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Theoretical Computer Science 321 (2004) 271–281
Theoretical Computer Science
www.elsevier.com/locate/tcs

On the least exponential growth admitting uncountably many closed permutation classes☆

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Received 26 July 2003; accepted 15 March 2004
Communicated by H. Prodinger

Abstract

We show that the least exponential growth of counting functions which admits uncountably many closed permutation classes lies between 2^n and $(2.33529\ldots)^n$.

Keywords: Permutation avoidance; Wqo; Exponential growth

1. Introduction

Let S_n be the set of $n!$ permutations of $[n] = \{1,2,\ldots,n\}$, $S = \bigcup_{n=0}^{\infty} S_n$ be the set of all finite permutations, and $\prec$ be the usual containment of permutations (defined below). It is well-known that the partial ordering $(S, \prec)$ has infinite antichains, see [11,13,16,18]. Equivalently, $(S, \prec)$ has uncountably many lower-order ideals $X \subset S$; these are called *closed permutation classes* or, for short, CPCs. In this article we want to localize the least exponential growth of the counting function $n \mapsto |X \cap S_n|$ which admits uncountably many CPCs X.

More precisely, if

$$K_\alpha = \{X : X \text{ is a CPC such that } |X \cap S_n| < \alpha^n \quad \text{for all } n > n_0\},$$

☆ ITI is supported by the project LN00A056 of the Ministry of Education of the Czech Republic.
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doi:10.1016/j.tcs.2004.03.056

what can be said about the number

$$\kappa = \inf\{\alpha > 1 : \text{the set } K_\alpha \text{ is uncountable}\}.$$

We prove the following bounds.

Theorem 1.1. *Let κ determine the least exponential growth of uncountably many CPC's, as defined above. Then*

$$2 \leqslant \kappa \leqslant 2.33529\ldots,$$

where the upper bound is the only real root of $x^5 - x^4 - 2x^3 - 2x^2 - x - 1$.

When the base α in α^n is increased, the "phase transition" from countably to uncountably many CPC's with growth majorized α^n, occurs somewhere in the interval $[2, 2.33529\ldots]$. It would be interesting to narrow it or to determine κ exactly.

In the proof of Theorem 1.1 we build on previously obtained results. In [10, Theorem 3.8] we have proved that the exponential growths of CPCs X such that $|X \cap S_n| < 2^{n-1}$ for at least one n form a discrete hierarchy α_i^n, $i = 2, 3, 4, \ldots$, where $\alpha_2 = 1.61803 \cdots < \alpha_3 < \alpha_4 < \cdots < 2$, $\alpha_i \uparrow 2$, and α_i is the largest positive real root of $x^i - x^{i-1} - \cdots - 1$. It follows from the proof, with some additional arguments from the wqo theory, that the structure of the corresponding CPCs is so restricted that each set $K_{2-\varepsilon}$ must be countable. In Section 2, we give a proof of this fact. On the other hand, Spielman and Bóna [16] constructed an infinite antichain $(R, \prec)$ such that $123 \not\prec \pi$ for every $\pi \in R$. Thus, denoting $S(123)$ the set of 123-avoiding permutations, there are uncountably many CPCs X with $X \subset S(123)$. Since $|S(123) \cap S_n| = (1/(n+1))\binom{2n}{n}$ ([14,15]...), we obtain the bound $\kappa \leqslant 4$. The enumeration of $S(123, 3214)$, due to West [20], and the infinite antichain U due to Atkinson et al. [5] give the improvement $\kappa \leqslant 2.61803\ldots$. In Section 3 we lower this further to the upper bound in Theorem 1.1.

Closed permutation classes and permutation avoidance (containment) are related to computer science mainly via sorting problems. The set of permutation π which, when inputed to some sorting device, can be sorted to the identical permutation, is often a CPC. Indeed, this was the very first motivation to introduce $\prec$ in the works of Pratt [13] and Tarjan [18]. Recent works on closed permutation classes and permutation containment with motivation in computer science (sorting, complexity of recognizing $\prec$) are, for example, [1–4,6–8].

Now we review the definition of $\prec$ and basic facts on CPCs. Further definitions will be given throughout next two sections.

For $\pi \in S_n$, n is the *length* of π and we define $|\pi| = n$. For $A, B \subset \mathbf{N} = \{1, 2, \ldots\}$ the notation $A < B$ means that $a < b$ for every $a \in A$ and $b \in B$. Interval $\{a, a+1, a+2, \ldots, b\}$, where $a, b \in \mathbf{N}$, is denoted $[a, b]$. Instead of $[1, n]$ we write $[n]$. Two m-term sequences $a_1 a_2 \ldots a_m$ and $b_1 b_2 \ldots b_m$ over $\mathbf{N}$ are *order-isomorphic* if $b_k < b_l \Leftrightarrow a_k < a_l$ for all $k, l \in [m]$. A permutation π is *contained* in another permutation ρ, written $\pi \prec \rho$, if ρ (as a sequence) has a subsequence that is order-isomorphic to π; in the opposite case ρ is π-*avoiding*. Visually, the graph of π (as a discrete function) can be obtained from that of ρ by omitting points. If $\pi \in S_n$ and $A \subset [n]$, the *restriction* $\pi|A$ is the permutation order-isomorphic to the corresponding subsequence of π. For $X \subset S$, $M(X)$ is the set of all $\prec$-minimal permutations not in X, and $S(X)$ is the set of all permutations not containing any member of X. We define $S_n(X) = S(X) \cap S_n$. For finite $X = \{\pi_1, \ldots, \pi_r\}$ we write $S(\pi_1, \ldots, \pi_r)$ and $S_n(\pi_1, \ldots, \pi_r)$ instead of $S(\{\pi_1, \ldots, \pi_r\})$ and $S_n(\{\pi_1, \ldots, \pi_r\})$. Clearly, each proper restriction of each $\pi \in M(X)$ lies in X. A set $X \subset S$ is a CPC (closed permutation class) if $\pi \prec \sigma \in X$ implies $\pi \in X$. Each $S(X)$ is a CPC and for each CPC X we have $X = S(M(X))$. Each $M(X)$ is an antichain (its elements are mutually incomparable by $\prec$) and for each antichain $X \subset S$ we have $X = M(S(X))$. Thus the mapping $X \mapsto M(X)$, with the inverse $X \mapsto S(X)$, is a bijection between the set of all CPC's and the set of all antichains of permutations.

2. The lower bound of Theorem 1.1

In this section we mostly follow the notation of [10]. A permutation σ is *alternating* if $\sigma(\{1,3,5,\ldots\}) > \sigma(\{2,4,6,\ldots\})$. For $\pi \in S$ we let $\mathrm{al}(\pi)$ be the maximum length of an alternating permutation σ such that $\sigma \prec \pi$ or $\sigma \prec \pi^{-1}$. For a set of permutations X we denote $\mathrm{al}(X) = \max\{\mathrm{al}(\pi) : \pi \in X\}$.

Lemma 2.1. *If X is a CPC with* $\mathrm{al}(X) = \infty$, *then* $|X \cap S_n| \geqslant 2^{n-1}$ *for every* $n \in \mathbf{N}$.

Proof. We suppose that X contains arbitrarily long alternating permutations; the other case with inverses is treated similarly. Using the closeness of X and the pigeonhole principle, we deduce that either for every $n \in \mathbf{N}$ there is an alternating $\pi \in X \cap S_n$ such that $\pi(1) < \pi(i)$ for every odd $i \in [2, n]$ or for every odd $n \in \mathbf{N}$ there is an alternating $\pi \in X \cap S_n$ such that $\pi(n) < \pi(i)$ for every odd $i \in [n-1]$. We assume that the former case occurs, the latter one is similar. It follows that for every $n \in \mathbf{N}$ and every subset $A \subset [2, n]$ there is a permutation $\pi_A \in X \cap S_n$ such that $\pi_A(i) < \pi_A(1) \Leftrightarrow i \in A$. For distinct subsets A we get distinct permutations π_A and $|X \cap S_n| \geqslant 2^{n-1}$. $\square$

If $\sigma \in S_n$ and $\tau \in S_m$, then $\pi = \sigma \oplus \tau \in S_{n+m}$ is the permutation defined by $\pi(i) = \sigma(i)$ for $i \in [n]$ and $\pi(i) = n + \tau(i-n)$ for $i \in [n+1, n+m]$. Similarly, $\pi = \sigma \ominus \tau$ is defined by $\pi(i) = m + \sigma(i)$ for $i \in [n]$ and $\pi(i) = \tau(i-n)$ for $i \in [n+1, n+m]$. Note that if $\sigma' \prec \sigma$ and $\tau' \prec \tau$, then $\sigma' \oplus \tau' \prec \sigma \oplus \tau$; similarly for $\ominus$. If $\pi \in S$ has no decomposition $\pi = \sigma \oplus \tau$ for any nonempty σ and τ, we say that π is up-indecomposable. The subset of up-indecomposable permutations in S_k is denoted Ind_k^+. Each $\pi \in S$ has a unique up-decomposition $\pi = \sigma_1 \oplus \sigma_2 \oplus \cdots \oplus \sigma_k$ where each σ_i is up-indecomposable; σ_i's are called *up-blocks*. The maximum size of an up-block in the up-decomposition of π is denoted $h^+(\pi)$. For the operation $\ominus$, the down-(in)decomposability, sets Ind_k^-, down-decompositions, down-blocks, and function $h^-(\cdot)$ are defined in an analogous way.

The proof of the next lemma is left to the reader as an exercise (or see [10, Lemma 3.7]).

Lemma 2.2. *For every* $\pi \in \mathrm{Ind}_n^+$, $n > 1$, *there is a* $\sigma \in \mathrm{Ind}_{n-1}^+$ *such that* $\sigma \prec \pi$. *The same holds for down-indecomposable permutations.*

Lemma 2.3. *If X is a CPC with the property that for every $k \in \mathbf{N}$ there is a permutation $\sigma \in \mathrm{Ind}_k^+$ such that $\sigma \oplus \sigma \oplus \cdots \oplus \sigma \in X$ (k summands), then $|X \cap S_n| \geqslant 2^{n-1}$ for every $n \in \mathbf{N}$. An analogous result holds for down-decompositions.*

Proof. Using the assumption and Lemma 2.2, we obtain that for every $n \in \mathbf{N}$ there is a set $\Sigma = \{\sigma_1, \sigma_2, \ldots, \sigma_n\}$ such that $\sigma_i \in \mathrm{Ind}_i^+$ and every permutation of the form $\pi = \rho_1 \oplus \rho_2 \oplus \cdots \oplus \rho_r$, where $\rho_i \in \Sigma$ and $r \leqslant n$, is in X. Since the up-decomposition uniquely determines π, there are exactly 2^{n-1} such permutations π in $X \cap S_n$ (as compositions of n) and $|X \cap S_n| \geqslant 2^{n-1}$. □

Let $H_k^+ = \{\pi \in S : h^+(\pi) < k\}$ and similarly for H_k^-. For $k \in \mathbf{N}$ and $\pi \in S_n$, we let $s_k(\pi)$ be the number r of intervals $I_1 < I_2 < \cdots < I_r$ in this unique decomposition of $[n] : I_1$ is the longest initial interval in $[n]$ such that $\pi | I_1 \in H_k^+ \cup H_k^-$, I_2 is the longest following interval such that $\pi | I_2 \in H_k^+ \cup H_k^-$ and so on. We call $I_1 < I_2 < \cdots < I_r$ the *k-decomposition* of π. Note that each restriction $\pi | I_i$ has up-decomposition or down-decomposition composed of blocks of lengths at most $k-1$ and that each restriction $\pi | I_i \cup I_{i+1}$ contains both an element from Ind_k^+ and an element from Ind_k^-. For $k \in \mathbf{N}$ and X a set of permutations we define $s_k(X) = \max\{s_k(\pi) : \pi \in X\}$. We let $s_1(\pi) = s_1(X) = \infty$ for every permutation π and set X.

Proposition 2.4. *If X is a CPC such that $|X \cap S_n| < 2^{n-1}$ for some $n \in \mathbf{N}$, then* $\mathrm{al}(X) < \infty$ *and, for some $k \in \mathbf{N}$, $s_k(X) < \infty$.*

Proof. If $\mathrm{al}(X) = \infty$, we have $|X \cap S_n| \geqslant 2^{n-1}$ for all $n \in \mathbf{N}$ by Lemma 2.1, which is a contradiction. Suppose that $s_k(X) = \infty$ for every $k \in \mathbf{N}$. By the remark after the definition of $s_k(\cdot)$, the pigeonhole principle and the closeness of X, for every $k \geqslant 2$ there are permutations $\sigma_k \in \mathrm{Ind}_k^+$, $\tau_k \in \mathrm{Ind}_k^-$ and $\pi_k \in X \cap S_r$, $k^2 \leqslant r \leqslant 2k^2$, with the property that $[r]$ can be decomposed into k intervals $I_{k,1} < I_{k,2} < \cdots < I_{k,k}$, $k \leqslant |I_{k,i}| \leqslant 2k$, so that each of the k restrictions $\pi_k | I_{k,i}$ contains both σ_k and τ_k. For $k \in \mathbf{N}$ and $1 \leqslant i \leqslant k$, we consider the interval

$$J_{k,i} = [\min\ \pi_k(I_{k,i}), \max\ \pi_k(I_{k,i})].$$

Using the Ramsey theorem and Lemma 2.2, we may assume that either for every $k \in \mathbf{N}$ the k intervals $J_{k,1}, \ldots, J_{k,k}$ intersect each other or for every $k \in \mathbf{N}$ these k intervals are mutually disjoint. In the former case, they must always have one point in common, and it follows that $\mathrm{al}(X) = \infty$. We have again the contradiction by Lemma 2.1. In the latter case, using again Ramsey theorem (or Erdős–Szekeres theorem) and Lemma 2.2, we may assume that either for every $k \in \mathbf{N}$ we have $J_{k,1} < J_{k,2} < \cdots < J_{k,k}$ or for every $k \in \mathbf{N}$ we have $J_{k,1} > J_{k,2} > \cdots > J_{k,k}$. Then for every $k \in \mathbf{N}$ we have $\sigma_k \oplus \sigma_k \oplus \cdots \oplus \sigma_k \in X$ (k summands) or for every $k \in \mathbf{N}$ we have $\tau_k \ominus \tau_k \ominus \cdots \ominus \tau_k \in X$ (k summands). By Lemma 2.3, we get the contradiction that $|X \cap S_n| \geqslant 2^{n-1}$ for all $n \in \mathbf{N}$. □

Every bijection $f : X \to Y$, where $X = \{x_1 < x_2 < \cdots < x_n\}$ and $Y = \{y_1 < y_2 < \cdots < y_n\}$ are subsets of $\mathbf{N}$, defines a unique $\pi \in S_n$ *order-isomorphic* to f: $\pi(i) = j \Leftrightarrow f(x_i) = y_j$. An interval in X is a subset of the form $\{x_i, x_{i+1}, \ldots, x_j\}$, $1 \leqslant i \leqslant j \leqslant n$.

Lemma 2.5. *Let* $X, Y \subset \mathbf{N}$ *be two* n*-element subsets,* $f : X \to Y$ *be a bijection, and* $\pi \in S_n$ *be order-isomorphic to* f*. Suppose* $\pi \in H_k^+ \cup H_k^-$*. Then every interval partition* $J_1 < J_2 < \cdots < J_r$ *of* X *can be refined by an interval partition* $I_1 < I_2 < \cdots < I_s$ *such that* $s \leqslant r + (k-1)(r-1)$ *and each image* $f(I_i)$ *is an interval in* Y*. Similarly, every partition of* Y *in* r *intervals can be refined by a partition in at most* $r + (k-1)(r-1)$ *intervals which under* f^{-1} *map to intervals in* X*.*

Proof. It suffices to prove only the first part because $\pi \in H_k^+ \cup H_k^-$ implies that $\pi^{-1} \in H_k^+ \cup H_k^-$. Without loss of generality we can assume that $X = Y = [n]$ and $f = \pi$. Let $\pi \in S_n \cap H_k^+$ (the case with H_k^- is similar) and $J_1 < J_2 < \cdots < J_r$ be an interval partition of $[n]$. We call an up-block in the up-decomposition $\pi = \sigma_1 \oplus \sigma_2 \oplus \cdots \oplus \sigma_t$ intact if its domain lies completely in some J_i and we call it split otherwise. Clearly, there are at most r maximal runs of intact up-blocks and at most $r-1$ split up-blocks. We partition $[n]$ in the intervals $I_1 < I_2 < \cdots < I_s$ so that each I_i is either the domain of a maximal run or a singleton in the domain of a split up-block. Since $|\sigma_i| < k$ for each i, we have $s \leqslant r + (k-1)(r-1)$. This is a refinement of the original interval partition and $\pi(I_i)$ is an interval for every i. □

We will need a continuity property of the functions $\mathrm{al}(\cdot)$ and $s_k(\cdot)$.

Lemma 2.6. *Let* $\sigma \in S_n$, $\tau \in S_{n+1}$, *and* $\sigma \prec \tau$. *Then* $\mathrm{al}(\tau) \leqslant \mathrm{al}(\sigma) + 2$ *and, for every* $k \in \mathbf{N}$, $s_k(\tau) \leqslant s_k(\sigma) + 2$.

Proof. Let $\rho \in S$ be alternating, $|\rho| = \mathrm{al}(\tau)$, and $\rho \prec \tau$ (the case $\rho \prec \tau^{-1}$ is similar). The permutation σ arises by deleting one point from the graph of τ. If this point does not lie in the embedding of ρ in τ, we have $\rho \prec \sigma$ and $\mathrm{al}(\sigma) \geqslant \mathrm{al}(\tau)$. If it does, we can delete one more point from the graph of ρ so that the resulting ρ' is alternating. But $\rho' \prec \sigma$ and $|\rho'| = |\rho| - 2$, so $\mathrm{al}(\sigma) \geqslant \mathrm{al}(\tau) - 2$.

Let $k \geqslant 2$ be given, $\pi \in S_n$ be arbitrary, and $I_1 < I_2 < \cdots < I_s$ be any decomposition of $[n]$ into s intervals satisfying, for every $i = 1, \ldots, s$, $\pi | I_i \in H_k^+ \cup H_k^-$; this can be called a weak k-decomposition of π. We claim that $s_k(\pi) \leqslant s$. This follows from the observation that each interval of the k-decomposition of π must contain the last element of some I_i. Now τ arises by inserting a new point p in the graph of σ. The domain $\{p_0\}$ of p is inserted in an interval J_j of the k-decomposition $J_1 < J_2 < \cdots < J_r$ of σ and splits it into three intervals J_j', $\{p_0\}$, and J_j'' (J_j' or J_j'' may be empty). Replacing J_j by J_j', $\{p_0\}$, and J_j'', we get a weak k-decomposition of τ with at most $r+2$ intervals. Thus $s_k(\tau) \leqslant r + 2 = s_k(\sigma) + 2$. □

Recall that a partial ordering $(Q, \leqslant_Q)$ is a *well partial ordering*, briefly wpo, if it has no infinite strictly descending chains and no infinite antichains. The first condition is in $(S, \prec)$ satisfied but the second one is not and therefore $(S, \prec)$ is not a wpo. Let $(Q, \leqslant_Q)$ be a partial ordering. The set $\mathrm{Seq}(Q)$ of all finite tuples $(q_1, q_2, \ldots, q_m)$ of elements from Q is partially ordered by the derived Higman ordering $\leqslant_H : (q_1, q_2, \ldots, q_m) \leqslant_H (r_1, r_2, \ldots, r_n) \Leftrightarrow$ there is an increasing mapping $f : [m] \to [n]$ such that $q_i \leqslant_Q r_{f(i)}$ for every $i \in [m]$. For the proof of the following theorem see [9] or [12].

Theorem 2.7 (Higman [9]). *If* $(Q, \leqslant_Q)$ *is a wpo then* $(\mathrm{Seq}(Q), \leqslant_H)$ *is a wpo as well.*

If $\sigma \in S_m$ and $\tau_i \in S_{n_i}$, $i = 1, \ldots, m$, the permutation $\pi = \sigma[\tau_1, \ldots, \tau_m] \in S_{n_1+\cdots+n_m}$ is defined, for $i \in [n_1 + \cdots + n_m]$ and setting $k = \max(\{j : n_1 + \cdots + n_j < i\} \cup \{0\})$ and $n_0 = 0$, by

$$\pi(i) = n_0 + n_1 + \cdots + n_k + \tau_{k+1}\ (i - n_0 - n_1 - \cdots - n_k).$$

Visually, for $i = 1, \ldots, m$ the ith point (counted from the left) in the graph of σ is replaced by a downsized copy of the graph of τ_i; the copies are small enough not to interfere horizontally and vertically each with the other. This operation generalizes $\oplus$ and $\ominus$: $\sigma \oplus \tau = 12[\sigma, \tau]$ and $\sigma \ominus \tau = 21[\sigma, \tau]$. If $\tau_i' \prec \tau_i$, $i = 1, \ldots, m$, then $\sigma[\tau_1', \ldots, \tau_m'] \prec \sigma[\tau_1, \ldots, \tau_m]$. If P and Q are sets of permutations, we define

$$P[Q] = \{\pi[\sigma_1, \ldots, \sigma_m] : m \in \mathbf{N}, \pi \in P \cap S_m, \sigma_i \in Q\}.$$

The next lemma is an immediate consequence of Higman's theorem or of the easier result that the Cartesian product of two wpo's also is a wpo.

Lemma 2.8. *Let* P *and* Q *be sets of permutations such that* P *is finite and* $(Q, \prec)$ *is a wpo. Then* $(P[Q], \prec)$ *is a wpo.*

Let $\pi \in S_n$ and $J_1 < J_2 < \cdots < J_r$ be an interval partition of $[n]$. Observe that if each image $\pi(J_i)$ is also an interval, then there is a permutations $\sigma \in S_r$ such that $\pi = \sigma[\pi|J_1, \ldots, \pi|J_r]$.

Lemma 2.9. *For every fixed* $k, K \in \mathbf{N}$ *there is a finite set of permutations* P *such that*

$$\{\pi \in S : \mathrm{al}(\pi) < K \ \&\ s_k(\pi) < K\} \subset P[H_k^+ \cup H_k^-].$$

Proof. We show that

$$P = S_1 \cup S_2 \cup \cdots \cup S_{kK^*}$$

works where $K^* = (K-1)\binom{K}{2} + 1$. Let $\pi \in S_n$ satisfy $\mathrm{al}(\pi) < K$ and $s_k(\pi) < K$. Since $s_k(\pi) < K$, $[n]$ can be partitioned in r intervals $J_1 < J_2 < \cdots < J_r$, $r < K$, so that always $\pi|J_i \in H_k^+ \cup H_k^-$ (we will not need the other property of k-decomposition of π). We show that $[n]$ can be partitioned in at most kK^* intervals so that their images under π^{-1} are intervals refining $J_1 < J_2 < \cdots < J_r$. Then we are done because $\pi|I \in H_k^+ \cup H_k^-$ for every interval (in fact, every subset) $I \subset J_i$.

We consider two words u and u' over $[K]$. The word $u = a_1 a_2 \ldots a_n$ is defined by $a_i = j \Leftrightarrow \pi^{-1}(i) \in J_j$ and u' arises from u by contracting each maximal run of one letter in one element. For example, if $u = 2221331111$ then $u' = 2131$. Let l be the length of u' which is also the number of maximal runs in u. Clearly, u' has no two consecutive identical letters. Since $\mathrm{al}(\pi) < K$, u and u' have no alternating subsequence $\ldots a \ldots b \ldots a \ldots b \ldots$, $a \neq b$, of length $K + 1$. A pigeonhole argument implies that $l \leqslant K^* = (K-1)\binom{K}{2} + 1$.

We partition $[n]$ in l intervals $L_1 < L_2 < \cdots < L_l$ according to the maximal runs in u. Each $\pi^{-1}(L_i)$ is a subset of some J_j but in general is not an interval. Let $j \in [r]$ and $M_j \subset [n]$ be the union of i_j intervals L_i corresponding to all i_j maximal runs of j in u; $\pi^{-1}(M_j) = J_j$. Applying Lemma 2.5 to the restricted mapping $\pi : J_j \to M_j$ and to the partition of M_j into i_j intervals L_i, we can refine the partition by at most $i_j + (k-1)(i_j - 1)$ intervals in M_j (but they are also intervals in $[n]$) whose images by π^{-1} are intervals in J_j (and so in $[n]$). Taking all these refinements for $j = 1, 2, \ldots, r$, we get a partition of $[n]$ in at most $\sum_{j=1}^{r} (i_j + (k-1)(i_j - 1)) < \sum_{j=1}^{r} k i_j = kl \leqslant kK^*$ intervals whose images by π^{-1} are intervals in $[n]$ refining the partition $J_1 < J_2 < \cdots < J_r$. □

Proposition 2.10. *For every fixed $k, K \in \mathbf{N}$, the set*

$$\{\pi \in S : \mathrm{al}(\pi) < K \ \& \ s_k(\pi) < K\}$$

is a wpo with respect to $\prec$.

Proof. In view of Lemmas 2.8 and 2.9, it suffices to show that $(H_k^+ \cup H_k^-, \prec)$ is a wpo. It is enough to show that $(H_k^+, \prec)$ is a wpo. Using k-decompositions, we represent each $\pi \in H_k^+$ by a word over $\Sigma = \mathrm{Ind}_1^+ \cup \cdots \cup \mathrm{Ind}_{k-1}^+$. Now, denoting $\leqslant_s$ the ordering by subsequence, it follows from Theorem 2.7 that $(\Sigma^*, \leqslant_s)$ is a wpo and this implies that $(H_k^+, \prec)$ is a wpo. □

Proposition 2.11. *For every $0 < \varepsilon \leqslant 1$, the set $K_{2-\varepsilon}$ is countable.*

Proof. Let an ε, $0 < \varepsilon \leqslant 1$, and a CPC $X \in K_{2-\varepsilon}$ be given. It suffices to show that the antichain of permutations $M(X)$ is finite. We have $|X \cap S_n| < 2^{n-1}$ for some $n > 1$ and, by Proposition 2.4, $\mathrm{al}(X) < K$ and $s_k(X) < K$ for some constants $k, K \in \mathbf{N}$. By Lemma 2.6, $\mathrm{al}(M(X)) < K + 2$ and $s_k(M(X)) < K + 2$. By Proposition 2.10, $M(X)$ is finite. □

This finishes the proof of the inequality $\kappa \geqslant 2$. In fact, we have proved that the set

$$\{X : X \text{ is a CPC such that } |X \cap S_n| < 2^{n-1} \text{ for some } n \in \mathbf{N}\}$$

is countable. It is likely that K_2 is countable.

3. The upper bound of Theorem 1.1

Atkinson et al. [5] introduced an infinite antichain of permutations

$$U = \{\mu_7, \mu_9, \mu_{11}, \ldots\},$$

where

$$\mu_7 = 4, 7, 6 | 1, 5, 3, 2;$$
$$\mu_9 = 6, 9, 8 | 4, 7 | 1, 5, 3, 2;$$

$$\mu_{11} = 8, 11, 10|6, 9, 4, 7|1, 5, 3, 2;$$
$$\vdots$$
$$\mu_{2k+5} = 2k+2, 2k+5, 2k+4|2k, 2k+3, 2k-2, 2k+1, \ldots, 6, 9, 4, 7|1, 5, 3, 2$$
$$\vdots$$

The initial segment in μ_{2k+5} is $2k+2, 2k+5, 2k+4$, the final segment is $1,5,3,2$, and in the middle segment the sequences $2k, 2k-2, \ldots, 4$ and $2k+3, 2k+1, \ldots, 7$ are interleaved. (In fact, we have reversed the permutations of [5].) We reprove, using a different argument than in Ref. [5], that μ_i form an antichain. We associate with $\pi \in S_n$ a graph $G(\pi)$ on the vertex set $\{(i, \pi(i)) : i \in [n]\}$, in which $(i, \pi(i))$ and $(j, \pi(j))$ are adjacent if and only if $i<j$ and $\pi(i)<\pi(j)$. It is clear that $\pi \prec \sigma$ implies $G(\pi) \leqslant_g G(\sigma)$ where $\leqslant_g$ is the subgraph relation (this holds even with the induced subgraph relation). A *double fork* F_i is the tree on i vertices, $i \geqslant 6$, that is obtained by appending pendant vertex both to the second and to the penultimate vertex of a path with $i-2$ vertices. It is easy to see that $(\{F_i : i \geqslant 6\}, \leqslant_g)$ is an antichain.

Lemma 3.1. *$(U, \prec)$ is an antichain. Moreover,*

$$(\{123, 3214, 2143, 15432\} \cup U, \prec)$$

is an antichain.

Proof. For every $i = 7, 9, 11, \ldots$, $G(\mu_i) = F_i$. Since double forks form an antichain to $\leqslant_g$, so do the permutations μ_i to $\prec$. It is clear that the four new short permutations form an antichain and none contains any μ_i. $G(123)$ is a triangle, $G(2143)$ is a quadrangle and $G(15432)$ has a vertex of degree 4, and therefore none of the three permutations is contained in any μ_i. That $3214 \not\prec \mu_i$ for every i is easily checked directly. □

Proposition 3.2. *Let $s_n = |S_n(123, 3214, 2143, 15432)|$. Then*

$$\sum_{n \geqslant 1} s_n x^n = \frac{x^5 + x^4 + x^3 + x^2 + x}{1 - x - 2x^2 - 2x^3 - x^4 - x^5}.$$

As $n \to \infty$, $s_n \sim c(2.33529\ldots)^n$ where $c>0$ is a constant and $2.33529\ldots$ is the only real root of $x^5 - x^4 - 2x^3 - 2x^2 - x - 1$.

Proof. We denote $S_n^* = S_n(123, 3214, 2143, 15432)$ and partition S_n^* in five sets $A_n, \ldots, E_n$ as follows. For $n \geqslant 2$ and $\pi \in S_n^*$, we let $\pi \in A_n \Leftrightarrow \pi(1) = n-1$, $\pi \in B_n \Leftrightarrow \pi(1) = n-2$, $\pi \in C_n \Leftrightarrow \pi(1) \leqslant n-3$, $\pi \in D_n \Leftrightarrow \pi(1) = n$ & $\pi(2) \geqslant n-3$, and $\pi \in E_n \Leftrightarrow \pi(1) = n$ & $\pi(2) \leqslant n-4$. We denote $|A_n| = a_n, \ldots, |E_n| = e_n$. Notice that for every $n \in \mathbf{N}$ and $\pi \in S_n^*$, $\pi^{-1}(n) \leqslant 3$. For if $\pi^{-1}(n) \geqslant 4$, the first three values of π have an ascend or all are descending, and $123 \prec \pi$ or $3214 \prec \pi$. Thus every $\sigma \in S_{n+1}^*$ arises from some $\pi \in S_n^*$ by inserting the value $n+1$ on one of the three sites: in front of the whole π (site 1), between the first two values of π (site 2) or between the second and the third value of π (site 3). We discuss the cases depending on in which set π lies.

In all five cases we can insert $n+1$ on site 1. With the exception of the case $\pi \in D_n$, we cannot insert $n+1$ on site 3 because this would give $123 \prec \sigma$ or $2143 \prec \sigma$ or $15432 \prec \sigma$. If $\pi \in C_n$, we cannot insert $n+1$ on site 2 because this would give $123 \prec \sigma$ or $15432 \prec \sigma$. One can check that there are no other restrictions on the insertion of $n+1$. Hence $\pi \in A_n$ produces two σ's, one in D_{n+1} and the other in B_{n+1}; $\pi \in B_n$ produces also two σ's, one in D_{n+1} and the other in C_{n+1}; $\pi \in C_n$ produces one σ in E_{n+1}; $\pi \in D_n$ produces three σ's, one in D_{n+1} and two in A_{n+1}; and $\pi \in E_n$ produces two σ's, one in D_{n+1} and the other in A_{n+1}. From this we obtain the recurrences $a_{n+1} = 2d_n + e_n$, $b_{n+1} = a_n$, $c_{n+1} = b_n$, $d_{n+1} = a_n + b_n + d_n + e_n$, and $e_{n+1} = c_n$.

We set $(a_1, b_1, c_1, d_1, e_1) = (0, 0, 0, 0, 1)$, which gives correctly $(a_2, b_2, c_2, d_2, e_2) = (1, 0, 0, 1, 0)$. Let $v = (0, 0, 0, 0, x)$ be the vector of initial conditions for $n = 1$ and M be the 5×5 transfer matrix

$$M = \begin{pmatrix} 0 & 0 & 0 & 2x & x \\ x & 0 & 0 & 0 & 0 \\ 0 & x & 0 & 0 & 0 \\ x & x & 0 & x & x \\ 0 & 0 & x & 0 & 0 \end{pmatrix}.$$

For the generating functions $A = \sum_{n \geqslant 1} a_n x^n, \ldots, E = \sum_{n \geqslant 1} e_n x^n$, the recurrences give relation

$$(A, B, C, D, E)^{\mathrm{T}} = (I + M + M^2 + \cdots)v^{\mathrm{T}} = (I - M)^{-1} v^{\mathrm{T}}.$$

From this, since $s_n = a_n + b_n + c_n + d_n + e_n$,

$$\sum_{n \geqslant 1} s_n x^n = A + B + C + D + E = (1, 1, 1, 1, 1)(I - M)^{-1} v^{\mathrm{T}}$$
$$= \frac{x(x^4 + x^3 + x^2 + x + 1)}{1 - x - 2x^2 - 2x^3 - x^4 - x^5}.$$

One can check that $2.33529\ldots$ is the dominant root of the reciprocal polynomial $x^5 - x^4 - 2x^3 - 2x^2 - x - 1$ of the denominator. The asymptotics of s_n follows from the standard facts on asymptotics of coefficients of rational functions. □

We obtain the recurrence $s_1 = 1$, $s_2 = 2$, $s_3 = 5$, $s_4 = 12$, $s_5 = 28$, and $s_n = s_{n-1} + 2s_{n-2} + 2s_{n-3} + s_{n-4} + s_{n-5}$ for $n \geqslant 6$. The first values of s_n are:

$$(s_n)_{n \geqslant 1} = (1, 2, 5, 12, 28, 65, 152, 355, 829, 1936, 4521, 10558, \ldots).$$

Proposition 3.3. *For every* $\varepsilon > 0$, *the set* $K_{2.33529\ldots+\varepsilon}$ *is uncountable.*

Proof. The set of CPCs

$$\{S(\{123, 3214, 2143, 15432\} \cup V) : V \subset U\}$$

is uncountable, due to Lemma 3.1 and the 1–1 correspondence between CPCs and antichains of permutations, and

$$|S_n(\{123, 3214, 2143, 15432\} \cup V)| \leqslant |S_n(123, 3214, 2143, 15432)| = s_n.$$

By Proposition 3.2 we know that for any $\varepsilon > 0$, $s_n < (2.33529\ldots + \varepsilon)^n$ for every $n > n_0$. $\square$

Thus $\kappa \leqslant 2.33529\ldots$ and the proof of Theorem 1.1 is complete. More restrictions can be added to the $\{123, 3214, 2143, 15432\}$-avoidance and the bound $\kappa \leqslant 2.33529\ldots$ can be almost surely improved but the question is by how much. It seems not very likely that one could prove this way that $\kappa \leqslant 2$.

We conclude with some comments on our choice of the four permutations 123, 3214, 2143, and 15432. By the results in [5], if $(S(\pi, \rho), \prec)$ is not a wpo, where $\pi \in S_3$, $\rho \in S_4$ and $\pi \not\prec \rho$, then (π, ρ) equals, up to obvious symmetries, to $(123, 3214)$ or $(123, 2143)$. In [5] it is also observed that $S(123, 3214, 2143) \supset U$ and so $(S(123, 3214, 2143), \prec)$ is not a wpo. We have employed one more restriction: From the 28 permutations in $S_5(123, 3214, 2143)$, only 15432 is not contained in infinitely many μ_i. The enumeration $|S_n(123)| = C_n$, where C_n is the nth Catalan number, is a classic result (see [17]); C_n have exponential growth 4^n. West [20] proved that $|S_n(123, 3214)| = |S_n(123, 2143)| = F_{2n}$ where $(F_n)_{n \geqslant 1} = (0, 1, 1, 2, 3, 5, 8, 13, \ldots)$ are Fibonacci numbers. F_{2n} grow as $((3 + \sqrt{5})/2)^n = (2.61803\ldots)^n$. Using simpler arguments than those in the proof of Proposition 3.2, we can prove that the numbers $t_n = |S_n(123, 3214, 2143)|$ follow the recurrence $t_1 = 1$, $t_2 = 2$ and $t_n = 2t_{n-1} + t_{n-2}$ for $n \geqslant 3$. Thus t_n grow as $(1 + \sqrt{2})^n = (2.41421\ldots)^n$.

In fact Murphy and Vatter [19] added four more restrictions, namely 625413, 526413, 625431, and 526431, and improved the upper bound to $\kappa \leqslant \gamma := 2.20556\ldots$ where γ is the dominant root of $x^3 - 2x^2 - 1$. They conjecture that $\kappa = \gamma$.

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