on $UD(2, 2)$

In this section we will use Theorem 1.4 to study the natural GL_m-action on the universal division algebra $UD(m,n)$, described in Example 3.4, for $m = n = 2$. Note that this case exhibits some special features that do not recur for other values of m and $n \geqslant 2$; see Proposition 13.1(a) (for $m \geqslant n^2$) and [RV5] (for $m \leqslant n^2 - 1$).

14.1. Proposition. *The fixed algebra $UD(2,2)^{\mathrm{GL}_2}$ is a non-central subfield of $UD(2,2)$ of transcendence degree 1 over k.*

Recall from Example 3.4 that the GL_2-action on $UD(2,2)$ is defined as follows. Denote by X and Y the two generic 2×2 matrices generating $UD(2,2)$. Then for $g \in \mathrm{GL}_2$, we have $g(X) = \alpha X + \beta Y$, and $g(Y) = \gamma X + \delta Y$, where $g^{-1} = \left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)$. Recall also that the associated variety for the GL_2-action on $UD(2,2)$ is $X = (\mathrm{M}_2)^2$. In order to use Theorem 1.4 to prove Proposition 14.1, we first need to determine the stabilizer in general position for the $\mathrm{GL}_2 \times \mathrm{PGL}_2$-action on $(\mathrm{M}_2)^2$.

14.2. Lemma. *For $x \in (\mathrm{M}_2)^2$ in general position,* $\mathrm{Stab}_{\mathrm{GL}_2 \times \mathrm{PGL}_2}(x)$ *is isomorphic to* $\mathbb{Z}/2\mathbb{Z}$.

Proof. By Lemma 7.1, $\mathrm{Stab}_{\mathrm{GL}_2 \times \mathrm{PGL}_2}(x)$ is isomorphic to $\mathrm{Stab}_{\mathrm{GL}_2}(y)$ for the GL_2-action on $W = X/\mathrm{PGL}_2$, which is a birational model for the GL_2-action on the center Z of $UD(2,2)$. In this case there is a particularly simple birational model, which we now describe.

It is well known that Z is freely generated (as a field extension of k) by the five elements $\mathrm{tr}(X)$, $\mathrm{tr}(Y)$, $\mathrm{tr}(X^2)$, $\mathrm{tr}(Y^2)$ and $\mathrm{tr}(XY)$; see [Pr$_1$, Theorem 2.2]. In other words, the categorical (and, hence, the rational) quotient for the PGL_2-action is $\mathbb{A}^5$. The group GL_2 acts on $\mathbb{A}^5$ linearly. In fact, the representation of GL_2 on $\mathbb{A}^5 = X/\mathrm{PGL}_2$ can be decomposed as $V_2 \oplus V_3$, where V_2 is the natural 2-dimensional representation (we can think of it as $\mathrm{Span}_k(\mathrm{tr}(X), \mathrm{tr}(Y))$) and V_3 is its symmetric square. (We can think of V_3 as $\mathrm{Span}_k(\mathrm{tr}(X^2), \mathrm{tr}(Y^2), \mathrm{tr}(XY))$.)

The question we are asking now reduces to the following: what is the stabilizer, in GL_2, of a pair (v, q), in general position, where v is a vector in k^2 and q is a quadratic form in 2 variables? Indeed, since GL_2 acts transitively on non-degenerate quadratic forms in two variables, we may assume that q is a fixed form of rank 2, e.g., $q = x^2 + y^2$. The stabilizer of q is thus the orthogonal group O_2, and our question further reduces to the following: what is the stabilizer in general position for the natural linear action of O_2 on k^2? The answer is easily seen to be $\mathbb{Z}/2\mathbb{Z}$, where the non-trivial element of $\mathrm{Stab}_{O_2}(v)$ is the orthogonal reflection in v; see Example 2.5. □

Proof of Proposition 14.1. Note that the GL_2-action on $X = (\mathrm{M}_2)^2$ is generically free (it is isomorphic to the direct sum of 4 copies of the natural 2-dimensional representation of GL_2). Thus the image of the stabilizer $\mathrm{Stab}_{\mathrm{GL}_2 \times \mathrm{PGL}_2}(x)$ under the natural projection to the second factor is $\mathbb{Z}/2\mathbb{Z}$. Since this image is non-trivial, Theorem 1.4(a) tells us that $UD(2,2)^{\mathrm{GL}_2}$ is not a division subalgebra of $UD(2,2)$ of degree 2. In other words, it is a subfield of $UD(2,2)$. On the other hand, Theorem 1.4(b) tells us that $UD(2,2)^{\mathrm{GL}_2}$ is not contained in the center Z of $UD(2,2)$. Indeed, every subgroup of PGL_2 of order 2 is contained in a torus. Hence, $\mathrm{Stab}_{\mathrm{GL}_2 \times \mathrm{PGL}_2}(x)$ is contained in $\mathrm{GL}_2 \times T_x$, where T_x is a maximal torus of PGL_2. It follows from Theorem 1.4(b) that the subfield $UD(2,2)^{\mathrm{GL}_2}$ is not central in $UD(2,2)$.

Finally, note that $UD(2,2)^{\mathrm{GL}_2}$ is algebraic over Z^{GL_2}, since the minimal polynomial of any element of $UD(2,2)^{\mathrm{GL}_2}$ over Z is unique, so must have coefficients in Z^{GL_2}. It follows from Lemmas 7.1 and 14.2 that the GL_2-action on $W = X/\mathrm{PGL}_2$ has a finite stabilizer in general position. Hence the transcendence degree of $Z^{\mathrm{GL}_2} = k(X/\mathrm{PGL}_2)^{\mathrm{GL}_2}$ (over k) is $\dim(X/\mathrm{PGL}_2) - \dim(\mathrm{GL}_2) = 1$. □

14.3. Remark. This argument also shows that $UD(2,2)^{\mathrm{SL}_2}$ is a division algebra of degree 2.

14.4. Remark. One can exhibit an explicit non-central GL_2-fixed element of $UD(2,2)$ as follows. Let

$$S_3(A_1, A_2, A_3) = \sum_{\sigma \in \mathrm{S}_3} (-1)^\sigma A_{\sigma(1)} A_{\sigma(2)} A_{\sigma(3)}$$

be the standard polynomial in three variables; cf. [Ro_1, p. 8]. Set $a = [X, Y] = XY - YX$ and $b = S_3(X, Y, a)$. Using the fact that $[A_1, A_2]$ and $S_3(A_1, A_2, A_3)$ are multilinear and alternating, it is easy to see that for $g \in \mathrm{GL}_2$, $g(a) = a/\det(g)$ and $g(b) = b/\det^2(g)$. Specializing X to $\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$ and Y to $\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)$, an elementary computation shows that a and b specialize to $\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}-4&0\\0&-2\end{smallmatrix}\right)$, respectively. This shows that $\det(a) \neq 0$ and that b is non-central. Now, $b/\det(a)$ is a non-central GL_2-fixed element of $UD(2,2)$.

Note also that a and b are non-commuting SL_2-invariant elements of $UD(2,2)$. This gives an explicit proof of Remark 14.3.

15. An example: A finite group action on a cyclic algebra

In this section we present an example of a finite group action on a cyclic algebra. This example illustrates Theorem 1.5 and, in particular, shows that the converse to Theorem 1.5(b) is false.

Let p be a prime integer, and ζ a primitive pth root of unity in k. Let $P = k\{x, y\}$ be the skew-polynomial ring with generators x and y, subject to the relation

$$xy = \zeta yx.$$

Let A be the division algebra of fractions of P; it is a central simple algebra of degree $n = p$. Note that A is the symbol algebra $(u, v)_p$ whose center is $\mathrm{Z}(A) = k(u, v)$, where $u = x^p$ and $v = y^p$ are algebraically independent over k.

For $(a, b) \in (\mathbb{Z}/p\mathbb{Z})^2$, define an automorphism $\sigma_{(a,b)}$ of A by

$$\sigma_{(a,b)}(x) = \zeta^a x \quad \text{and} \quad \sigma_{(a,b)}(y) = \zeta^b y. \tag{15.1}$$

These automorphisms of A form a group K which is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^2$. Next, we define an automorphism τ of A by

$$\tau(x) = y \quad \text{and} \quad \tau(y) = x^{-1}y^{-1}. \tag{15.2}$$

Note that τ is well defined since

$$\tau(x)\tau(y) - \zeta\tau(y)\tau(x) = y(yx)^{-1} - \zeta x^{-1} = y\big(\zeta^{-1}xy\big)^{-1} - \zeta x^{-1} = 0.$$

Elementary calculations show that τ has order three, and that $\tau^{-1}\sigma_{(a,b)}\tau = \sigma_{(b,-a-b)}$. Consequently, the subgroup G of automorphisms of A generated by K and τ is a semidirect product $G = K \rtimes H$, where $K \simeq (\mathbb{Z}/p\mathbb{Z})^2$ and $H = \langle\tau\rangle \simeq \mathbb{Z}/3\mathbb{Z}$.

One easily checks that sending τ to the matrix $\left(\begin{smallmatrix}0&-1\\1&-1\end{smallmatrix}\right)$ defines a representation

$$\phi_p : H \to \mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z}),$$

and thus an action of H on $(\mathbb{Z}/p\mathbb{Z})^2$.

Let X be the $G \times \mathrm{PGL}_n$-variety associated to the action of G on the central simple algebra A of degree $n = p$. That is, X is an irreducible $G \times \mathrm{PGL}_n$-variety which is PGL_n-generically free, and A is G-equivariantly isomorphic to $k_n(X)$.

15.3. Proposition.

(a) *For $x \in X$ in general position, there exists a maximal torus T_x of* PGL_n *such that* $\mathrm{Stab}_{G\times \mathrm{PGL}_n}(x) \subseteq G \times \mathrm{N}(T_x)$.
(b) *A has a G-invariant maximal subfield if and only if the* 2*-dimensional representation $\phi_p : H \to \mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})$ is reducible over $\mathbb{Z}/p\mathbb{Z}$.*
(c) *The converse to Theorem* 1.5(b) *is false.*

Before we proceed with the proof, two remarks are in order. First of all, every finite group action on a central simple algebra is automatically geometric (and algebraic).

Secondly, an explicit model for X is not immediately transparent (a description of X as a PGL_n-variety can be found in [RY, Lemma 5.2]). On the other hand, the G-variety W associated to the G-action on the center of A (see the beginning of Section 7) is easy to describe: we can take W to be the two-dimensional torus $W = (k^*)^2 = \mathrm{Spec}(k[u, v, u^{-1}, v^{-1}])$, where as before, $u = x^p$ and $v = y^p$. It follows from (15.1) and (15.2) that the K-action on W is trivial, and that the action of τ is induced from $\tau(u) = v$, $\tau(v) = (x^{-1}y^{-1})^p = \epsilon \cdot u^{-1}v^{-1}$, where $\epsilon = 1$ if $p > 2$ and $\epsilon = -1$ if $p = 2$.

We now proceed with the proof of Proposition 15.3.

Proof. (a) Since G is a finite group, $\mathrm{Stab}_G(w)$, for $w \in W$ in general position, is precisely the kernel of the G-action on W. We claim that the kernel is equal to K. That it contains K is immediate from (15.1), since every element of K preserves both $u = x^p$ and $v = y^p$. On the other hand, the H-action on W is faithful, because H is a simple group acting nontrivially on $\mathrm{Z}(A) = k(W)$. We have thus shown that $\mathrm{Stab}_G(w) = K$ for $w \in W$ in general position.

By Lemma 7.1, $\mathrm{Stab}_{G\times \mathrm{PGL}_n}(x) \simeq K \simeq (\mathbb{Z}/p\mathbb{Z})^2$ for x in general position in X. In particular, the projection of this group to PGL_n is a finite abelian subgroup of PGL_n. By [SS, II.5.17], every finite abelian subgroup of PGL_n lies in the normalizer of a maximal torus T_x. Thus

$$\mathrm{Stab}_{G\times \mathrm{PGL}_n}(x) \subseteq G \times \mathrm{N}(T_x),$$

as claimed.

(b) First we will describe the K-invariant maximal subfields of A, then determine which ones of them are also invariant under H. Note that since A is a division algebra of prime degree p, every non-trivial field extension L of the center $\mathrm{Z}(A)$ is a maximal subfield of A.

The group $K \simeq (\mathbb{Z}/p\mathbb{Z})^2$ acts trivially on $\mathrm{Z}(A)$; its action on A decomposes as a direct sum of p^2 one-dimensional character spaces $\mathrm{Span}_{\mathrm{Z}(A)}(x^i y^j)$, where $0 \leqslant i, j \leqslant p - 1$. These spaces are associated to the p^2 distinct characters of $(\mathbb{Z}/p\mathbb{Z})^2$; hence, every K-invariant $\mathrm{Z}(A)$-vector subspace L contains $x^i y^j$ for some $0 \leqslant i, j \leqslant p - 1$. Moreover, if L is a K-invariant maximal subfield of A then $\mathrm{Z}(A)(x^i y^j) \subseteq L$, where $0 \leqslant i, j \leqslant p - 1$ and

$(i, j) \neq (0,0)$. Since $[L\colon \mathrm{Z}(A)] = p$ and $x^i y^j \notin \mathrm{Z}(A)$, we conclude that $L = \mathrm{Z}(A)(x^i y^j)$. We will denote $\mathrm{Z}(A)(x^i y^j)$ by $L_{(i,j)}$.

Now suppose (i, j) and (r, s) are non-zero elements of $(\mathbb{Z}/p\mathbb{Z})^2$. We claim that $L_{(i,j)} = L_{(r,s)}$ if and only if (i, j) and (r, s) are proportional, i.e., if and only if they lie in the same 1-dimensional $\mathbb{Z}/p\mathbb{Z}$-subspace of $(\mathbb{Z}/p\mathbb{Z})^2$. Indeed, if (i, j) and (r, s) are proportional then up to a multiple from $\mathrm{Z}(A)$, $x^i y^j$ and $x^r y^s$ are powers of one another. Since neither one is central, they generate the same maximal subfield. Conversely, since a maximal subfield has dimension p over $\mathrm{Z}(A)$, it can contain only $p-1$ distinct $x^i y^j$ with $(0,0) \neq (i, j) \in (\mathbb{Z}/p\mathbb{Z})^2$. Since there are $p-1$ non-zero $\mathbb{Z}/p\mathbb{Z}$-multiples of (i, j), this proves the claim.

We have thus shown that the K-invariant maximal subfields of A are in bijective correspondence with 1-dimensional $\mathbb{Z}/p\mathbb{Z}$-subspaces of $(\mathbb{Z}/p\mathbb{Z})^2$: a 1-dimensional subspace V corresponds to the maximal subfield $L_V = \mathrm{Z}(A)(x^i y^j)$, where (i, j) is a non-zero element of V.

It is clear from (15.2) that $\tau(L_V) = L_{\tau(V)}$, where τ acts on $(\mathbb{Z}/p\mathbb{Z})^2$ via the representation ϕ_p. To sum up: A has a maximal G-invariant subfield $\Leftrightarrow$ τ preserves one of the K-invariant maximal subfields L_V $\Leftrightarrow$ $(\mathbb{Z}/p\mathbb{Z})^2$ has a τ-invariant 1-dimensional $\mathbb{Z}/p\mathbb{Z}$-subspace $V \subset (\mathbb{Z}/p\mathbb{Z})^2$ $\Leftrightarrow$ the representation ϕ_p of H is reducible.

(c) In view of parts (a) and (b) it suffices to show that the representation ϕ_p of H is irreducible if and only if $p \equiv 2 \pmod 3$. If $p = 3$, ϕ_p is reducible, since in this case $\left(\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right)$ is an eigenvector for the matrix $\left(\begin{smallmatrix} 0 & -1 \\ 1 & -1 \end{smallmatrix}\right)$. Now assume that $p \neq 3$. Then Maschke's theorem implies that $\phi_p(\tau)$ is diagonalizable over the algebraic closure of $\mathbb{Z}/p\mathbb{Z}$. The eigenvalues of $\phi_p(\tau)$ are then necessarily third roots of unity, including at least one primitive third root of unity.

Thus the action of H on $(\mathbb{Z}/p\mathbb{Z})^2$ is reducible $\Leftrightarrow$ $\phi_p(\tau)$ is diagonalizable over $\mathbb{Z}/p\mathbb{Z}$ $\Leftrightarrow$ the eigenvalues of $\phi_p(\tau)$ belong to $\mathbb{Z}/p\mathbb{Z}$ $\Leftrightarrow$ $\mathbb{Z}/p\mathbb{Z}$ contains a primitive third root of unity $\Leftrightarrow$ $3 \mid p-1$.

Consequently, the representation ϕ_p irreducible if and only if $p \equiv 2 \pmod 3$. □

Appendix A. Inner actions on division algebras

In this appendix we continue to assume that k is an algebraically closed base field of characteristic zero, and that every division algebra is finite-dimensional over its center, which in turn is a finitely generated field extension of k. (Some of the lemmas below hold in greater generality; see Remark A.5.) Our main result is the following theorem.

A.1. Theorem. *Let G be an algebraic group acting on a division algebra D of degree n by inner automorphisms. Then the kernel N of this action contains the connected component G^0 of G, and G/N is a finite abelian n-torsion group.*

Here the algebraic group G is treated as an abstract group; in particular, the (inner) action of G on D is not assumed to be algebraic or geometric. Consequently, our proof has a rather different flavor from the other arguments in this paper. Instead of using algebraic geometry, we exploit, in the spirit of [RV_1], the fact that connected algebraic groups are generated, as abstract groups, by their divisible subgroups. Note that the special case of Theorem A.1, where G is a torus is proved in [RV_1, Corollary 5.6].

Before we prove Theorem A.1, we deduce an easy consequence.

A.2. Corollary. *Let G be an algebraic group acting faithfully and geometrically on a division algebra D of degree n. Then the normal subgroup of G acting by inner automorphisms is a finite abelian n-torsion group.*

Proof. Since G acts geometrically, the normal subgroup H of G consisting of the elements acting by inner automorphisms (i.e., acting trivially on $Z(D)$) is closed, so itself an algebraic group. Now apply Theorem A.1 to the faithful action of H on D. □

We now turn to the proof of Theorem A.1, beginning with two lemmas.

A.3. Lemma. *The group of inner automorphisms of a division algebra contains no divisible subgroups.*

Proof. Assume to the contrary that there is a non-trivial divisible group H acting faithfully on a division algebra D by inner automorphisms. By [RV_1, Corollary 3.2], the torsion subgroup of H acts trivially on D, so it must be trivial. Hence H is a torsion-free divisible group, i.e., a direct sum of copies of $(\mathbb{Q}, +)$; cf. [Sc, 5.2.7]. By [RV_1, Lemma 3.3(a)], there is a subfield L of D containing the center K of D such that H embeds into L^*/K^*. Thus $(\mathbb{Q}, +)$ embeds into L^*/K^*. By [RV_1, Lemma 5.5],[5] this implies that K is not finitely generated over the algebraically closed field k, a contradiction. □

A.4. Lemma. *Let D be a division algebra of degree n whose center K contains all roots of unity.*

(a) *Suppose $x \in D$ has the following properties*: $\det(x) = 1$, *and $x^m \in K$ for some integer $m \geqslant 1$. Then $x \in K$.*
(b) *If G is a finite group acting faithfully on D by conjugation, then G is an abelian n-torsion group.*

As the statement of the lemma implies, here K is not assumed to contain an algebraically closed base field.

Proof. (a) Suppose $x^m = a$ for some $a \in K$. Taking the determinant (i.e., reduced norm) on both sides, we obtain $a^n = 1$. Thus, after replacing m by mn, we may assume $x^m = 1$. Since the polynomial $f(t) = t^m - 1$ splits over K, we conclude that $x \in K$.

[5] We take the opportunity to correct an error in the proof of [RV_1, Lemma 5.5]. The third paragraph of that proof should read: "If $\pi_i \circ \phi$ is not injective, its image is a torsion group. Since $\pi \circ \phi$ is injective, $\pi_i(\phi(\mathbb{Q}))$ is not torsion for some i. Hence, for this i, $\psi = \pi_i \circ \phi$ is injective, so that $\psi(\mathbb{Q})$ is non-trivial. Thus by the argument in the previous paragraph, $\psi(\mathbb{Q})$ is not contained in K^*."

(b) Suppose $g \in G$ acts by conjugation by d_g. Then for every $g, h \in G$, the commutator $x = d_g d_h d_g^{-1} d_h^{-1}$ satisfies the conditions of part (a), where m can be taken to be the order of $ghg^{-1}h^{-1}$ in G. Thus $x \in K$ and consequently, g and h commute in G. This shows that G is abelian.

To prove that G is n-torsion, choose $g \in G$ and consider the element $x = (d_g)^n / \det(d_g)$. Once again, x satisfies the conditions of part (a), with m the order of g^n in G. Thus $x \in K$, and consequently, $g^n = 1$ in G, as claimed. □

Proof of Theorem A.1. Let S be a torus of G, or a closed subgroup which is isomorphic to $(k, +)$. We claim that $S \subseteq N$. Since S is a divisible group, so is $S/N \cap S$; cf. [Sc, 5.2.19]. Since $S/N \cap S$ acts faithfully on D, Lemma A.3 tells us that $S/N \cap S = \{1\}$, i.e., $S \subseteq N$, as claimed.

Now recall that every element $g \in G^0$ has the Jordan decomposition $g = g_s g_u$, where g_s is semisimple and g_u is unipotent; cf., e.g., [H, Theorem 15.3]. Since g_s lies in a torus of G, $g_s \in N$. Similarly, $g_u \in N$; cf., e.g., [H, Lemma 15.1C]. Thus $G^0 \subseteq N$, as claimed. The desired conclusion now follows from Lemma A.4. □

A.5. Remark. Lemmas A.3 and A.4 also hold in prime characteristic, and so does Theorem A.1, provided G is reductive (since then G^0 is generated as abstract group by the tori it contains).

Appendix B. Regular actions on prime PI-algebras

It is a consequence of Posner's theorem that every prime PI-algebra R of PI-degree n can be realized as a subalgebra of $n \times n$ matrices over some commutative domain C. Given an action of a group G on R, it is natural to ask whether one can always find such an embedding $R \hookrightarrow \mathrm{M}_n(C)$ which is G-equivariant for some action of G on $\mathrm{M}_n(C)$. We now deduce from Theorem 1.7 a rather strong affirmative answer in the case of regular actions of algebraic groups (see Definition 5.1) on prime PI-algebras. Such actions were extensively studied in [V_1,V_2].

B.1. Proposition. *Let R be a prime PI-algebra of PI-degree n, which is finitely generated as k-algebra. Let G be an algebraic group acting regularly on R. Then there is a finitely generated commutative k-algebra C which is a domain, and a regular action of G on C such that R embeds G-equivariantly into $\mathrm{M}_n \otimes_k C$. Here G acts trivially on M_n.*

In the case where G is a torus, this assertion was proved in [V_3, Corollary 9].

Proof. Let A be the total ring of fractions of R; it is a central simple algebra of degree n, and G acts algebraically on A. Note that since R is finitely generated as k-algebra, the center of A is a finitely generated field extension of k. By Theorem 1.7, there is a G-splitting field $L = k(X_0)$ for A, where X_0 is an affine G-variety, i.e., the G-action on L is algebraic; cf. Definition 5.2. This gives rise to a G-equivariant embedding $\varphi : R \to \mathrm{M}_n \otimes_k L = A'$. Hence G also acts algebraically on A', so that A' contains a unique largest subalgebra $S_{A'}$ on which G acts regularly, and which contains every subalgebra of A' on which G acts regularly. Denote by S_L the corresponding subalgebra of L. Since $S_{A'}$ contains $\mathrm{M}_n \otimes_k k$, it follows that $S_{A'} = \mathrm{M}_n \otimes_k S_L$. Since G acts regularly on $\varphi(R)$, $\varphi(R) \subseteq \mathrm{M}_n \otimes_k S_L$. Since R is finitely generated, and since G acts regularly on S_L, there is a finitely generated G-invariant subalgebra C of S_L such that $\varphi(R) \subseteq \mathrm{M}_n \otimes_k C$. □

References

[B] N. Bourbaki, Algebra II, Chapters 4–7, Translated from the French by P.M. Cohn and J. Howie, Springer, Berlin, 1990.

[H] J. Humphreys, Linear Algebraic Groups, corrected second printing, Springer, New York, 1981.

[J] N. Jacobson, Basic Algebra II, Freeman, New York, 1980.

[lB] L. Le Bruyn, The Artin–Schofield theorem and some applications, Comm. Algebra 14 (8) (1986) 1439–1455.

[LR] D. Luna, R.W. Richardson, A generalization of the Chevalley restriction theorem, Duke Math. J. 46 (3) (1979) 487–496.

[M] S. Montgomery, Fixed Rings of Finite Automorphism Groups of Associative Rings, Lecture Notes in Math., vol. 818, Springer, Berlin, 1980.

[P] V.L. Popov, Sections in invariant theory, in: The Sophus Lie Memorial Conference, Oslo, 1992, Scand. Univ. Press, Oslo, 1994, pp. 315–361.

[PV] V.L. Popov, E.B. Vinberg, Invariant theory, in: Algebraic Geometry IV, Encyclopaedia Math. Sci., vol. 55, Springer, 1994, pp. 123–284.

[Pr_1] C. Procesi, Non-commutative affine rings, Atti Accad. Naz. Lincei Mem. Cl. Sci. Fis. Mat. Natur. Sez. I (8) 8 (1967) 237–255.

[Pr_2] C. Procesi, The invariant theory of $n \times n$ matrices, Adv. Math. 19 (3) (1976) 306–381.

[Re] Z. Reichstein, On the notion of essential dimension for algebraic groups, Transform. Groups 5 (3) (2000) 265–304.

[RV_1] Z. Reichstein, N. Vonessen, Torus actions on rings, J. Algebra 170 (1994) 781–804.

[RV_2] Z. Reichstein, N. Vonessen, Rational central simple algebras, Israel J. Math. 95 (1996) 253–280.

[RV_3] Z. Reichstein, N. Vonessen, Stable affine models for algebraic group actions, J. Lie Theory 14 (2) (2004) 563–568.

[RV_4] Z. Reichstein, N. Vonessen, Polynomial identity rings as rings of functions, J. Algebra, in press. Preprint available at http://arxiv.org/math.RA/0507152.

[RV_5] Z. Reichstein, N. Vonessen, Group actions and invariants in algebras of generic matrices, Adv. Appl. Math., in press. Preprint available at http://arxiv.org/math.RA/0507548.

[RY] Z. Reichstein, B. Youssin, Conditions satisfied by characteristic polynomials in fields and division algebras, J. Pure Appl. Algebra 166 (1–2) (2002) 165–189.

[Ri] R.W. Richardson Jr., Deformations of Lie subgroups and the variation of isotropy subgroups, Acta Math. 129 (1972) 35–73.

[Ro_1] L.H. Rowen, Polynomial Identities in Ring Theory, Academic Press, New York, 1980.

[Ro_2] L.H. Rowen, Ring Theory, vol. II, Academic Press, New York, 1988.

[Sa] D.S. Sage, Group actions on central simple algebras, J. Algebra 250 (2002) 18–43.

[Sc] W.R. Scott, Group Theory, second ed., Dover, New York, 1987.

[SS] T.A. Springer, R. Steinberg, Conjugacy classes, in: Seminar on Algebraic Groups and Related Finite Groups, The Institute for Advanced Study, Princeton, NJ, 1968/69, in: Lecture Notes in Math., vol. 131, Springer, Berlin, 1970, pp. 167–266.

[St] R. Steinberg, Torsion in reductive groups, Adv. Math. 15 (1975) 63–92.

[V_1] N. Vonessen, Actions of linearly reductive groups on affine PI-algebras, Mem. Amer. Math. Soc., vol. 414, 1989.

[V_2] N. Vonessen, Actions of linearly reductive groups on PI-algebras, Trans. Amer. Math. Soc. 335 (1993) 425–442.

[V_3] N. Vonessen, Actions of solvable algebraic groups on central simple algebras, Algebr. Represent. Theory, in press. Preprint available at http://www.math.umt.edu/vonessen/publications.html.

[Vos] V.E. Voskresenskiĭ, Algebraic Groups and Their Birational Invariants, Amer. Math. Soc., Providence, RI, 1998.