Note

\( \ast \mu \)-semirings and \( \ast \lambda \)-semirings

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Abstract

We introduce and study \( \ast \mu \)-semirings and \( \ast \lambda \)-semirings which generalize inductive \( \ast \)-semirings and weak inductive \( \ast \)-semirings, respectively. Also, we discuss the semiring of formal power series with coefficients in such a semiring and prove that the semiring of formal power series with coefficients in a weak inductive \( \ast \)-semiring \([\ast \mu \text{-semiring}, \ast \lambda \text{-semiring}, \ast \ast \lambda \text{-semiring}]\) is a weak inductive \( \ast \)-semiring \([\mu \text{-semiring}, \lambda \text{-semiring}, \ast \ast \lambda \text{-semiring}, \text{respectively}]\). This gives a positive answer to one of Ésik and Kuich’s open problems.

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1. Introduction

A semiring [5,6] \( S \) is said to be ordered if it is also equipped with a partial order such that the operations on \( S \) are monotonic. A \( \ast \)-semiring is a semiring \( S \) equipped with a star operation \( \ast : S \rightarrow S \). A ordered \( \ast \)-semiring means an ordered semiring equipped with a star operation (the star operation need not be monotonic).

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An ordered \(\ast\)-semiring is called an \emph{inductive} \(\ast\)-semiring \cite{4} if it satisfies the fixed point inequation
\[
 \quad aa^* + 1 \leq a^* \tag{1}
\]
and the fixed point induction rule
\[
 \quad ax + b \leq x \Rightarrow a^*b \leq x. \tag{2}
\]

It is also proved by Ésik and Kuich in \cite{4} that an inductive \(\ast\)-semiring satisfies
\[
 \quad aa^* + 1 = a^* \quad \text{(fixed point equation)} \tag{3},
\]
\[
 \quad a^*a + 1 = a^*, \tag{4}
\]
\[
 \quad (ab)^* = 1 + a(ba)^*b \quad \text{(product-star equation)} \tag{5},
\]
\[
 \quad (ab)^*a = a(ba)^*, \tag{6}
\]
\[
 \quad (a + b)^* = (a^*b)^*a^* \quad \text{(sum-star equation)} \tag{7}
\]
and the star operation on an inductive \(\ast\)-semiring is monotonic.

An ordered \(\ast\)-semiring is called an \emph{weak inductive} \(\ast\)-semiring \cite{4} if it satisfies the fixed point equation, the sum-star equation and the \emph{weak fixed point induction rule}
\[
 \quad ax + b = x \Rightarrow a^*b \leq x. \tag{8}
\]

In Section 2, we introduce and study \(\ast\)-\(\mu\)-semirings and \(\ast\)-\(\lambda\)-semirings which generalize inductive \(\ast\)-semirings and weak inductive \(\ast\)-semirings, respectively. In Section 3, we discuss the semiring of formal power series with coefficients in such a semiring and prove that the semiring of formal power series with coefficients in a weak inductive \(\ast\)-semiring \([\mu\text{-semiring, } \lambda\text{-semiring, } \ast\lambda\text{-semiring}]\) is also a weak inductive \(\ast\)-semiring \([\mu\text{-semiring, } \lambda\text{-semiring, } \ast\lambda\text{-semiring}, \ast\\lambda\text{-semiring}, \text{ respectively}]\). This gives a positive answer to one of Ésik and Kuich’s open problems.

### 2. \(\ast\)-\(\mu\)-semirings and \(\ast\)-\(\lambda\)-semirings

Let \(S\) be an ordered semiring. It is easily seen that every linear function \(f : x \mapsto ax + b\) over \(S\) is monotonic. For a monotonic function \(f\) on \(S\), an element \(p\) of \(S\) is called a \emph{prefixed} \([\text{fixed}]\) \emph{point} of \(f\) if \(f(p) \leq p\) \([f(p) = p]\). A prefixed \([\text{fixed}]\) point \(\mu_f\) \([\lambda_f]\) of \(f\) is called the \emph{least prefixed} \([\text{fixed}]\) \emph{point} of \(f\) if it is the least element of the set of all the prefixed \([\text{fixed}]\) points of \(f\). It is obvious that a monotonic function \(f\) has at most one least prefixed \([\text{fixed}]\) point.
Now, we introduce and study the \( \mu \)-semirings, \( \lambda \)-semirings, \( *\mu \)-semirings and \( *\lambda \)-semirings.

**Definition 1.** A \( \mu \)-semiring \([ \lambda \)-semiring] is an ordered semiring on which every linear function \( f_{a,b} : x \mapsto ax + b \) has the least prefixed [fixed] point. A \( *\mu \)-semiring \([ *\lambda \)-semiring] is a \( \mu \)-semiring \([ \lambda \)-semiring] equipped with a star operation such that \( \mu f_{a,b} [\lambda f_{a,b}] = a^*b \) for any \( a, b \in S \) and the linear function \( f_{a,b} : x \mapsto ax + b \) over \( S \).

From one of Tarski’s results in [9]: the least prefixed point of a function \( f \) is a fixed point of \( f \), it follows that the least prefixed point of a function \( f \) is the least fixed point of \( f \). Hence, we immediately have

**Lemma 2.** A \( \mu \)-semiring \([ *\mu \)-semiring] \( S \) is a \( \lambda \)-semiring \([ *\lambda \)-semiring] and \( \mu f_{a,b} = \lambda f_{a,b} [\lambda f_{a,b}] = a^*b \) for any linear function \( f_{a,b} (x) = ax + b \) over \( S \).

An ordered semiring \( S \) is called a cpo-semiring if its underlying poset is a complete poset. We will show that cpo-semirings are \( \mu \)-semirings. For this, we need the following lemma.

**Lemma 3** (Abian and Brown [1] Markowsky [7]). Let \( A \) be a cpo and \( k, p \geq 0 \). Then for any monotonic function \( f : A^{n+p} \rightarrow A^{n} \) and any \( y = (y_1, \ldots, y_p) \in A^p \), the function \( f_y \) defined by

\[
f_y : A^n \rightarrow A^n, \quad z \mapsto f(z, y)
\]

has the least prefixed point.

**Proposition 4.** A cpo-semiring \( S \) is a \( \mu \)-semiring.

**Proof.** Suppose that \( S \) is a cpo-semiring. Take \( n = 1, p = 2, A = S, f(x_1, x_2, x_3) = x_2 x_1 + x_3 \) and \( y = (a, b) \in S^2 \), respectively in Lemma 3. Then it is easily seen that \( f \) is monotonic. Also, by Lemma 3, \( f_y (x) = ax + b = f_{a,b} (x) \) has the least prefixed point. Hence, \( S \) is a \( \mu \)-semiring. \( \square \)

Inductive \( *\)-semirings are examples of \( *\mu \)-semirings. In fact, we have

**Proposition 5.** The following conditions on an ordered \( *\)-semiring \( S \) are equivalent:

(i) \( S \) is an inductive \( *\)-semiring;
(ii) \( S \) is a \( *\mu \)-semiring;
(iii) \( S \) is both a \( \mu \)-semiring and a \( *\lambda \)-semiring;
(iv) \( S \) is both a \( \mu \)-semiring and a weak inductive \( *\)-semiring.

\[2\] Note that a different notion of \( \mu \)-semiring already exists in the literature (cf. Ésik, Leiss: Greibach normal form and algebraically complete semirings, in: CSL 2002, Lecture Notes in Computer Science, vol. 2471, 135–150).
Proof. Let $S$ be an ordered $\ast$-semiring and $f_{a,b}(x) = ax + b$ a linear function over $S$.

(i) $\Rightarrow$ (ii): Suppose that (i) is true, i.e., $S$ is an inductive $\ast$-semiring. Then, by the fixed point equation (3), we have

$$f_{a,b}(a^* b) = a(a^* b) + b = (aa^* + 1)b = a^* b.$$ 

Thus, $a^* b$ is a (pre)fixed point of $f_{a,b}$. Further, it follows from the fixed point induction rule (2) that $a^* b$ is indeed the least prefixed point of $f_{a,b}$. This shows that (ii) is true.

(ii) $\Rightarrow$ (iii): Suppose that (ii) is true, i.e., $S$ is a $\ast\mu$-semiring. It follows immediately from the definition of $\ast\mu$-semirings that $S$ is a $\mu$-semiring. On the other hand, by Lemma 2, we have that $S$ is also a $\ast\lambda$-semiring. This show that (iii) is true.

(iii) $\Rightarrow$ (i): Suppose that (iii) is true, i.e., $S$ is both a $\mu$-semiring and a $\ast\lambda$-semiring. Then, by Lemma 2, it follows that $\mu_{f_{a,b}} = \lambda_{f_{a,b}} = a^* b$ for any linear function $f_{a,b}(x) = ax + b$ on $S$. That is to say, $S$ satisfies the fixed point inequation and the fixed point induction rule, i.e., $S$ is an inductive $\ast$-semiring.

(i) $\Rightarrow$ (iv): It is clear.

(iv) $\Rightarrow$ (i): Suppose that (iv) is true, i.e., $S$ is both a $\mu$-semiring and weak inductive $\ast$-semiring. Then, by Lemma 2 and the definition of the weak inductive $\ast$-semirings, it follows immediately that $\mu_{f_{a,b}} = \lambda_{f_{a,b}} = a^* b$ for any linear function $f_{a,b}(x) = ax + b$ on $S$. That is to say, $S$ satisfies the fixed point inequation and the fixed point induction rule, i.e., $S$ is an inductive $\ast$-semiring. The proof of this proposition is completed. $\square$

Weak inductive $\ast$-semirings are examples of $\ast\lambda$-semiring. In particular, we have

**Theorem 6.** The following two conditions on an ordered $\ast$-semiring $S$ with a commutative multiplication are equivalent:

(i) $S$ is a weak inductive $\ast$-semiring;

(ii) $S$ is a $\ast\lambda$-semiring.

**Proof.** (i) $\Rightarrow$ (ii): It is clear.

(ii) $\Rightarrow$ (i): Suppose that (ii) is true, i.e., $S$ is a $\ast\lambda$-semiring. Then it follows from the definition of $\ast\lambda$-semirings that $S$ satisfies the fixed point equation (3) and the weak fixed point induction rule (8). So it suffices to prove that $S$ also satisfies the sum-star equation (7). In fact, it is shown in [4] that (3) and (8) imply the product-star equation (5), Eqs. (4) and (6). Hence, all of these equations are true in the $\ast\lambda$-semiring $S$. By Eqs. (3), (5) and (6), we have

\[(a + b)(a^* b)^* a^* + 1 = a(a^* b)^* a^* + b(a^* b)^* a^* + 1 = aa^* (ba^*)^* + (ba^*)^* + 1 = aa^* (ba^*)^* + (ba^*)^* = (aa^* + 1)(ba^*)^* = (a^* (ba^*)^* = (a^* b)^* a^*.\]

Hence,

\[(a + b)^* \leq (a^* b)^* a^* \quad (9)\]
by the weak fixed point induction rule (8). Next, we prove that the reverse inequation of (9) is also true. In fact, by fixed point equation (3),

\[(a + b)(a + b)^* + 1 = a(a + b)^* + b(a + b)^* + 1 = (a + b)^*\]

and so

\[a^*(b(a + b)^* + 1) \leq (a + b)^*\]  \hspace{1cm} (10)

by the weak fixed point induction rule (8). Further, since the multiplication on \(S\) is commutative, we have

\[(a + b)(a^*(b(a + b)^* + 1)) + 1 = (a + b)(a^*b(a + b)^* + a^* + ba^*) + 1\]
\[= a^*b(a + b)(a + b)^* + a^* + a^*b\]
\[= a^*b((a + b)(a + b)^* + 1) + a^*\]
\[= a^*b(a + b)^* + a^*\]
\[= a^*(b(a + b)^* + 1).\]

Hence, by the weak fixed point induction rule (8), it follows that

\[(a + b)^* \leq a^*(b(a + b)^* + 1).\]  \hspace{1cm} (11)

From inequations (10) and (11), we immediately have

\[a^*b(a + b)^* + a^* = a^*(b(a + b)^* + 1) = (a + b)^*.\]

Hence, we have

\[(a^*b)^*a^* \leq (a + b)^*\]  \hspace{1cm} (12)

by the weak fixed point induction rule (8). Now, it follows from inequations (9) and (12) that \(S\) satisfies the sum-star equation (7). This shows that \(S\) is a weak inductive \(*\)-semiring, as required. \(\square\)

### 3. Semirings of formal power series

Suppose that \(S\) is a semiring and \(A\) is a set. A formal power series over \(A\) with coefficients in \(S\) is a function \(r : A^* \rightarrow S\),

usually denoted by

\[r = \sum_{u \in A^*} (r, u)u,\]

where \((r, u)\) is just the value of \(r\) on the word \(u\), and \(A^*\) the free monoid of all words over \(A\), including the empty word \(\varepsilon\). Equipped with the operations of pointwise sum and Cauchy product, all of the formal power series over \(A\) with coefficients in \(S\),
form a semiring, which we denote $S\langle\langle A^*\rangle\rangle$. In this semiring the two nullary operations, i.e.,

constants are the series 0 whose coefficients are all zero, and the series 1 such that the

coefficient of the empty word is 1 and all other coefficients are 0. When $S$ is an ordered

semiring, $S\langle\langle A^*\rangle\rangle$ turns into an ordered semiring by the pointwise order.

Let $S$ be a $\ast$-semiring. For any $r \in S\langle\langle A^*\rangle\rangle$ and any $u \in A^*$, if $u = \varepsilon$, define

$$(r^\ast, \varepsilon) = (r, \varepsilon)^\ast,$$

otherwise, define

$$(r^\ast, u) = (r, \varepsilon)^\ast \sum_{v w = u, v \neq \varepsilon} (r, v)(r^\ast, w).$$

This gives a star operation on $S\langle\langle A^*\rangle\rangle$.

It is known that if $S$ is an inductive $\ast$-semiring, then so is the semiring of formal power

series $S\langle\langle A^*\rangle\rangle$ [4, Theorem 57]. Similarly, for $\mu$-semirings, we have

**Theorem 7.** If $S$ is a $\mu$-semiring, then so is $S\langle\langle A^*\rangle\rangle$.

**Proof.** Suppose that $S$ is a $\mu$-semiring. For any linear function $f_{r, s} : x \mapsto rx + s$ over $S\langle\langle A^*\rangle\rangle$, we define $r_0 : A^* \to S$ recursively as follows:

(i) If $u = \varepsilon$, then

$$(r_0, \varepsilon) = \mu f_\varepsilon,$$

where $f_\varepsilon(x) = (r, \varepsilon)x + (s, \varepsilon)$ is a linear function over $S$.

(ii) If $u \neq \varepsilon$, then

$$(r_0, u) = \mu f_u,$$

where $f_u(x) = (r, \varepsilon)x + \sum_{v w = u, v \neq \varepsilon} (r, v)(r_0, w) + (s, u)$ is a linear function over $S$.

Since $S$ is a $\mu$-semiring, it is clear that $r_0$ is indeed a function from $A^*$ to $S$ and so $r_0$

is an element in $S\langle\langle A^*\rangle\rangle$. Next, we will prove that $r_0$ is just the least prefixed point of the function $f_{r, s}$.

Note first that for any $u \in A^*$ and $u \neq \varepsilon$,

$$(f_{r, s}(r_0), u) = (rr_0 + s, u)$$

$$= (rr_0, u) + (s, u)$$

$$= (r, \varepsilon)(r_0, u) + \sum_{v w = u, v \neq \varepsilon} (r, v)(r_0, w) + (s, u)$$

$$= f_u((r_0, u)) \quad \text{(by the definition of $f_u$)}$$

$$= (r_0, u) \quad \text{(by the definition of $(r_0, u)$)}.$$  

Similarly,

$$(f_{r, s}(r_0), \varepsilon) = (rr_0 + s, \varepsilon)$$

$$= (r, \varepsilon)(r_0, \varepsilon) + (s, \varepsilon)$$

$$= f_\varepsilon((r_0, \varepsilon)) \quad \text{(by the definition of $f_\varepsilon$)}$$

$$= (r_0, \varepsilon) \quad \text{(by the definition of $(r_0, \varepsilon)$)}.$$
Hence, we have
\[ f_{r_0}(r_0) = r_0 + s = r_0, \]
i.e., \( r_0 \) is a fixed point of the function \( f_{r,s} \).

In the following, we prove that \( r_0 \) is the least prefixed point of the linear function \( f_{r,s} \).

Given any prefixed point \( x_1 \) of \( f_{r,s} \). Since \( f_{r,s}(x_1) = rx_1 + s \leq x_1 \), for any \( u \in A^* \), we have
\[ (f_{r,s}(x_1), u) \leq (x_1, u). \]
That is to say, when \( u = \varepsilon \),
\[ (r, \varepsilon)(x_1, \varepsilon) + (s, \varepsilon) \leq (x_1, \varepsilon) \]
and when \( u \neq \varepsilon \)
\[ (r, \varepsilon)(x_1, u) + \sum_{v \in u, v \neq \varepsilon} (r, v)(x_1, w) + (s, u) \leq (x_1, u). \]
Thus, it follows from the definition of \( r_0 \) that for any \( u \in A^* \),
\[ (r_0, u) \leq (x_1, u). \]
This shows that \( r_0 \) is the least prefixed point of the linear function \( f_{r,s} \). Hence, \( S\langle A^* \rangle \) is a \( \mu \)-semiring, as required.

By substituting the least fixed point for the least prefixed point in the proof of Theorem 7, we can similarly prove that

**Theorem 8.** If \( S \) is a \( \lambda \)-semiring, then so is \( S\langle A^* \rangle \).

Since an inductive \( *\)-semiring \( S \) is a \( \mu \)-semiring, by Theorem 7, we have that \( S\langle A^* \rangle \) is also a \( \mu \)-semiring. Further, by Proposition 5, \( S \) is indeed a \( \ast \)-\( \mu \)-semiring. Thus, it follows that
\[ \mu_{f_\varepsilon} = (r, \varepsilon)^\ast(s, \varepsilon) \]
and
\[ \mu_{f_u} = (r, \varepsilon)^\ast \left( \sum_{v \in u, v \neq \varepsilon} (r, v)(r_0, w) + (s, u) \right). \]
On the other hand, by [4, Lemma 58], we have for all \( u \in A^* \),
\[ (r, \varepsilon)^\ast \left( \sum_{v \in u, v \neq \varepsilon} (r, v)(r^\ast s, w) + (s, u) \right) = (r^\ast s, u). \]
Hence, for any linear function \( f_{r,s} : x \mapsto rx + s \) over \( S\langle A^* \rangle \), the least prefixed point \( r_0 \) of \( f_{r,s} \) constructed in the proof of Theorem 7, coincides with \( r^\ast s \). This fact supports our construction of \( r_0 \). Moreover, we also have the following theorem.

**Theorem 9.** If \( S \) is a \( \ast\lambda \)-semiring, then so is \( S\langle A^* \rangle \).
Proof. Suppose that $S$ is a $\ast$-$\lambda$-semiring. For any linear function $f_{r,s} : x \mapsto rx + s$ over $S\langle A^* \rangle$, we define $r_0 : A^* \rightarrow S$ recursively as follows:

(i) If $u = \varepsilon$, then

$$(r_0, \varepsilon) = \lambda f_\varepsilon,$$

where $f_\varepsilon(x) = (r, \varepsilon)x + (s, \varepsilon)$ is a linear function over $S$.

(ii) If $u \neq \varepsilon$, then

$$(r_0, u) = \lambda f_u,$$

where $f_u(x) = (r, \varepsilon)x + \sum_{v \in u, v \neq \varepsilon} (r, v)(r_0, w) + (s, u)$ is a linear function over $S$. Since $S$ is a $\ast$-$\lambda$-semiring, we immediately have

$$(r_0, \varepsilon) = \lambda f_\varepsilon = (r, \varepsilon)^\ast (s, \varepsilon)$$

and

$$(r_0, u) = \lambda f_u = (r, \varepsilon)^\ast \left( \sum_{v \in u, v \neq \varepsilon} (r, v)(r_0, w) + (s, u) \right).$$

Moreover, since a $\ast$-$\lambda$-semiring is a $\lambda$-semiring, it follows from Theorem 8 that $S\langle A^* \rangle$ is a $\lambda$-semiring and $r_0$ is the least fixed point of the function $f_{r,s}$. Now, in order to prove that $S\langle A^* \rangle$ is a $\ast$-$\lambda$-semiring, we only need to show that $r_0 = r^\ast s$ for the function $f_{r,s}$. We prove this by induction on $|u|$, the length of the word $u \in A^*$. First, note that for $u = \varepsilon$, we already have

$$(r_0, \varepsilon) = \lambda f_\varepsilon = (r, \varepsilon)^\ast (s, \varepsilon).$$

Next, suppose that $u \neq \varepsilon$ and the assertion holds for every word $w \in A^*$ with $|w| < |u|$. Then, we have

$$(r_0, u) = \lambda f_u$$

$$= (r, \varepsilon)^\ast \left( \sum_{v \in u, v \neq \varepsilon} (r, v)(r_0, w) + (s, u) \right)$$

$$= (r, \varepsilon)^\ast \left( \sum_{v \in u, v \neq \varepsilon} (r, v)(r^\ast s, w) + (s, u) \right).$$

Finally, by Lemma 58 in [4], we have

$$(r_0, u) = (r^\ast s, u).$$

Hence, $r_0 = r^\ast s$ and $S$ is a $\ast$-$\lambda$-semiring, as required. □

A Conway semiring is a $\ast$-semiring satisfying the sum-star equation (7) and the product-star equation (5). It is proved that any weak inductive $\ast$-semiring is a Conway semiring in [4]. And it is shown in [2,3] that the power series semiring $S\langle A^* \rangle$ with coefficients in a Conway semiring $S$ is also a Conway semiring. Combining these results with Theorem 9, we immediately have the following result.
Theorem 10. If $S$ is a weak inductive $\ast$-semiring, then so is $S\langle\langle A^*\rangle\rangle$.

Proof. Suppose that $S$ is a weak inductive $\ast$-semiring. Then $S$ is a $\ast\lambda$-semiring satisfying the sum-star equation (7). Hence, by Theorem 9, we have that $S\langle\langle A^*\rangle\rangle$ is a $\ast\lambda$-semiring. On the other hand, since any weak inductive $\ast$-semiring is a Conway semiring, we have that $S\langle\langle A^*\rangle\rangle$ is also a Conway semiring. It follows that $S\langle\langle A^*\rangle\rangle$ is, in particular, a $\ast\lambda$-semiring satisfying the sum-equation (7). Hence, $S\langle\langle A^*\rangle\rangle$ is a weak inductive $\ast$-semiring, as required. □

The last theorem gives a positive answer to one of Ésik and Kuich’s open problems.

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