A Constructive Proof of a Whitney Extension Theorem in One Variable

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We prove a new stability property of the classical Hermite interpolation scheme in one variable. This stability property refines the usual convergence property of this interpolation scheme. Consequently, we obtain a natural proof of a Whitney extension theorem in one variable. © 1992 Academic Press, Inc.

1. INTRODUCTION

Whitney has proved an extension theorem in \( \mathbb{R}^d \) which deals with Hermite interpolation ([7, 8], see also [3, 1]) and an extension theorem in \( \mathbb{R} \) which deals with Lagrange interpolation ([6], see also [2, 4]). We are interested here with the former. We shall refer to it as the Whitney–Hermite extension theorem.

First recall the Whitney–Hermite theorem for \( d = 1 \). For \( m \) a positive integer we note \( \mathbb{R}_m[x] \) the space of polynomials of degree at most \( m \) and real coefficients.

Given an open non-empty interval \( C \) of \( \mathbb{R} \) and \( f \in C^m(C, \mathbb{R}) \), we naturally associate the field \( Tf: C \to \mathbb{R}_m[x] \)

\[
a \to (Tf)_a (x) \quad \text{with } f,
\]

where \( (Tf)_a (x) := \sum_{k=0}^{m} ((x - a)^k/k! \right) f^{(k)}(a) \) is the \( m \)-Taylor expansion of \( f \) in \( a \).

Let then \( K \) be a non-empty compact subset of \( C \) and let \( T: K \to \mathbb{R}_m[x] \)

\[
a \to T_a(x)
\]

be a field of polynomials of degree at most \( m \) defined on \( K \). The
Whitney–Hermite extension theorem in one variable states that there is $f \in C^m(K)$ such that $Tf$ and $T$ coincide in restriction to $K$ if and only if a modulus of continuity $\omega$ exists such that
\[
\forall k = 0, \ldots, m, \quad \forall a, b \in K, \quad |T_a^{(k)}(a) - T_b^{(k)}(a)| \leq |b - a|^{m-k} \omega(|b - a|).
\]

When $K = X$ is finite, the Whitney–Hermite extension problem reduces clearly to a Hermite interpolation problem (the hypothesis on $\omega$ is, in this case, useless). This problem is solved by (among many others) the classical Hermite interpolant denoted $H_X$ that we briefly describe below.

For $X = \{x_0, \ldots, x_n\}$, $x_0 < x_1 < \cdots < x_n$, $H_X$ is defined by
\[
H_X(x) = T_{x_0}(x) \quad \text{if} \quad x \leq x_0
\]
\[
= T_{x_n}(x) \quad \text{if} \quad x \geq x_n
\]
\[
= \text{the unique polynomial of degree at most } 2m + 1 \text{ whose } m\text{-Taylor expansion at } x_i \text{ (resp. } x_{i+1}) \text{ coincides with } T_{x_i} \text{ (resp. } T_{x_{i+1}}) 
\]
\[
\quad \text{if} \quad x_i \leq x \leq x_{i+1}, \quad i = 0, \ldots, n - 1.
\]

In the general case ($K$ compact infinite) we prove in this article the uniform continuity of the map $H: X \to H_X$ defined on the set $\mathcal{F}(K)$ of the finite non-empty subsets of $K$.

We will then apply the classical extension theorem which allows us to extend to the whole space (for us the compact metric space $\mathcal{K}(K)$ of the non-empty compact subsets of $K$) a uniformly continuous function (for us $X \to H_X$) on a dense subset of this whole space (for us $\mathcal{F}(K)$) and then to give sense to $H_K$ which solves the Whitney–Hermite extension problem in one variable.

Thus, the above classical extension theorem extends the Hermite interpolation scheme (valid for finite sets) to an extension scheme (valid for general compact sets) which has the same property of uniform continuity. Our result strengthens, from this point of view, the Whitney–Hermite extension theorem in one variable. Moreover, it will be shown that the well-known minimisation property of the classical Hermite interpolant $H_X$ extends to $H_K$.

From a numerical point of view, the uniform continuity of $X \to H_X$ must be viewed as a stability result: if we alter the set $X$ of the abscissas of interpolation to $X^*$ close to $X$ (according to Hausdorff distance) $X$ and $X^*$ both subsets of $K$, $H_X$ is altered to $H_{X^*}$ close to $H_X$ in the appropriate metric space. Our result is thus, in the case of the classical Hermite
interpolation, a proof of a very natural stability property which we can expect from an interpolation scheme.

The notion of stability we study here should not be confused with the usual notion (where, for a fixed set of abscissas, we alter the values of the Taylor polynomials at points of \( X \)). It can be viewed as a generalisation of the classical notion of convergence of an interpolation scheme: the bounds of stability that we establish in this article contain, as particular cases, the usual bounds of convergence of the classical Hermite interpolation scheme.

It must be noted that it is easy to define \( H_K \) without using the technique of this article: since \( \mathbb{R} - K \) has the remarkable property of being a disjointed union of non-empty open intervals \( ]a, b[ \) with \( a < b, a, b \in K \) (and \( ] - \infty, \min K[, ]\max K, + \infty[ \)), we can define \( H_K \) in restriction to each of these \( ]a, b[ \) as the unique polynomial of degree at most \( 2m + 1 \) whose \( m \)-Taylor expansion in a (resp. \( b \)) coincides with \( T_a \) (resp. \( T_b \)). But, in this case, the smoothness of \( H_K \) still has to be proved. Constructive proofs (that is, involving explicit bounds in terms of the data \( \omega \)) can be found in [2] (in the slightly different context of the Whitney–Lagrange problem and with perfect splines instead of classical Hermite splines as basic interpolants).

These kinds of definitions and proofs make extensive use of the above remarkable structure of the compacts of \( \mathbb{R} \): they seem unable to be extended to \( \mathbb{R}^d \) for \( d > 1 \). On the contrary, the technique we use in this article extends naturally to \( \mathbb{R}^d \) for \( d > 1 \). Let us, indeed, assume the existence of a field \( T \)

\[
K \to \mathbb{R}_m[X_1, \ldots, X_d]
\]

\[
x \to T_x
\]
of polynomials of total degree at most \( m \) defined on a compact \( K \) of \( \mathbb{R}^d \) and satisfying the Whitney hypothesis. The knowledge of a uniformly stable interpolation scheme \( X \to I_X \) will immediately give a new proof of the Whitney–Hermite theorem in \( \mathbb{R}^d \). It must be noted that all known proofs of the Whitney–Hermite extension theorem in many variables produce extension schemes which are not uniformly stable in the above sense.

2. Precise Statements and Proofs

For \( A, B \) relatively compact non-empty subsets of \( \mathbb{R} \), we denote by \( \delta(A, B) \) the classical Hausdorff distance between \( A \) and \( B \):

\[
\delta(A, B) := \sup_{x \in A, y \in B} (d(x, B), d(A, y)),
\]
where
\[ d(x, B) := \inf_{y \in B} |y - x|, \quad d(A, y) := \inf_{x \in A} |y - x|. \]

**THEOREM.** Let \( K \) be a non-empty compact subset of \( \mathbb{R} \), \( T \) a field (defined on \( K \)) of real polynomials of degree at most \( m \) and \( \omega \) a modulus of continuity. Assume the Whitney–Hermite hypothesis
\[
\forall k = 0, \ldots, m \quad \forall a, b \in K, \quad |T^{(k)}_a(a) - T^{(k)}_b(a)| \leq |b - a|^{m-k}\omega(|b - a|).
\]

Then the previously described classical Hermite interpolation scheme \( L \to H_L \) defined on \( \mathcal{F}(K) \) has a unique extension (still denoted \( L \to H_L \)) defined on \( \mathcal{H}(K) \) with the following properties

(i) \( \forall L \in \mathcal{H}(K), H_L \in \mathcal{C}^m(\mathbb{R}). \)

(ii) \( \forall L \in \mathcal{H}(K), TH_L \) and \( T \) coincide on \( L \).

(iii) There are numerical constants \( \varphi(m, k) \) depending only on \( m \) and \( k \) such that
\[
\forall L_1, L_2 \in \mathcal{H}(K), \quad \forall k = 0, \ldots, m
\]
\[
\sup_{x \in C} |H^{(k)}_{L_1}(x) - H^{(k)}_{L_2}(x)| \leq \varphi(m, k) \sup(\delta^{m-k}(L_1, C), \delta^{m-k}(L_2, C))\omega(\delta(L_1, L_2)),
\]
where \( C \) denotes any bounded open interval containing \( K \).

**COROLLARY.** Let \([a, b]\) be a closed interval of \( \mathbb{R} \), \( f \in \mathcal{C}^m([a, b]) \), \( \omega \) the modulus of continuity of \( f^{(m)} \), \( X \in \mathcal{F}([a, b]) \), \( H \) the classical Hermite interpolant which interpolates the field \( Tf \) on \( X \).

Then, setting \( h := \delta(X, [a, b]) \), we have
\[
\sup_{x \in [a, b]} |H^{(k)}(x) - f^{(k)}(x)| \leq \varphi(m, k) h^{m-k}\omega(h), \quad k = 0, \ldots, m,
\]
where \( \varphi(m, k) \) is the constant occurring in the conclusion of the theorem.

**Proof of the Corollary.** It is clear, from the Taylor formula, that \((Tf, \omega)\) satisfies the Whitney–Hermite hypothesis. Thus, we can apply the stability property (iii) of the theorem with \( K = [a, b] \), \( T = Tf \), \( L_1 = X \), \( L_2 = K \), \( C = [a - \varepsilon, b + \varepsilon] \), where \( 0 < \varepsilon < \delta(X, K) \).

The result follows since \( \lim_{\varepsilon \to 0} \delta(X, C) = \delta(X, K) = h \).

This corollary means that the stability property of the theorem refines the usual convergence property of the classical Hermite interpolation scheme.
We now turn to the proof of the theorem.

Let \( T_a(x) = \sum_{i=0}^{m} y_a^i ((x-a)^i/i!) \) and, for \( 0 \leq k \leq m \), let \( T_{a,k}(x) \) be the truncation of \( T_a \).

\[
T_{a,k}(x) := \sum_{i=0}^{k} y_a^i \frac{(x-a)^i}{i!}.
\]

For \( a, b \in \mathbb{K} \), let \( H_{a,b} = H_{\lfloor a, b \rfloor} \).

We divide the proof into 5 lemmas. Lemma 1 gives an explicit form of \( H_{a,b} \) in terms of \( T_{a,k}, T_{b,k}, k = 0, \ldots, m \). Lemma 2 gives bounds for \( |H_{a,b}^{(k)}(c) - y_{a,k}^{(c)}| \) \( (c \in \mathbb{K}, a \leq c \leq b) \) in terms of \( \omega \) (and of the Whitney differences). Lemma 5 gives the key stability property of the theorem.

We shall assume from now on that the modulus of continuity \( \omega \) has the following extra-property (P):

(P) the function \( h \rightarrow \omega(h)/h \) is decreasing.

There is no loss of generality because it is known that for any modulus of continuity \( \omega_0 \) there is a modulus of continuity \( \omega \) such that:

1. \( \omega_0 \leq \omega \leq 2\omega_0 \)
2. \( \omega \) is concave (and thus \( h \rightarrow \omega(h)/h \) is decreasing).

**Lemma 1.** Let \( a, b \in \mathbb{R}, a < b \), and \( T_a, T_b \in \mathbb{R}_m[x] \). Then, we have the following explicit formula

\[
H_{a,b}(x) = \left( \frac{b-x}{b-a} \right)^{m+1} \sum_{k=0}^{m} \left( \frac{x-a}{b-a} \right)^k \binom{m+k}{m} T_{a,m-k}(x)
\]

\[
+ \left( \frac{x-a}{b-a} \right)^{m+1} \sum_{k=0}^{m} \left( \frac{x-b}{a-b} \right)^k \binom{m+k}{m} T_{b,m-k}(x).
\]

**Proof:** The polynomial which appears in the right-hand side of the expression is of degree at most \( 2m+1 \).

The polynomial

\[
g_a(x) = \left( \frac{b-x}{b-a} \right)^{m+1} \sum_{k=0}^{m} \left( \frac{x-a}{b-a} \right)^k \binom{m+k}{m} T_{a,m-k}(x)
\]

satisfies \( g_a^{(k)}(b) = 0 \) for \( k = 0, \ldots, m \).

Likewise,

\[
g_b(x) = \left( \frac{x-a}{b-a} \right)^{m+1} \sum_{k=0}^{m} \left( \frac{x-b}{b-a} \right)^k \binom{m+k}{m} T_{b,m-k}(x)
\]

satisfies \( g_b^{(k)}(a) = 0 \) for \( k = 0, \ldots, m \).
It is then sufficient to check that \( g^{(k)}_a(a) = y^k_a \) and \( g^{(k)}_b(b) = y^k_b \), \( k = 0, \ldots, m \). By linearity, it is sufficient to check that for \( T_a(x) = (x-a)^i/i! \) and \( T_b(x) = (x-b)^i/i! \), \( i = 0, \ldots, m \), the associated polynomials denoted by \( g_{a,i} \) and \( g_{b,i} \) satisfy

\[
g^{(k)}_a(a) = g^{(k)}_b(b) = \delta_{i,k} \quad (k = 0, \ldots, m).
\]

We have

\[
g_{a,i}(x) = \frac{(x-a)^i}{i!} \sum_{l=0}^{m-i} \binom{m}{l} \frac{(x-a)^l}{l!} \frac{(b-x)^{m+1}}{(b-a)^{m+1}}
\]

and

\[
g_{b,i}(x) = \frac{(x-b)^i}{i!} \sum_{l=0}^{m-i} \binom{m}{l} \frac{(b-x)^l}{l!} \frac{(x-a)^{m+1}}{(b-a)^{m+1}}.
\]

Clearly

\[
g^{(k)}_{a,i}(a) = g^{(k)}_{b,i}(b) = 0 \quad \text{for} \quad k < i
\]

and

\[
g^{(i)}_{a,i}(a) = g^{(i)}_{b,i}(b) = 1.
\]

For \( k > i \), using the Leibniz differentiation formula, we have, with \( N := k - i \),

\[
g^{(k)}_{a,i}(a) = \frac{m+1}{(b-a)^N} (-1)^{m+1-N} \binom{N+i}{i} \left( \sum_{l=0}^{N} (-1)^l \binom{N}{l} \frac{(m+l)!}{(m+l-N+1)!} \right)
\]

and

\[
g^{(k)}_{a,i}(b) = \frac{m+1}{(b-a)^N} \binom{N+i}{i} \left( \sum_{l=0}^{N} (-1)^l \binom{N}{l} \frac{(m+l)!}{(m+l-N+1)!} \right).
\]

Now, for \( m \geq N \), by development of \( (1-t)^N \) and differentiation, we have

\[
(t^m(1-t)^N)^{(N-1)} = \sum_{l=0}^{N} (-1)^l \binom{N}{l} \frac{(m+l)!}{(m+l-N+1)!} t^{m+l-N+1}.
\]

Since \( (1-t) \) is a factor of the left hand side, it follows that

\[
0 = \sum_{l=0}^{N} (-1)^l \binom{N}{l} \frac{(m+l)!}{(m+l-N+1)!}.
\]

This finishes the proof of Lemma 1.

In order to prove Lemmas 2, 3, 4, and 5 we systematically highlight Whitney differences and Hausdorff distance.
**Lemma 2.** Let \( a, c, b \in K, a \leq c \leq b, a < b, \) and assume the conditions of the theorem. Let \( \delta_{a,b}(c) := \inf(c - a, b - c). \)

Then there are numerical constants \( \psi(m,k) \) \((k = 0, \ldots, m)\) depending only on \( m \) and \( k \) such that

\[
|H^{(k)}_{a,b}(c) - y^{(k)}_c| \leq \psi(m,k) \delta^{m-k}_{a,b}(c) \omega(\delta_{a,b}(c)), \quad k = 0, \ldots, m.
\]

**Proof.** The polynomial \( T_c \) induces the constant field (equal to \( T_c \)) that we denote by \( U \). By the unicity of the classical Hermite interpolant, this polynomial coincides with the Hermite interpolant which interpolates \( U_a \) in \( a \) and \( U_b \) in \( b \).

Thus, from Lemma 1

\[
T_c(x) = \left( \frac{b-x}{b-a} \right)^{m+1} \sum_{k=0}^{m} \frac{(m+k)}{m} \left( \frac{x-a}{b-a} \right)^k U_{a,m-k}(x)
+ \left( \frac{x-a}{b-a} \right)^{m-1} \sum_{k=0}^{m} \frac{(m+k)}{m} \left( \frac{b-x}{b-a} \right)^k U_{b,m-k}(x).
\]

Therefore, by difference

\[
H^{(l)}_{a,b}(c) - y^{(l)}_c = \left[ \sum_{k=0}^{m} \frac{(m+k)}{m} \left( \frac{x-a}{b-a} \right)^k \left( \frac{b-x}{b-a} \right)^{m+1} \right]_{x=c}^{(l)}
+ \left[ \sum_{k=0}^{m} \frac{(m+k)}{m} \left( \frac{b-x}{b-a} \right)^k \left( \frac{x-a}{b-a} \right)^{m+1} \right]_{x=c}^{(l)},
\]

where

\[
W_{a,c,k}(x) := T_{a,k}(x) - U_{a,k}(x)
W_{b,c,k}(x) := T_{b,k}(x) - U_{b,k}(x)
\]

From the Leibniz differentiation formula we obtain

\[
H^{(l)}_{a,b}(c) - y^{(l)}_c = \left[ \sum_{k=0}^{m} \frac{(m+k)}{m} \sum_{q=0}^{l} \binom{l}{q} W_{a,c,m-k}(x)
\times \left( \frac{x-a}{b-a} \right)^k \left( \frac{b-x}{b-a} \right)^{m+1} \right]_{x=c}^{(l-q)}
+ \left[ \sum_{k=0}^{m} \frac{(m+k)}{m} \sum_{q=0}^{l} \binom{l}{q} W_{b,c,m-k}(x)
\times \left( \frac{b-x}{b-a} \right)^k \left( \frac{x-a}{b-a} \right)^{m+1} \right]_{x=c}^{(l-q)}.
\]
Now
\[ W_{a,c,k}^{(l)}(c) = T_{a,k}^{(l)}(c) - U_{a,k}^{(l)}(c) = \sum_{i=0}^{k-l} \frac{(y_{a}^{l+i} - T_{c}^{(l+i)}(a))}{i!} \] for \( k \geq l \)
\[ = 0 \] for \( k < l. \)

Since \( y_{a}^{l+i} = T_{a}^{(l+i)}(a), \) we see Whitney differences \( T_{a}^{(l)}(a) - T_{c}^{(l)}(a) \) appear. Thus, using the hypothesis, we have
\[ |W_{a,c,k}^{(l)}(c)| \leq E(k-l)(c-a)^{m-l-1}(c-a), \]
(2.1)
where
\[ E(k-l) := 1 + \frac{1}{1!} + \cdots + \frac{1}{(k-l)!} \]
for \( k \geq l \)
\[ := 0 \] for \( k < l. \)

In the same way
\[ |W_{b,c,k}^{(l)}(c)| \leq E(k-l)(b-c)^{m-l-1}(b-c). \]
(2.2)

Now, from (2.1) and (2.2)
\[ |H_{a,b}^{(l)}(c) - y_{c}^{l}| \leq \sum_{k=0}^{m} \sum_{q=0}^{l} \binom{m+k}{m} \binom{l}{q} E(m-k-q)(c-a)^{m-q} \omega(c-a) \]
\[ \times \left| \left( \frac{x-a}{b-a} \right)^{k} \left( \frac{b-x}{b-a} \right)^{m+1} \right|^{(l-q)}_{x=c} \]
\[ + \sum_{k=0}^{m} \sum_{q=0}^{l} \binom{m+k}{m} \binom{l}{q} E(m-k-q)(b-c)^{m-q} \omega(b-c) \]
\[ \times \left| \left( \frac{b-x}{b-a} \right)^{k} \left( \frac{x-a}{b-a} \right)^{m+1} \right|^{(l-q)}_{x=c}. \]

Assume \( \delta_{a,b}(c) = c-a \) (i.e., \( b-a < b-c \)).

Setting \( t := (x-a)/(b-a) \) and \( u := (c-a)/(b-a) \) we have
\[ \left( \frac{x-a}{b-a} \right)^{k} \left( \frac{b-x}{b-a} \right)^{m+1} \left( \frac{c-a}{b-a} \right)^{l-q} = (b-a)^{q-l}(t^{k}(1-t)^{m+1})^{(l-q)}_{t=u}. \]

Thus we can factorize directly \( (c-a)^{m-q} \omega(c-a) \) in each term of the first double summation of the previous inequality. We can also factorize this quantity in each term of the second as follows. We can write
\[ (b-c)^{m-q} \omega(b-c) \left| \left( \frac{b-x}{b-a} \right)^{k} \left( \frac{x-a}{b-a} \right)^{m+1} \right|^{(l-q)}_{x=c} \]
\[ = (c-a)^{m-l} \left( \frac{c-a}{b-a} \right) \omega(b-c) \left( \frac{1-u}{u^{m-l+1}} \right)^{(l-q)}_{t=u}. \]
From the property (P) of \( \omega \), we have
\[
\frac{c-a}{b-a} \omega(b-c) \leq \omega \left[ \frac{(c-a)(b-c)}{b-a} \right].
\]
Since \( (c-a)(b-c)/(b-a) \leq c-a \) and \( \omega \) increases, we have \( (c-a)/(b-a)) \omega(b-c) \leq \omega(c-a) \). Thus the factor \((c-a)^{m-l} \omega(c-a)\) appears as required.

Finally we have
\[
|H_{a,b}(c) - y_c^{(l)}| \leq \psi(m, l) \delta_{a,b}^{m-l}(c) \omega(\delta_{a,b}(c)),
\]
where
\[
\psi(m, l) := \sum_{k=0}^{m} \sum_{q=0}^{l} \binom{m+k}{m} \binom{l}{q} E(m-k-q) 
\times \sup_{0 \leq t \leq 1} \left| t^{l-q}((1-t)^{m+1} t^k)(l-q) \right|
+ \left| \frac{(1-t)^{m-q}}{t^{m-l+1}} (t^{m+1}(1-t)^k(l-q)) \right|.
\]
The case \( \delta_{a,b}(c) = b-c \) is similar and leads to the same bound.

Remark. \( \psi(m, l) \) is finite because \( ((1-t)^{m-q}/t^{m-l+1})(t^{m+1}(1-t)^k(l-q)) \) is, as we see by development, a polynomial.

**Lemma 3.** Let \( a, b \in K, a < b \), and assume the conditions of the theorem. Then a numerical constant \( \chi(m) \) exists depending only on \( m \) such that
\[
\sup_{a \leq x \leq b} |H_{a,b}^{(m+1)}(x)| \leq \chi(m) \frac{\omega(b-a)}{b-a}.
\]

**Proof.** Let \( f(x) := H_{a,b}(x) - T_a(x) \). Since \( T_a \in \mathbb{R}_m[x] \) we have \( T_a^{(m+1)} = 0 \). It follows that \( f^{(m+1)} = H_{a,b}^{(m+1)} \). Thus it is sufficient to estimate \( |f^{(m+1)}(x)| \). We have
\[
f^{(k)}(a) = 0, \\
f^{(k)}(b) = y_b^k - T_a^{(k)}(b) = T_h^{(k)}(b) - T_a^{(k)}(b)
\]
k = 0, ..., m.
Therefore, using the hypothesis, \( |f^{(k)}(b)| \leq (b-a)^m \omega(b-a) \).

Now \( f \) is a polynomial of degree at most \( 2m+1 \) and induces an \( m \)-Taylorian field \( F \) with \( F_a = 0 \) and \( F_b = T_b - T_a \).

Thus, by the unicity of the Hermite interpolant and Lemma 1 we have
\[
f(x) - \left( \frac{x-a}{b-a} \right)^{m+1} \sum_{k=0}^{m} \binom{m+k}{m} \binom{b-x}{b-a} F_{b,m-k}(x), \quad a \leq x \leq b.
\]
From the Taylor formula and the Leibniz differentiation formula

\[ f^{(m+1)}(x) = \sum_{k=0}^{m} \sum_{i=0}^{m-k} \sum_{q=0}^{k+i} \binom{m+k}{m} \binom{m+1}{q} \binom{k+i}{q} \times \frac{(m+1)!}{i!} (-1)^{i+q} \frac{(x-a)^q (b-x)^{k+i-q}}{(b-a)^{m+1+k}} f^{(i)}(b). \]

Therefore

\[ |H_{a,b}^{(m+1)}(x)| = |f^{(m+1)}(x)| \leq \chi(m) \frac{\omega(b-a)}{b-a} \quad \text{for} \quad a \leq x \leq b, \]

where

\[ \chi(m) := \sum_{k=0}^{m} \sum_{i=0}^{m-k} \sum_{q=0}^{k+i} \binom{m+k}{m} \binom{m+1}{q} \binom{k+i}{q} \frac{(m+1)!}{i!}. \]

**Lemma 4.** Let \( a, b, c, d \in \mathbb{K}, a \leq c < d \leq b, \) and assume the conditions of the theorem. Let \( X := \{c, d\} \) and \( Z := \{a, b, c, d\}. \) (We notice that \( \delta(X, Z) = \sup(\delta_{a,b}(c), \delta_{a,b}(d)).) \)

Then numerical constants \( \rho(m, k) \) \( (k = 0, \ldots, m) \) depending only on \( m \) and \( k \) exist such that

\[ \sup_{c \leq x \leq d} |H_{a,b}^{(k)}(x) - H_{c,d}^{(k)}(x)| \leq \rho(m, k)(b-a)^{m-k} \omega(\delta(X, Z)). \]

**Proof.** We distinguish two cases.

**Case 1.** \( d-c \geq \delta(X, Z). \) Set

\[ A_{c,k}(x) := \sum_{q=0}^{k} \frac{(x-c)^q}{q!} (H_{a,b}^{(q)}(c) - y_c^q) \]

and

\[ A_{d,k}(x) := \sum_{q=0}^{k} \frac{(x-d)^q}{q!} (H_{a,b}^{(q)}(d) - y_d^q). \]

Now \( H_{a,b} - H_{c,d} \) is, in restriction to \([c, d]\), a polynomial of degree at most \(2m+1\). Thus, from the unicity of the Hermite interpolant once again and Lemma 1, we have, for \( x \in [c, d]\)

\[ H_{a,b}(x) - H_{c,d}(x) = \left(\frac{d-x}{d-c}\right)^{m+1} \sum_{k=0}^{m} \binom{m+k}{k} \left(\frac{x-c}{d-c}\right)^k A_{c,m-k}(x) \]

\[ + \left(\frac{x-c}{d-c}\right)^{m+1} \sum_{k=0}^{m} \binom{m+k}{k} \left(\frac{d-x}{d-c}\right)^k A_{d,m-k}(x). \]

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Now we use Lemma 2 to bound $|A_{c,k}(x)|$ and $|A_{d,k}(x)|$ for $x \in [c, d]$.

$$|A_{c,k}(x)| \leq \sum_{q=0}^{k} \psi(m, q) \frac{(x-c)^q}{q!} \delta_{a,b}^{m-q}(c) \omega(\delta_{a,b}(c)).$$

Since, in this case, $\delta_{a,b}(c) \leq d - c$, we have

$$|A_{c,k}(x)| \leq e(m, k)(d-c)^m \omega(\delta_{a,b}(c)),$$

where $e(m, k) := \sum_{q=0}^{k} (\psi(m, q)/q!)$.

A similar bound holds for $|A_{d,k}(x)|$.

Now, applying Lemma 1 to any constant function, we have

$$\left(\frac{d-x}{d-c}\right)^{m+1} \sum_{k=0}^{m} \binom{m+k}{k} \frac{(x-c)^k}{(d-c)} + \left(\frac{d-x}{d-c}\right)^{m+1} \sum_{k=0}^{m} \binom{m+k}{k} \frac{(d-x)^k}{(d-c)} = 1.$$ Therefore

$$\sup_{c \leq x \leq d} |H_{a,b}(x) - H_{c,d}(x)| \leq e(m, m) (d-c)^m \omega(\delta(X, Z)).$$

We finish this first case by applying the Markov inequality for polynomials

$$\sup_{c \leq x \leq d} |H_{a,b}^{(k)}(x) - H_{c,d}^{(k)}(x)| \leq \frac{2^k [(2m+1)(2m) \cdots (2m-k+2)]^2 (d-c)^k}{s(m, k)(d-c)^{m-k}} \omega(\delta(X, Z)),$$

where $s(m, k) := 2^k [(2m+1)(2m) \cdots (2m-k+2)]^2 e(m, m)$. Since $d-c \leq b-a$, the first case is done.

**Case 2.** $d-c \leq \delta(X, Z)$. For $x \in [c, d]$ we have

$$|H_{a,b}^{(m)}(x) - H_{c,d}^{(m)}(x)| \leq |H_{a,b}^{(m)}(x) - H_{a,b}^{(m)}(c)| + |H_{a,b}^{(m)}(c) - H_{c,d}^{(m)}(c)| \leq \delta(X, Z)
+ |H_{c,d}^{(m)}(c) - H_{c,d}^{(m)}(x)|.$$

From Lemma 2 we know that $|H_{a,b}^{(m)}(c) - H_{c,d}^{(m)}(c)| \leq \psi(m, m) \omega(\delta_{a,b}(c))$. Using $H_{c,d}^{(m)}(x) - H_{c,d}^{(m)}(c) = \int_c^x H_{c,d}^{(m+1)}(t) \, dt$ and Lemma 3 we obtain

$$|H_{c,d}^{(m)}(x) - H_{c,d}^{(m)}(c)| \leq \chi(m) \omega(d-c).$$

From $H_{a,b}^{(m)}(x) - H_{a,b}^{(m)}(c) = \int_c^x H_{a,b}^{(m+1)}(t) \, dt$ and Lemma 3 again, we obtain

$$|H_{a,b}^{(m)}(x) - H_{a,b}^{(m)}(c)| \leq \chi(m) \frac{d-c}{b-a} \omega(b-a).$$
From the property (P) of \( \omega \), we have
\[
|H^{(m)}_{b,d}(x) - H^{(m)}_{c,d}(c)| \leq \chi(m) \omega (d-c).
\]

Since \( d-c \leq \delta(X,Z) \) in this second case, we obtain \( \sup_{c \leq x \leq d} |H^{(m)}_{b,d}(x) - H^{(m)}_{c,d}(x)| \leq r(m, m) \omega (\delta(X,Z)) \), where \( r(m, m) := 2 \chi(m) + \psi(m, m) \). Now we have
\[
H^{(k)}_{a,b}(x) - H^{(k)}_{c,d}(x) = \int_{c}^{x} \left( H^{(k+1)}_{a,b}(t) - H^{(k+1)}_{c,d}(t) \right) \, dt
+ H^{(k)}_{a,b}(c) - H^{(k)}_{c,d}(c), \quad k = 0, \ldots, m-1.
\]

Thus, by successive integrations and from Lemma 2, we obtain
\[
\sup_{c \leq x \leq d} |H^{(k)}_{a,b}(x) - H^{(k)}_{c,d}(x)| \leq r(m, k)(b-a)^{m-k} \omega (\delta(X,Z)),
\]
where \( r(m, k) := r(m, k+1) + \psi(m, k), k = 0, \ldots, m-1. \)

Now set \( \rho(m, k) := \sup(s(m, k), r(m, k)) \) to finish the proof of the lemma.

**Lemma 5.** Assuming the conditions of the theorem, numerical constants \( \varphi(m, k) \) depending only on \( m \) and \( k = 0, \ldots, m \) exist such that:

- For any bounded open interval \( C \) containing \( K \)
- For any \( X, Z \in \mathcal{F}(K) \)
- For any \( k = 0, \ldots, m \)

\[
\sup_{x \in C} |H^{(k)}_{X}(x) - H^{(k)}_{Z}(x)|
\leq \varphi(m, k) \sup(\delta^{m-k}(X, C), \delta^{m-k}(Z, C)) \omega (\delta(X,Z)).
\]

**Proof.** Since \( \delta(X,Z) = \sup(\delta(X, X \cup Z), \delta(Z, X \cup Z)) \) and \( \omega \) increases, it is sufficient to consider the case \( X \subset Z \) and to prove that
\[
\sup_{x \in C} |H^{(k)}_{X}(x) - H^{(k)}_{Z}(x)|
\leq \frac{1}{2} \varphi(m, k) \cdot \delta^{m-k}(X, C) \omega (\delta(X,Z)), \quad k = 0, \ldots, m.
\]

Let \( X := \{x_i : i = 0, \ldots, N\}, x_0 < x_1 < \cdots < x_i < \cdots < x_N \).

**Case 1.** \( x \in [x_i, x_{i+1}] \) for an \( i \in \{0, 1, \ldots, N-1\} \). If \( [x_i, x_{i+1}] \cap Z = \{x_i, x_{i+1}\} \), then, by definition, we have \( H_{X} = H_{Z} \) restricted to \( [x_i, x_{i+1}] \).

Otherwise denote by \( z_0, z_1, \ldots, z_i \) the points of \( Z \cap [x_i, x_{i+1}] \). \( x \) belongs
to \([z_j, z_{j+1}]\) for a \(j \in \{0, \ldots, l-1\}\). Now applying Lemma 4 with \(a = x_i, b = x_{i+1}, c = z_j, d = z_{j+1}\), we obtain
\[
|H_x^k(x) - H_z^k(x)| \leq \rho(m, k) |x_{i+1} - x_i|^{m-k} \\
\times \delta\left(\{x_i, x_{i+1}\}, \{x_i, z_j, z_{j+1}, x_{i+1}\}\right).
\]

Since \(\delta(\{x_i, x_{i+1}\}, \{x_i, z_j, z_{j+1}, x_{i+1}\}) \leq \delta(X, Z)\) and \(|x_{i+1} - x_i| \leq 2\delta(X, C)\), we obtain in this case the desired bound with
\[
\frac{1}{2} \varphi_1(m, k) := 2^{m-k} \rho(m, k).
\]

**Case 2.** \(x \leq x_0\) or \(x \geq x_N\). These subcases are similar. Thus, without loss of generality, we suppose \(x \leq x_0\). Set \(C := ]c_0, c_1[\). We have only to consider the situation
\[
Z \cap ]c_0, x_0[ = \{z_0, \ldots, z_l\} \quad \text{with} \quad l > 0 \quad \text{and} \quad z_l = x_0.
\]

If \(x \in [z_j, z_{j+1}]\) we have, by definition of the Hermite interpolant in this case, \(H_x = T_{x_0}\) restricted to \(] - \infty, x_0[\), therefore, to \([z_j, z_{j+1}]\). Thus \(H_x^{(m+1)} = 0\) in this interval and from Lemma 3
\[
|H_z^{(m+1)}(x) - H_x^{(m+1)}(x)| \leq \chi(m) \frac{\omega(z_{j+1} - z_j)}{z_{j+1} - z_j}.
\]

By integration we have
\[
|H_z^{(m)}(x) - H_x^{(m)}(x)| \leq \left|H_z^{(m)}(z_j) - H_x^{(m)}(z_j)\right| \\
+ \left|\int_{z_j}^{x} (H_z^{(m+1)}(t) - H_x^{(m+1)}(t)) \, dt\right|.
\]

Since \(H_z^{(m)}(z_j) - H_x^{(m)}(z_j) = T_{x_0}^{(m)}(z_j) - T_{x_0}^{(m)}(z_j)\), we recognize a Whitney difference. Since \(|x - z_j| \leq |z_{j+1} - z_j| \leq \delta(X, Z)\), it follows that
\[
|H_z^{(m)}(x) - H_x^{(m)}(x)| \leq \frac{1}{2} \varphi_2(m, m) \omega(\delta(X, Z)),
\]
where \(\varphi_2(m, m) := 2(1 + \chi(m))\).

As in the end of Lemma 4, we have
\[
H_z^{(k)}(x) - H_x^{(k)}(x) = \int_{z_j}^{x} (H_z^{(k+1)}(t) - H_x^{(k+1)}(t)) \, dt \\
+ H_z^{(k)}(z_j) - H_x^{(k)}(z_j), \quad k = 0, \ldots, m - 1.
\]

Again \(H_z^{(k)}(z_j) - H_x^{(k)}(z_j)\) is a Whitney difference.
Thus, from the hypothesis and $|x_0 - z_j| \leq |x_0 - c_0| \leq \delta(X, C)$, we obtain by successive integrations

$$\sup_{z_j \leq x \leq z_{j+1}} |H^{(k)}_Z(x) - H^{(k)}_X(x)| \leq \frac{1}{2} \varphi_2(m, k) \delta^{m-k}(X, C) \omega(\delta(X, Z)),$$

where $\varphi_2(m, k) := \psi_2(m, k + 1) + 2 = \psi_2(m, m) + 2(m - k)$, $k = 0, \ldots, m - 1$. The case $x \in \overline{c_0, z_0}$ is similar and leads to the same bounds. We finish the proof of the lemma by setting $\varphi(m, k) := \sup(\varphi_1(m, k), \varphi_2(m, k))$.

**Proof of the Theorem.** It is known that $(\mathcal{K}(K), \delta)$ $(\delta =$ Hausdorff distance) is a compact complete metric space and that $\mathcal{F}(K)$ is dense in $\mathcal{K}(K)$. We shall make use of the following version of this last property: the subset of $\mathcal{F}(K)$ whose elements contain a selected point $x_0 \in K$ is dense in the subset of $\mathcal{K}(K)$ whose elements contain $x_0$.

Now let $C$ be a given bounded open interval containing $K$. Denote its length by $l(C)$. For $f \in C^m(C)$ define

$$\|f\|_{m, C} := \sup \{|f^{(k)}(x)| : k = 0, \ldots, m, x \in C\}$$

and

$$E(m, C) := \{f \in C^m(C) : \|f\|_{m, C} < +\infty \}.$$

Clearly $(E(m, C), \| \cdot \|_{m, C})$ is a Banach space. Now let $X$ and $Z$ belong to $\mathcal{F}(K)$. Remark that $\delta(X, C) \leq l(C)$ we have by Lemma 5

$$\|H_Z - H_X\|_{m, C} \leq p(m, l(C)) \omega(\delta(X, Z)),$$

where $p(m, l(C)) := \sup_{k = 0, \ldots, m} \varphi(m, k)(l(C))^{m-k}$.

Thus the map $X \to H_X$ is uniformly continuous from $(\mathcal{F}(K), \delta)$ to $(E(m, C), \| \cdot \|_{m, C})$, with modulus of (uniform) continuity $\omega_0 = p(m, l(C)) \omega$. It follows that this map extends to a unique uniformly continuous map denoted $f_C$ (apparently this map depends on $C$) defined on the whole space $\mathcal{K}(K)$ with the same modulus of continuity. Now, if $C_1 \subset C_2$ are two bounded intervals containing $K$, and $L$ is a non-empty compact subset of $K$, it is clear that $f_{C_1}(L)$ and $f_{C_2}(L)$ coincide on $C_1$. This allows us to define $H_L = f_C(L)$ as the unique function whose restriction to any bounded open interval $C$ containing $K$ is $f_C(L)$.

The required properties must now be checked.

Property (i) follows from the fact that $(E(m, C), \| \cdot \|_{m, C})$ is a Banach space. Property (iii) for elements of $\mathcal{F}(K)$ is exactly Lemma 5. It follows for elements of $\mathcal{K}(K)$ by density of $\mathcal{F}(K)$ in $\mathcal{K}(K)$.

In order to check property (ii), let $L \in \mathcal{K}(K)$ and $x_0 \in L$. As mentioned above, we can choose a sequence $(X_n)$ such that $X_n \in \mathcal{F}(K), x_0 \in X_n$

$$\lim_{n \to \infty} \delta(X_n, L) = 0.$$
Since $x_0 \in X$, the $m$-Taylorian field of $H_{X_0}$ coincides with $T$ at $x_0$. From the continuity of map $H$, the same holds for $H_L$.

This finishes the proof of the theorem.

Remark. $H_K$ defined by the theorem coincides with the one described in the Introduction. To see this, let $a, b \in K$, $a < b$, $\lfloor a, b \cap K = \emptyset$, and let $(X_n)_{n \in \mathbb{N}}$ be an increasing sequence with $X_0 = \{a, b\}$, $X_n \subset K$, and $\lim_{n \to \infty} \delta(X_n, K) = 0$.

By construction each $H_{X_n}$, and therefore $H_K$, coincides with $H_{a,b}$ on $[a, b]$.

We close this paper by establishing the previously announced minimisation property of $H_K$. Let

$$D^{-(m+1)} = \{ f \in \mathcal{S}'(\mathbb{R}) : f^{(m-1)} \in \mathcal{L}^2(\mathbb{R}) \}.$$ 

For $f \in D^{-(m+1)}$ let $|f| := (\int_{\mathbb{R}} (f^{(m+1)}(x))^2 \, dx)^{1/2}$. ($\mathcal{S}'(\mathbb{R})$ is the well-known space of Schwartz tempered distributions.)

By the Kryloff imbedding theorem [5, p. 181], we have $D^{-(m+1)} \subset \mathcal{C}^m(\mathbb{R})$. Thus any $f \in D^{-(m+1)}$ induces an $m$-Taylorian field $Tf$. We remark that the Hermite interpolants $H_x$, where $X$ is finite, are in $D^{-(m+1)}$.

It is known and easy to show by integration by parts that these interpolants $H_x$ have the following minimisation property: For $g \in D^{-(m+1)}$ with $Tg$ and $T$ coinciding on $X$, we have the key formula

$$|g|^2 = |H_x|^2 + |g - H_x|^2.$$ 

From (2.3) it is clear that if $\sup_{X \in \mathcal{F}(K)} |H_X| = +\infty$, then there is no $f \in D^{-(m+1)}$ such that $Tf$ extends $T$.

In the other case we have the following minimisation property of $H_K$:

**Proposition.** Assume the conditions of the theorem and $\sup_{X \in \mathcal{F}(K)} |H_X| < +\infty$. Then $H_K$ defined previously is in $D^{-(m+1)}$ and for any $f \in D^{-(m+1)}$ with $Tf$ extending $T$ we have $|H_K| \leq |f|$.

**Proof.** Let $(X_n)_{n \in \mathbb{N}}$ be an increasing sequence of elements of $\mathcal{F}(K)$ with $\lim_{n \to \infty} \delta(X_n, K) = 0$. Applying (2.3) with $H_{X_p}$ and $H_{X_q}$, $p \leq q$, we see that the sequence $(|H_{X_n}|)_{n \in \mathbb{N}}$ increases. It follows from the hypothesis that this sequence converges and, therefore, is a Cauchy sequence. From (2.3) again the sequence $(H_{X_n})_{n \in \mathbb{N}}$ is itself a Cauchy sequence in the space $(D^{-(m+1)} \mathcal{L}^1)$. Now let $C$ be a bounded open interval containing $K$. From the stability property of the theorem we have

$$\sup_{0 \leq k \leq m} \left( \int_C (H^{(k)}_{X_p}(x) - H^{(k)}_{X_q}(x))^2 \, dx \right)^{1/2} \leq (l(C))^{1/2} p(m, l(C)) \omega(\delta(X_p, X_q)).$$
In other words, the sequence \((H_{x_n})_{n \in \mathbb{N}}\) is a Cauchy sequence in the Sobolev space \(\mathcal{H}_m(C)\). Putting these two results together \((H_{x_n})_{n \in \mathbb{N}}\) is thus a Cauchy sequence in \(\mathcal{H}_{m+1}(C)\). It follows that this sequence converges in \(\mathcal{H}_{m+1}(C)\) to an element denoted \(g_K\) which, by the Sobolev imbedding theorem, is in \(C^m(C)\). By the Sobolev imbedding theorem again, the sequence \((H_{x_n})_{n \in \mathbb{N}}\) also converges in \(E(m, C)\). Since, by our theorem, this sequence converges to \(H_K\) in \(E(m, C)\), we deduce that \(g_K = H_K\). Thus \(H_K^{(m+1)} \in \mathcal{L}^2(C)\).

We remark now that \(H_K^{(m+1)}(x) = 0\) for \(x \leq \inf_{z \in K} z\) and \(x \geq \sup_{z \in K} z\). Thus \(H_K^{(m+1)} \in \mathcal{L}^2(\mathbb{R})\); that is, \(H_K \in D^{-(m+1)}\).

The minimisation property must now be proved. From (2.3) we have

\[
|f|^2 = |H_{x_n}|^2 + |f - H_{x_n}|^2.
\]

Thus, for \(n \to \infty\), \(|f|^2 = |H_K|^2 + |f - H_K|^2\), that is, \(|H_K| \leq |f|\).

**Remark.** This proposition can also be established in a form which is not so close to approximation theory: For \(K\) infinite (there is nothing new to prove in the other case) we can write \(\mathbb{R} - K = \bigcup_{n \in \mathbb{N}} \]a_n, b_n[\) where

\[
a_0 = -\infty, \quad a_n \in K \text{ for } n \neq 0
\]

\[
b_1 = +\infty, \quad b_n \in K \text{ for } n \neq 1
\]

(this union is countable because \(\mathbb{R}\) has a countable dense subset).

For \(f \in D^{-(m+1)}\) and \(A\) a measurable subset of \(\mathbb{R}\) let

\[
|f|_A := \left( \int_A (f^{(m+1)}(x))^2 \, dx \right)^{1/2}.
\]

Thus

\[
|f|^2 = |f|_\mathbb{R}^2 = \sum_{n=0}^{\infty} |f|_{]a_n, b_n[}^2.
\]

Since \(Tf\) extends \(T, f^{(m+1)}\) and \(H_K^{(m+1)}\) coincide on \(K\). It follows that \(|f|_K = |H_K|_K\). Since \(H_K\) and \(H_{a_n, b_n[}\) coincide on \(]a_n, b_n[\) (see the Remark following the theorem), we have

\[
|H_K|_{]a_n, b_n[} = |H_{a_n, b_n[} = |H_{a_n, b_n[}^2.
\]

From (2.3) in restriction to \(]a_n, b_n[\) we have

\[
|f|_{]a_n, b_n[}^2 = |H_{a_n, b_n[}^2 + |f - H_{a_n, b_n[}|_{]a_n, b_n[}^2.
\]

Thus \(|f|_{]a_n, b_n[} \geq |H_K|_{]a_n, b_n[}\). The result follows.
Final Remark. The functions (polynomials resp.) of this article need not have their range (their coefficients resp.) in $\mathbb{R}$. They can have their range (their coefficients resp.) in any Banach space (any Hilbert space for the minimisation property): the same statements and proofs hold.

REFERENCES