Computers Math. Applic. Vol. 18, No. 4, pp. 357–364, 1989
Printed in Great Britain.

0097-4943/89 $3.00 + 0.00

EIGENVALUES OF DISCONTINUOUS STURM–LIOUVILLE PROBLEMS WITH SYMMETRIC POTENTIALS

M. Kobayashi
IBM Research, Tokyo Research Laboratory, 5–19 Sanban-cho, Chiyoda-ku, Tokyo 102, Japan

(*Received 7 November* 1988)

Abstract—In this paper we consider three examples of discontinuous Sturm–Liouville problems with symmetric potentials. The eigenvalues of the systems were determined using the classical fourth order Runge–Kutta method. These eigenvalues are used to reconstruct the potential function using an algorithm presented in Kobayashi [1, 2]. The results of our numerical experiments are discussed.

1. A MATTHIEU EQUATION

In this section we generate the first fifteen eigenvalues of two discontinuous Sturm–Liouville systems with symmetric boundary and jump conditions, then we try to reconstruct the potential function of the second system, a Matthieu potential, using the fifteen eigenvalues and algorithm from Kobayashi [1, 2]. Begin with the Sturm–Liouville system with potential $q \equiv 0$:

System 1

$$-u'' = \lambda u,$$

with boundary conditions:

$$u'(0) = u'(\pi) = 0$$

and symmetric jump conditions:

$$u(d_1+) = au(d_1-), \quad u'(d_1+) = a^{-1}u'(d_1-) + bu(d_1-),$$
$$u(d_2-) = au(d_2+), \quad u'(d_2-) = a^{-1}u'(d_2+) - bu(d_2-),$$

where $0 \leqslant x \leqslant \pi$, the discontinuities d_1 and d_2 satisfy $0 < d_1 < \pi/2 < d_2 < \pi$ and $d_2 = \pi - d_1$, and a and b are jump constants satisfying $|a-1| + |b| > 0$. Let $\alpha = (a - a^{-1})/(a + a^{-1})$. Choose a so that $2|\alpha| + |\alpha^2| < 1$. We determine the eigenvalues for system 1 from the Volterra integral equations of u [2]. Since the potential function is identically zero, the integral terms in the Volterra equations vanish, and we are left with a simple expression for u. The IMSL subroutine ZBRENT is used to determine the zeros of $u'(\pi, \lambda)$. The eigenvalues are the points λ_i where $u_3'(\pi, \lambda_i) = 0$. A different technique is required to determine the eigenvalues of the Matthieu equation:

System 2

$$-u'' + (2\cos 2x)u = \lambda u,$$

with the boundary conditions:

$$u'(0) = u'(\pi) = 0$$

and symmetric jump conditions:

$$u(d_1+) = au(d_1-), \quad u'(d_1+) = a^{-1}u'(d_1-) + bu(d_1-),$$
$$u(d_2-) = au(d_2+), \quad u'(d_2-) = a^{-1}u'(d_2+) - bu(d_2-),$$

where a, b, d_1 and d_2 satisfy the conditions described above; $q = 2\cos 2x$ so that the integral terms in the Volterra integral equations do not vanish. We solve the system

$$\begin{pmatrix} u_j \\ u_j' \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ q - \lambda_j & 0 \end{pmatrix} \begin{pmatrix} u_j \\ u_j' \end{pmatrix},$$

with the initial conditions

$$\begin{pmatrix} u_j \\ u_j' \end{pmatrix}_{x=0} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and the jump conditions

$$\begin{pmatrix} u_j \\ u_j' \end{pmatrix}_{x=d_1+} = \begin{pmatrix} a & 0 \\ b & a^{-1} \end{pmatrix} \begin{pmatrix} u_j \\ u_j' \end{pmatrix}_{x=d_1-}$$

and

$$\begin{pmatrix} u_j \\ u_j' \end{pmatrix}_{x=d_2+} = \begin{pmatrix} a^{-1} & 0 \\ b & a \end{pmatrix} \begin{pmatrix} u_j \\ u_j' \end{pmatrix}_{x=d_2-}$$

using the classical fourth order Runge–Kutta method. The IMSL subroutine ZBRENT is used to find the points λ_i such that $u_3'(\pi, \lambda_i) = 0$. The λ_i are the eigenvalues.

To determine the fifteenth eigenvalue to eleven decimal places, a gridsize of at most $\pi/10{,}000$ should be used, where π is the length of the interval. The accuracy of the eigenvalues can be found by beginning with gridsize $\pi/20$. As the gridsize is halved we gain one to two decimal places of accuracy. In the continuous problem this rate of convergence is expected. Our experiment shows that the method is also a fourth order method in the discontinuous problem. The expression for the calculated eigenvalues using the classical fourth order method is

$$\lambda_{j\,\text{calculated}} = \lambda_{j\,\text{exact}} + C_j \cdot h^4 + \cdots$$

where h is the gridsize. We determined C_j for system 2 when $a = 1.5, b = 0.5$ and $d = \pi/20$. Calculations for the eleventh eigenvalue with varying meshsizes show that C_{11} varies between 21.99 and 22.85 for meshsizes $\pi/320$ to $\pi/20480$. From our calculations for various eigenvalues for meshsize $\pi/640$ we find that C_j is proportional to λ_j^p where $2 < p < 3$ and $p \approx 2.5$. We note that other higher order methods may be used to find the eigenvalues [3]. The Prince–Dormand Runge–Kutta order 7–8 method [4] was used by the author. Extra work is required in feeding the coefficients into the routine, and it is not clear whether there is a significant saving in computation time. In the fourth order Runge–Kutta method a very small step size is needed to achieve high

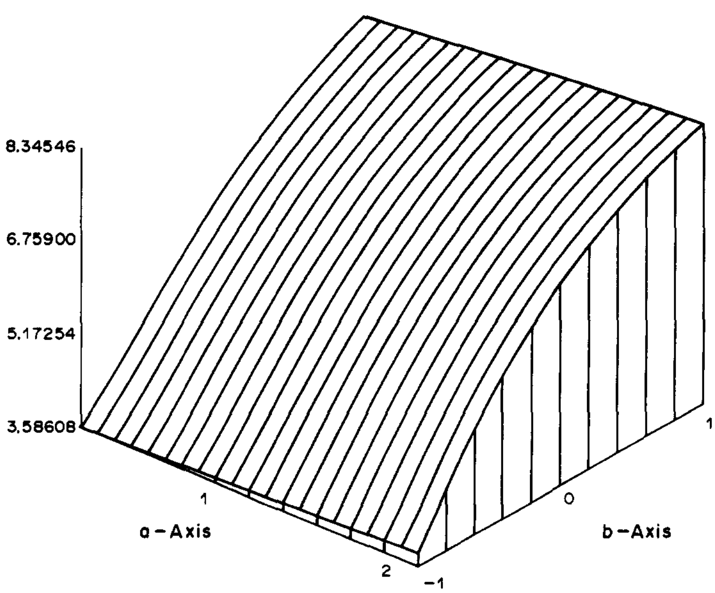

Fig. 1. Second eigenvalue of Matthieu's equation.

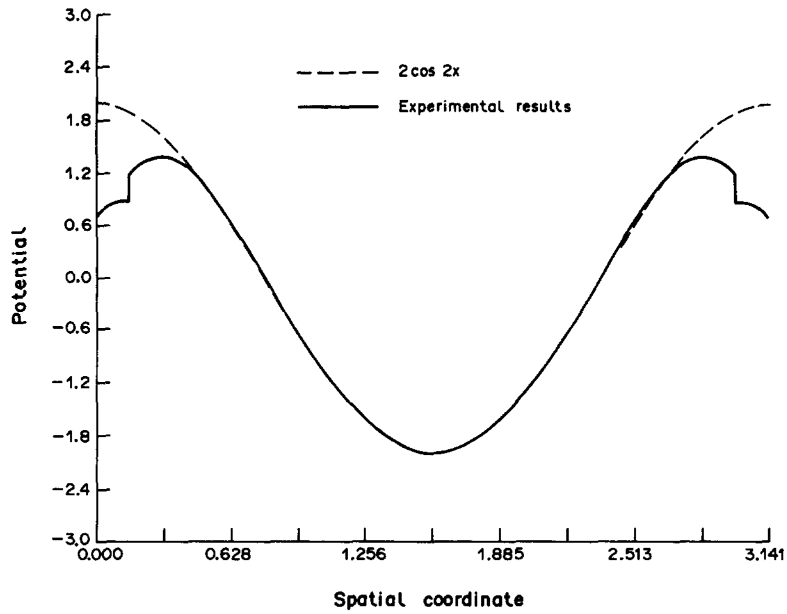

Fig. 2. Five eigenvalues, $a = 1.5$, $b = 0.5$, $d = \pi/20$.

accuracy, whereas in the Prince–Dormand routine a large number of sums and products must be computed for each step. To illustrate the dependence of the eigenvalues on the jump constants a and b, we select the second eigenvalue from System 2 and show how it varies as a and b range from 0.1 to 2.1 and -1.0 to 1.0 respectively (see Fig. 1). The graph shows that for a given value of λ_i there is an associated level set of pairs $\{(a, b)\}$, i.e. the value of λ_2 does not uniquely determine (a, b). In addition we note that the eigenvalue increases with an increase in either or both a and b.

We have tried to reconstruct the potential function $q = 2 \cos 2x$ using the eigenvalues generated by the methods outlined above and an algorithm from Kobayashi [1, 2]. The results are given in Figs 2–4. Calculated values of the jump constant b for the Matthieu system using the zeroth through the fourteenth eigenvalues are given in Fig. 5. The changes in the L_1-error, L_2-error and L_∞-error with respect to the number of eigenvalues used in the reconstruction are illustrated in

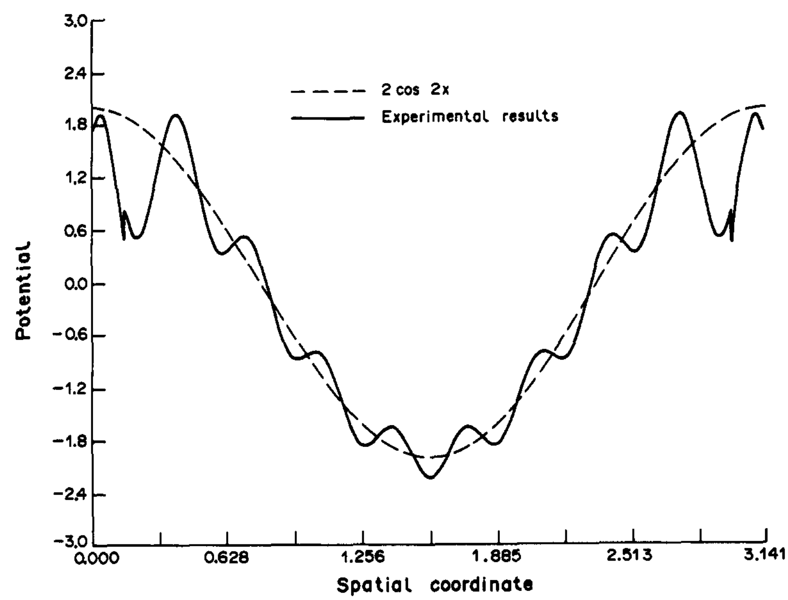

Fig. 3. Ten eigenvalues, $a = 1.5$, $b = 0.5$, $d = \pi/20$.

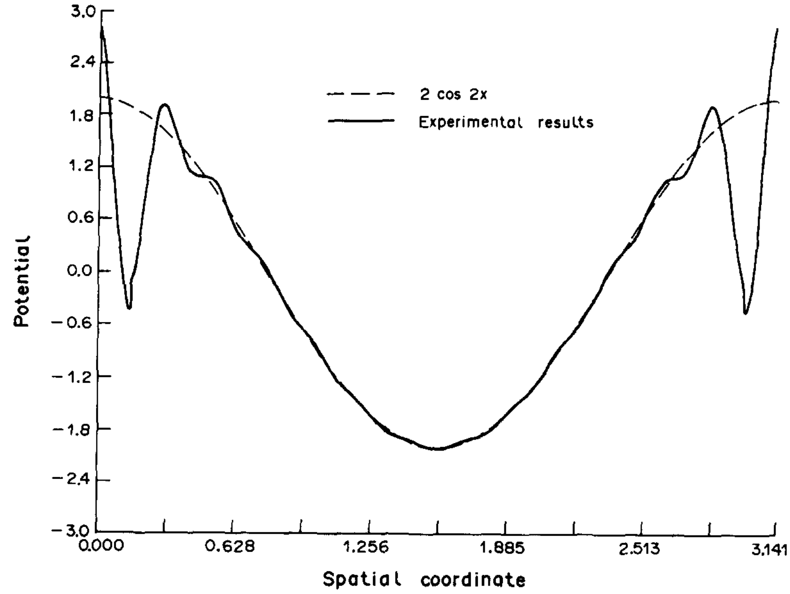

Fig. 4. Fifteen eigenvalues, $a = 1.5$, $b = 0.5$, $d = \pi/20$.

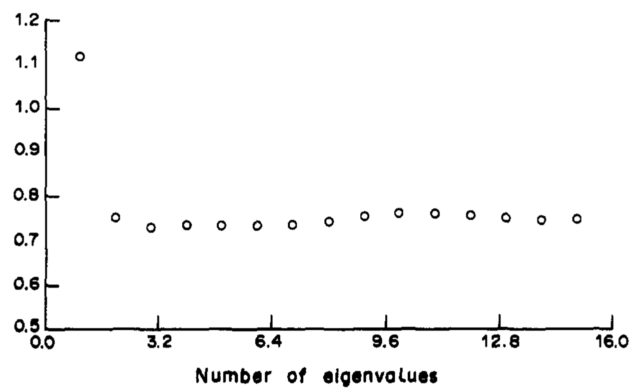

Fig. 5. New value of b ($a = 1.5$, $b = 0.5$, $d = \pi/20$).

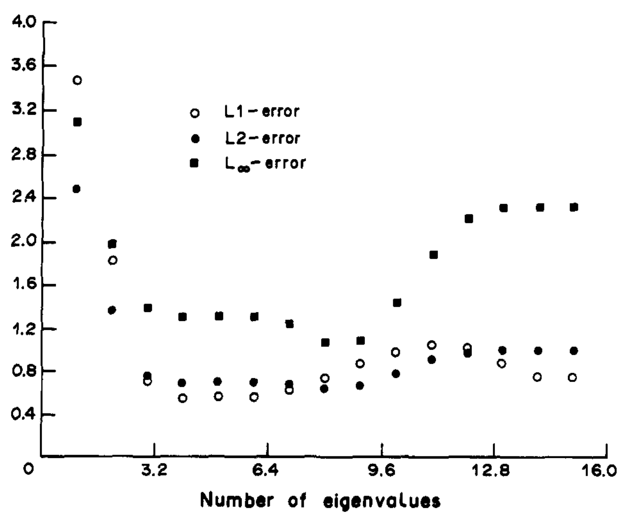

Fig. 6. Error in $Q = 2\cos 2x$ ($a = 1.5$, $b = 0.5$, $d = \pi/20$).

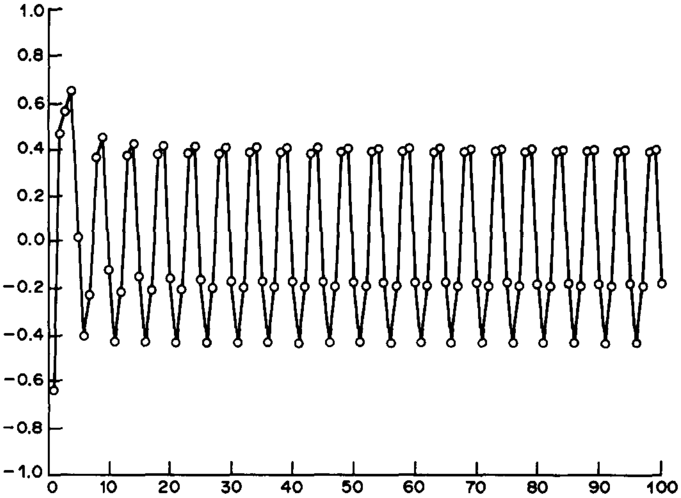

Fig. 7. $a = 1.5$, $b = -0.5$, $d = \pi/5$.

Fig. 6. Detailed numerical results can be found in Kobayashi [2]. As the number of eigenvalues used in the reconstruction increases to 4 or 5, the experimental results appear to converge toward the potential $q = 2\cos 2x$. However as we pass to 6 or more eigenvalues the experimental results begin to oscillate about $q = 2\cos 2x$. The cause for this behaviour is unclear, and several explanations have been suggested. First, the eigenvalues we use for the Matthieu system are generated assuming that $b = 0.5$. In the reconstruction algorithm we determine a new value for b. We see from Fig. 5 that the value of b does not equal 0.5 and furthermore does not approach 0.5 as higher eigenvalues are considered. Next, note that the oscillations for six or more eigenvalues appear to be related to the Gibbs phenomenon which has been observed in other similar numerical experiments [5]. Two tall, thin spikes are observed in the reconstructed potential function. The spikes occur at the points where the eigenfunctions are discontinuous and become more pronounced as the number of eigenvalues used in the reconstruction increases. Ching-ju Lee at the University of California, Berkeley first noted the discrepancy between the asymptotic behavior of the eigenvalues. She generated the first 100 eigenvalues for Systems 1 and 2 to five or six digit accuracy and calculated the difference between the corresponding eigenvalues of the two systems. This difference did not converge to zero; it oscillated in a periodic manner about zero. The locations of the discontinuities and the jump constants were varied to see how the oscillatory patterns were changed. Her results are summarized in Fig. 7. These findings show that the assumption that the eigenvalues for systems 1 and 2 are equal for $j > n$ is violated. In a follow-up experiment C.-J. Lee reconstructed the potential $q(x) = \frac{1}{2}\cos 16x$ from $q \equiv 0$. "The reconstruction algorithm turns out to be quite successful (for this choice)", she notes, "(and) compared to other pairs of (potential) functions the asymptotic difference between the spectrum for $q \equiv 0$ and the one for $q(x) = \frac{1}{2}\cos 16x$ turns out to be quite small." (Private communication.)

2. A DISCONTINUOUS POTENTIAL

In the previous section we tried to reconstruct the smooth potential $q(x) = 2\cos 2x$ from the zero potential. In this section we present an example to study how the same algorithm reconstructs a discontinuous, symmetric potential function. Consider the two systems given below:

System 3

$$-u'' = \lambda u,$$

with symmetric boundary conditions:

$$u'(0) = u'(\pi) = 0$$

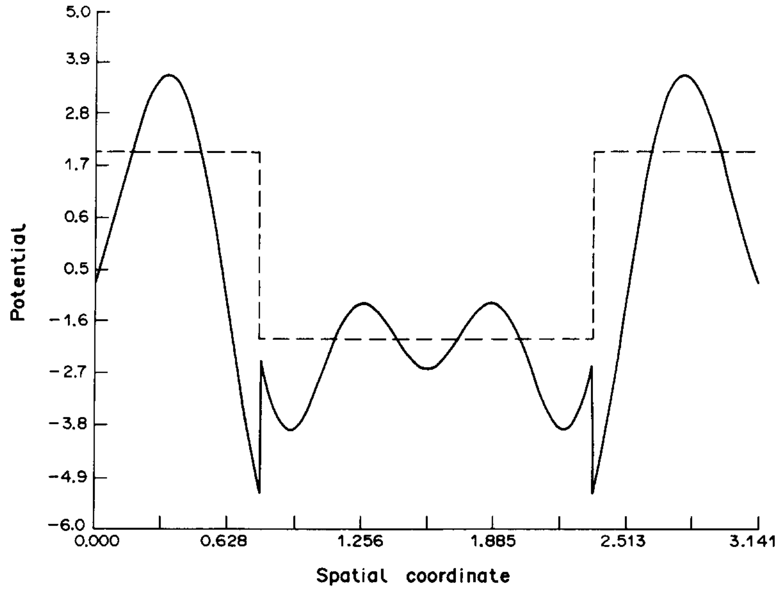

Fig. 8. Five eigenvalues, $a = 1.5$, $b = 0.5$, $d = \pi/4$.

and symmetrically located discontinuities d_1, d_2 satisfying symmetric jump conditions:

$$u(d_1+) = au(d_1-), \quad u'(d_1+) = a^{-1}u'(d_1-) + bu(d_1-),$$

$$u(d_2-) = au(d_2+), \quad u'(d_2-) = a^{-1}u'(d_2+) - bu(d_2-).$$

System 4

$$-u'' + qu = \lambda u,$$

with symmetric boundary conditions:

$$u'(0) = u'(\pi) = 0$$

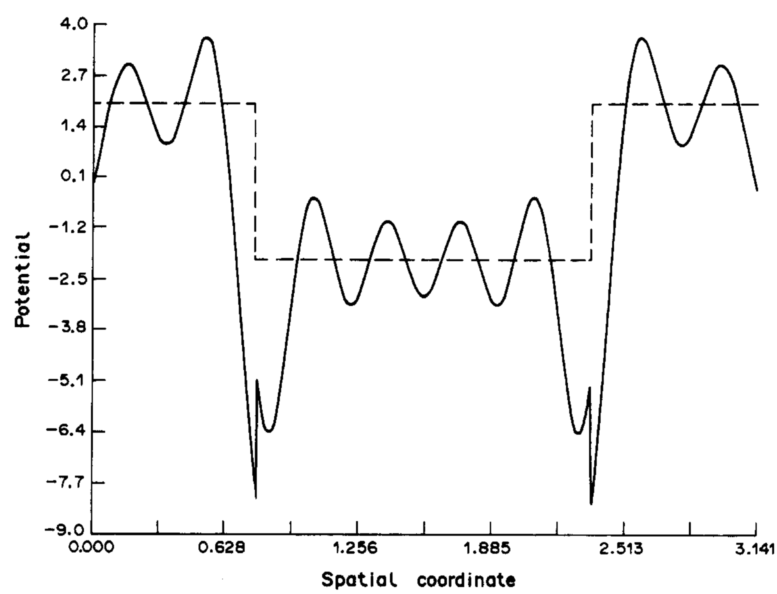

Fig. 9. Ten eigenvalues, $a = 1.5$, $b = 0.5$, $d = \pi/4$.

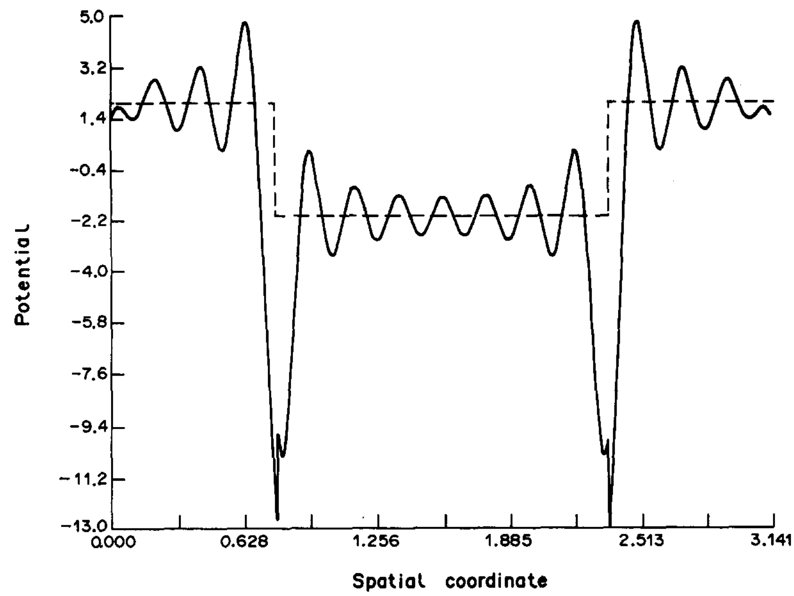

Fig. 10. Fifteen eigenvalues, $a = 1.5$, $b = 0.5$, $d = \pi/4$.

and symmetrically located discontinuities d_1, d_2 satisfying symmetric jump conditions:

$$u(d_1+) = au(d_1-), \quad u'(d_1+) = a^{-1}u'(d_1-) + bu(d_1-)$$

$$u(d_2-) = au(d_2+), \quad u'(d_2-) = a^{-1}u'(d_2+) - bu(d_2-),$$

where $0 < x < \pi$ and the potential to be reconstructed is described by

$$q = \begin{cases} -2 \text{ for } 0 \leqslant x < \pi/4 \\ +2 \text{ for } \pi/4 < x < 3\pi/4. \\ -2 \text{ for } 3\pi/4 < x \leqslant \pi. \end{cases}$$

In both systems a, b, d_1 and d_2 satisfy the conditions given in the previous section. We choose eigenfunctions with discontinuities at $x = \pi/4$ and $x = 3\pi/4$ and jump constants $a = 1.5$, $b = 0.5$. Note that the discontinuities in the eigenfunctions coincide with those of the desired potential $q(x)$. We investigate this problem to determine whether this match in the discontinuities reduces the error. First generate the eigenvalues $\{\lambda_i\}_{i=0}^{i=14}$ for Systems 3 and 4 using the methods described in the previous section. Then we use this data to reconstruct the potential q. Results from our implementation of the algorithm are given in Figs 8–10.

3. CONCLUSIONS

The eigenvalue generating techniques given in this paper are simple and can be applied to many eigenvalue systems. We have presented two examples of the implementation of a reconstruction algorithm for discontinuous, inverse Sturm–Liouville problems from Kobayashi [1, 2]. The data we used did not meet all of the specified requirements of the algorithm so that the computed result was poor in some locations and excellent in others. The information acquired from the algorithm can be considerable value in scientific and engineering problem-solving when data from a variety of sources will be used to analyze a situation.

Acknowledgements—I would like to thank Professor Ole Hald for his patient help and for being my thesis advisor. I would also like to thank Professors Chorin and Concus for making the facilities and resources at the Lawrence Berkeley Laboratory available to me. Special thanks to Bill Behrman at U.C. Berkeley for his advice on the implementation of a Prince–Dormand routine and to C.-J. Lee also at U.C. Berkeley for her eigenvalue calculations and for many helpful discussions. This work is a part of the author's Ph.D. Thesis work at the University of California, Berkeley. The author is currently a member of the Foundations of Systems Group at the IBM Tokyo Research Laboratory. This work was supported in part by the Applied Mathematical Sciences Subprogram of the Office of Energy Research, U.S. Department of Energy under contract DE-AC03-76SF00098. The revising and editing of this paper was made possible by IBM Japan.

REFERENCES

1. M. Kobayashi, An algorithm for discontinuous inverse Sturm–Liouville problems with symmetric potentials. *Computers Math. Applic.* **18**, 349–356 (1989).
2. M. Kobayashi, Discontinuous inverse Sturm–Liouville problems with symmetric potentials. Ph.D. Thesis, U.C. Berkeley (1988).
3. E. Hairer *et al.*, *Solving Ordinary Differential Equations*, Vol. 1. *Nonstiff Problems*. Springer, Berlin (1987).
4. P. J. Prince and J. R. Dormand, High order embedded Runge–Kutta formulae. *J. comp. appl. Math.* **7**, 67–75 (1981).
5. O. H. Hald, The inverse Sturm–Liouville problem and the Rayleigh Ritz method. *Math. Comput* **32**, 687–705 (1978).