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## The 2-adic valuation of a sequence arising from a rational integral

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### ABSTRACT

We analyze properties of the 2-adic valuation of an integer sequence that originates from an explicit evaluation of a quartic integral. We also give a combinatorial interpretation of the valuations of this sequence.

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## 1. Introduction

Wallis's formula

$$\int_0^{\infty} \frac{dx}{(x^2 + 1)^{m+1}} = \frac{\pi}{2^{2m+1}} \binom{2m}{m} \quad (1.1)$$

is one of the earlier instances of evaluation of definite integrals where the result contains interesting arithmetical and combinatorial properties. In this paper we examine such connection for the integral

$$N_{0,4}(a; m) = \int_0^{\infty} \frac{dx}{(x^4 + 2ax^2 + 1)^{m+1}}. \quad (1.2)$$

The condition  $a > -1$  is imposed for convergence. The evaluation

$$N_{0,4}(a, m) = \frac{\pi}{2} \frac{P_m(a)}{[2(a+1)]^{m+\frac{1}{2}}} \quad (1.3)$$

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where

$$P_m(a) = \sum_{l=0}^m d_l(m) a^l \quad (1.4)$$

with

$$d_l(m) = 2^{-2m} \sum_{k=l}^m 2^k \binom{2m-2k}{m-k} \binom{m+k}{m} \binom{k}{l}, \quad 0 \leq l \leq m, \quad (1.5)$$

appeared in [4]. The reader will find in [2] a survey of the different proofs of (1.3) and an introduction to the many issues involved in the evaluation of definite integrals in [8].

The study of combinatorial aspects of the sequence  $d_l(m)$  was initiated in [3] where the authors show that  $d_l(m)$  form a *unimodal* sequence, that is, there exists an index  $l^*$  such that  $d_0(m) \leq \dots \leq d_{l^*}(m)$  and  $d_{l^*}(m) \geq \dots \geq d_m(m)$ . The fact that  $d_l(m)$  satisfies the stronger condition of *logconcavity*  $d_{l-1}(m)d_{l+1}(m) \leq d_l^2(m)$  has been recently established in [6]. We consider here arithmetical properties of the sequence  $d_{l,m}$ . It is more convenient to analyze the auxiliary sequence

$$A_{l,m} = l!m!2^{m+l}d_{l,m} = \frac{l!m!}{2^{m-l}} \sum_{k=l}^m 2^k \binom{2m-2k}{m-k} \binom{m+k}{m} \binom{k}{l} \quad (1.6)$$

for  $m \in \mathbb{N}$  and  $0 \leq l \leq m$ . The integral (1.2) is then given explicitly as

$$N_{0,4}(a; m) = \frac{\pi}{\sqrt{2}m!(4(2a+1))^{m+1/2}} \sum_{l=0}^m A_{l,m} \frac{a^l}{l!}. \quad (1.7)$$

In [5] it is shown that  $A_{l,m} \in \mathbb{N}$ . Observe that the computation of  $A_{l,m}$  using (1.6) is more efficient if  $l$  is close to  $m$ . For instance,

$$A_{m,m} = 2^m(2m)! \quad \text{and} \quad A_{m-1,m} = 2^{m-1}(2m-1)!(2m+1). \quad (1.8)$$

A second method to compute  $A_{l,m}$ , efficient now when  $l$  is small, has been discussed in [5]. There, it is shown that  $A_{l,m}$  is a linear combination (with polynomial coefficients) of

$$\prod_{k=1}^m (4k-1) \quad \text{and} \quad \prod_{k=1}^m (4k+1). \quad (1.9)$$

For example,

$$A_{0,m} = \prod_{k=1}^m (4k-1) \quad \text{and} \quad A_{1,m} = (2m+1) \prod_{k=1}^m (4k-1) - \prod_{k=1}^m (4k+1). \quad (1.10)$$

The results described in this paper started with some empirical observations on the behavior of the 2-adic valuation of  $A_{l,m}$ , i.e.  $v_2(A_{l,m})$ . Recall that, for  $x \in \mathbb{N}$ , the 2-adic valuation  $v_2(x)$  is the highest power of 2 that divides  $x$ . This is extended to  $x = a/b \in \mathbb{Q}$  via  $v_2(x) = v_2(a) - v_2(b)$ . From (1.10) it follows that  $A_{0,m}$  is odd, so  $v_2(A_{0,m}) = 0$ . Moreover,

$$v_2(A_{1,m}) = v_2(m(m+1)) + 1, \quad (1.11)$$

i.e., the main result of [5]. We present as Theorem 2.1, an expression for  $v_2(A_{l,m})$  that generalizes (1.11).

The study of the sequence

$$X(l) := \{v_2(A_{l,l+m-1}) : m \geq 1\} \quad (1.12)$$

requires the introduction of two operators,  $F$  and  $T$ , defined in (4.1) and (4.2), respectively. The iteration of these operators creates an integer vector

$$\Omega(l) := \{n_1, n_2, n_3, \dots, n_{\omega(l)}\}, \quad \text{with } n_i \in \mathbb{N}, \quad (1.13)$$

associated to the index  $l \in \mathbb{N}$ . We call  $\Omega(l)$  the *reduction sequence* of  $l$ . See (4.2) for the precise definition of the integers  $n_j$ . The structure of  $X(l)$  motivates the following definition.

**Definition 1.1.** Let  $s \in \mathbb{N}$ ,  $s \geq 2$ . We say that a sequence  $\{a_j: j \in \mathbb{N}\}$  is *simple of length  $s$*  (or  *$s$ -simple*) if  $s$  is the largest integer such that for each  $t \in \{0, 1, 2, \dots\}$ , we have

$$a_{st+1} = a_{st+2} = \dots = a_{s(t+1)}. \quad (1.14)$$

The sequence  $\{a_j: j \in \mathbb{N}\}$  is said to have a *block structure* if it is  $s$ -simple for some  $s \geq 2$ .

Section 2 presents two proofs of the expression for  $v_2(A_{l,m})$ . Section 3 shows that  $X(l)$  is a simple sequence of length  $2^{1+v_2(l)}$ . In Section 4 an algorithm generating the vector  $\Omega(l)$  is described in detail. A combinatorial interpretation of  $\Omega(l)$ , as the composition of  $l$ , is provided in Section 5. Theorem 5.5 gives  $\Omega(l)$  in terms of the dyadic expansion of  $l$ . More precisely, if  $\{k_1, \dots, k_n: 0 \leq k_1 < k_2 < \dots < k_n\}$  is the unique collection of distinct nonnegative integers such that  $l = \sum_{i=1}^n 2^{k_i}$ , then the reduction sequence  $\Omega(l)$  of  $l$  is  $\{k_1 + 1, k_2 - k_1, \dots, k_n - k_{n-1}\}$ . Finally, the last section contains a conjecture on symmetries of the graph of  $v_2(A_{l,m})$ .

## 2. The 2-adic valuation of $A_{l,m}$

In this section we prove that  $v_2(A_{l,m})$  agrees with  $v_2((m+1-l)_{2l}) + l$ . The first proof actually produces the latter term in a natural way starting from the former. The second proof employs the WZ-machinery [9] to prove the identity (2.1).

**Theorem 2.1.** *The 2-adic valuation of  $A_{l,m}$  satisfies*

$$v_2(A_{l,m}) = v_2((m+1-l)_{2l}) + l, \quad (2.1)$$

where  $(a)_k = a(a+1) \dots (a+k-1)$  is the Pochhammer symbol for  $k \geq 1$ . For  $k = 0$ , we define  $(a)_0 = 1$ .

**Proof.** *First proof.* We have

$$v_2(A_{l,m}) = l + v_2\left(\sum_{k=l}^m T_{m,k} \frac{(m+k)!}{(m-k)!(k-l)!}\right), \quad (2.2)$$

where

$$T_{m,k} = \frac{(2m-2k)!}{2^{m-k}(m-k)!}. \quad (2.3)$$

The identity

$$T_{m,k} = \frac{(2(m-k))!}{2^{m-k}(m-k)!} = (2m-2k-1)(2m-2k-3) \dots 3 \cdot 1 \quad (2.4)$$

shows that  $T_{m,k}$  is an odd integer. Then (2.2) can be written as

$$v_2(A_{l,m}) = l + v_2\left(\sum_{k=0}^{m-l} T_{m,l+k} \frac{(m+k+l)!}{(m-k-l)!k!}\right) = l + v_2\left(\sum_{k=0}^{m-l} T_{m,l+k} \frac{(m-k-l+1)_{2k+2l}}{k!}\right).$$

The term corresponding to  $k = 0$  is singled out as we write

$$v_2(A_{l,m}) = l + v_2\left(T_{m,l}(m-l+1)_{2l} + \sum_{k=1}^{m-l} T_{m,l+k} \frac{(m-k-l+1)_{2k+2l}}{k!}\right).$$

The claim

$$v_2\left(\frac{(m-k-l+1)_{2k+2l}}{k!}\right) > v_2((m-l+1)_{2l}) \quad (2.5)$$

for any  $k$ ,  $1 \leq k \leq m-l$ , will complete the proof.

To prove (2.5) we use the identity

$$\frac{(m-k-l+1)_{2k+2l}}{k!} = (m-l+1)_{2l} \cdot \frac{(m-l-k+1)_k(m+l+1)_k}{k!}$$

and the fact that the product of  $k$  consecutive numbers is always divisible by  $k!$ . This follows from the identity

$$\frac{(a)_k}{k!} = \binom{a+k-1}{k}. \quad (2.6)$$

Now if  $m+l$  is odd,

$$v_2\left(\frac{(m-l-k+1)_k}{k!}\right) \geq 0 \quad \text{and} \quad v_2((m+l+1)_k) > 0, \quad (2.7)$$

and if  $m+l$  is even

$$v_2\left(\frac{(m+l+1)_k}{k!}\right) \geq 0 \quad \text{and} \quad v_2((m-l-k+1)_k) > 0. \quad (2.8)$$

This proves (2.5) and establishes the theorem.

*Second proof.* Define the numbers

$$B_{l,m} := \frac{A_{l,m}}{2^l(m+1-l)_{2l}}. \quad (2.9)$$

We need to prove that  $B_{l,m}$  is odd. The WZ-method [9] provides the recurrence

$$B_{l-1,m} = (2m+1)B_{l,m} - (m-l)(m+l+1)B_{l+1,m}, \quad 1 \leq l \leq m-1.$$

Since the initial values  $B_{m,m} = 1$  and  $B_{m-1,m} = 2m+1$  are odd, it follows that  $B_{l,m}$  is an odd integer.  $\square$

### 3. Properties of the function $v_2(A_{l,m})$

Let  $l \in \mathbb{N} \cup \{0\}$  be fixed. In this section we describe properties of the function  $v_2(A_{l,m})$ . In particular, we show that each of these sequences has a block structure.

**Theorem 3.1.** *Let  $l \in \mathbb{N} \cup \{0\}$  be fixed. Then for  $m \geq l$ , we have*

$$v_2(A_{l,m+1}) - v_2(A_{l,m}) = v_2(m+l+1) - v_2(m-l+1). \quad (3.1)$$

**Proof.** From (2.1) and  $(a)_k = (a+k-1)!/(a-1)!$ , we have

$$v_2(A_{l,m}) = v_2\left(\frac{(m+l)!}{(m-l)!}\right) + l. \quad (3.2)$$

This implies

$$\begin{aligned} v_2(A_{l,m+1}) - v_2(A_{l,m}) &= v_2\left(\frac{(m+l+1)!}{(m-l+1)!}\right) - v_2\left(\frac{(m+l)!}{(m-l)!}\right) \\ &= v_2\left(\frac{(m+l+1)!(m-l)!}{(m-l+1)!(m+l)!}\right) \\ &= v_2(m+l+1) - v_2(m-l+1). \quad \square \end{aligned}$$

The next corollary is a special case of Theorem 3.1.

**Corollary 3.2.** *The sequence  $v_2(A_{l,m})$  satisfies*

- (1)  $v_2(A_{l,l+1}) = v_2(A_{l,l})$ .
- (2) For  $l$  even,

$$v_2(A_{l,l+3}) = v_2(A_{l,l+2}) = v_2(A_{l,l+1}) = v_2(A_{l,l}).$$

- (3) The sequence  $v_2(A_{1,m})$  is 2-simple; i.e.,  $v_2(A_{1,m+1}) = v_2(A_{1,m})$  for  $m$  odd. In fact,

$$A_{1,m} = \{2, 2, 3, 3, 2, 2, 4, 4, 2, 2, \dots\}.$$

Fix  $k, l \in \mathbb{N}$  and let  $\mu := 1 + v_2(l)$ . Define the following sets

$$C_{k,l} := \{l + k \cdot 2^\mu + j : 0 \leq j \leq 2^\mu - 1\}, \quad (3.3)$$

which will be instrumental in proving the main result of this section; i.e.,  $\{v_2(A_{l,m})\}$  is  $2^{1+v_2(l)}$ -simple.

We begin by showing that these sets form a partition of  $\mathbb{N}$ . Moreover, for fixed  $k, l \in \mathbb{N}$  the set  $C_{k,l}$  has cardinality  $2^\mu$  and the 2-adic valuation of  $\{A_{l,m} : m \in C_{k,l}\}$  is constant. For example, if  $l \in \mathbb{N}$  is odd, then  $\mu = 1$  and

$$C_{k,l} = \{l + 2k, l + 2k + 1\}. \quad (3.4)$$

The next result is immediate.

**Lemma 3.3.** *Let  $l \in \mathbb{N}$  be fixed. The sets  $\{C_{k,l} : k \geq 0\}$  form a disjoint partition of  $\mathbb{N}$ ; namely,*

$$\{m \in \mathbb{N} : m \geq l\} = \bigcup_{k \geq 0} C_{k,l}, \quad (3.5)$$

and  $C_{r,l} \cap C_{t,l} = \emptyset$ , whenever  $r \neq t$ .

**Lemma 3.4.** *Fix  $l \in \mathbb{N}$  and let  $\mu = v_2(2l)$ .*

- (1) The sequence  $\{v_2(A_{l,m}) : m \in C_{k,l}\}$  is constant. We denote this value by  $v_2(C_{k,l})$ .
- (2) For  $k \geq 0$ ,  $v_2(C_{k+1,l}) \neq v_2(C_{k,l})$ .

**Proof.** Suppose  $0 \leq j \leq 2^\mu - 2$ . Since  $v_2(2l) = \mu \leq v_2(k \cdot 2^\mu)$ , then

$$v_2(2l + k \cdot 2^\mu) \geq v_2(2l) = \mu > v_2(j + 1), \quad (3.6)$$

because  $j + 1 < 2^\mu$ . Therefore

$$v_2(2l + k \cdot 2^\mu + j + 1) = v_2(j + 1) = v_2(k \cdot 2^\mu + j + 1). \quad (3.7)$$

Using these facts and (3.1), we obtain

$$\begin{aligned} v_2(A_{l,l+k \cdot 2^\mu + j + 1}) - v_2(A_{l,l+k \cdot 2^\mu + j}) &= v_2(2l + k \cdot 2^\mu + j + 1) - v_2(k \cdot 2^\mu + j + 1) \\ &= v_2(j + 1) - v_2(j + 1) = 0 \end{aligned}$$

for consecutive values in  $C_{k,l}$ . This proves part (1). To prove part (2), it suffices to take elements  $l + k \cdot 2^\mu + 2^\mu - 1 \in C_{k,l}$  and  $l + (k + 1) \cdot 2^\mu \in C_{k+1,l}$  and compare their 2-adic values. Again by (3.1), we have

$$\begin{aligned} v_2(A_{l,l+(k+1) \cdot 2^\mu}) - v_2(A_{l,l+(k+1) \cdot 2^\mu - 1}) &= v_2(2l + (k + 1) \cdot 2^\mu) - v_2((k + 1) \cdot 2^\mu) \\ &= \mu + v_2(2l \cdot 2^{-\mu} + k + 1) - \mu - v_2(k + 1) \\ &= v_2(2l \cdot 2^{-\mu} + k + 1) - v_2(k + 1) \neq 0. \end{aligned}$$

The last step follows from  $2l \cdot 2^{-\mu}$  being odd and thus  $2l \cdot 2^{-\mu} + k + 1$  and  $k + 1$  having opposite parities. This completes the proof.  $\square$

**Theorem 3.5.** For each  $l \geq 1$ , the set  $\{v_2(A_{l,m}); m \geq l\}$  is an  $s$ -simple sequence, with  $s = 2^{1+v_2(l)}$ .

**Proof.** From Lemmas 3.3 and 3.4, we know that  $v_2(\cdot)$  maintains a constant value on each of the disjoint sets  $C_{k,l}$ . The length of each of these blocks is  $2^{1+v_2(l)}$ .  $\square$

#### 4. The algorithm and its combinatorial interpretation

In this section we describe an algorithm that extracts from the sequence  $X(1) := \{v_2(A_{1,m}); m \geq 1\}$  its combinatorial information. We begin with the definition of the operators  $F$  and  $T$  mentioned in the Introduction.

**Definition 4.1** (The maps  $F$  and  $T$ ). These are defined by

$$F(\{a_1, a_2, a_3, \dots\}) := \{a_1, a_1, a_2, a_3, \dots\}, \quad (4.1)$$

and

$$T(\{a_1, a_2, a_3, \dots\}) := \{a_1, a_3, a_5, a_7, \dots\}. \quad (4.2)$$

We employ the notation

$$c := \{v_2(m); m \geq 1\} = \{0, 1, 0, 2, 0, 1, 0, 3, 0, \dots\}. \quad (4.3)$$

**The algorithm:**

- (1) Start with the sequence  $X(l) := \{v_2(A_{l,l+m-1}); m \geq 1\}$ .
- (2) Find  $n \in \mathbb{N}$  so that the sequence  $X(l)$  is  $2^n$ -simple. Define  $Y(l) := T^n(X(l))$ . At the initial stage, Theorem 3.5 ensures that  $n = 1 + v_2(l)$ .
- (3) Introduce the shift  $Z(l) := Y(l) - c$ .
- (4) Define  $W(l) := F(Z(l))$ .

If  $W(l)$  is a constant sequence, then STOP; otherwise go to step (2) with  $W$  instead of  $X$ . Define  $X_k(l)$  as the new sequence at the end of the  $(k-1)$ th cycle of this process, with  $X_1(l) = X(l)$ .

Section 5 contains the justification for the steps of this algorithm. In particular, we prove that the sequences  $X_k(l)$  have a block structure, so they can be used back in step (1) after each cycle. Theorem 5.3 states that the algorithm finishes in a finite number of steps and that  $W(l)$  is essentially  $X(j)$ , for some  $j < l$ .

**Definition 4.2.** Let  $\omega(l)$  be the number of cycles required for the algorithm to yield a constant sequence and denote by  $n_j$  the integers appearing in step (2) of the algorithm. The integer vector

$$\Omega(l) := \{n_1, n_2, n_3, \dots, n_{\omega(l)}\} \quad (4.4)$$

is called the *reduction sequence* of  $l$ . The number  $\omega(l)$  will be called the *reduction length* of  $l$ . The constant sequence obtained after  $\omega(l)$  cycles is called the *reduced constant*.

In Corollary 5.8 we enumerate  $\omega(l)$  as the number of ones in the binary expansion of  $l$ . Therefore the algorithm yields a constant sequence in a finite number of steps. In fact, the algorithm terminates after  $O(\log_2(l))$  cycles as will follow directly from Corollary 5.8. Table 1 shows the results of the algorithm for  $4 \leq l \leq 15$ .

We now provide a combinatorial interpretation of  $\Omega(l)$ . This requires the composition of the index  $l$ .

**Table 1**Reduction sequence for  $1 \leq l \leq 15$ 

$l$	Binary form	$\Omega(l)$
4	100	3
5	101	1, 2
6	110	2, 1
7	111	1, 1, 1
8	1000	4
9	1001	1, 3
10	1010	2, 2
11	1011	1, 1, 2
12	1100	3, 1
13	1101	1, 2, 1
14	1110	2, 1, 1
15	1111	1, 1, 1, 1

**Definition 4.3.** Let  $l \in \mathbb{N}$ . The *composition* of  $l$ , denoted by  $\Omega_1(l)$ , is defined as follows: write  $l$  in binary form. Read the sequence from right to left. The first part of  $\Omega_1(l)$  is the number of digits up to and including the first 1 read in the corresponding binary sequence; the second one is the number of additional digits up to and including the second 1 read, and so on.

**Example 4.4.** Reading off the values from Table 1, we obtain  $\Omega_1(13) = \{1, 2, 1\}$  and  $\Omega_1(14) = \{2, 1, 1\}$ . Therefore  $\Omega_1(13) = \Omega(13)$  and  $\Omega_1(14) = \Omega(14)$ . Corollary 5.6 shows that this is always true.

The next result describes the formation of  $\Omega_1(l)$  from  $\Omega_1(\lfloor l/2 \rfloor)$ .

**Lemma 4.5.** Given the values of  $\Omega_1(l)$  for  $2^j \leq l \leq 2^{j+1} - 1$ , the list for  $2^{j+1} \leq l \leq 2^{j+2} - 1$  is formed according to the following rule:

$l$  is even: add 1 to the first part of  $\Omega_1(l/2)$  to obtain  $\Omega_1(l)$ ;

$l$  is odd: prepend a 1 to  $\Omega_1(\frac{l-1}{2})$  to obtain  $\Omega_1(l)$ .

**Proof.** Let  $x_1x_2 \cdots x_t$  be the binary representation of  $l$ . Then  $x_1x_2 \cdots x_t0$  corresponds to  $2l$ . Thus, the first part of  $\Omega_1(2l)$  is increased by 1, due to the extra 0 on the right. The relative position of the remaining 1s stays the same. A similar argument takes care of  $\Omega_1(2l+1)$ . The extra 1 that is placed at the end of the binary representation gives the first 1 in  $\Omega_1(2l+1)$ .  $\square$

We now relate the 2-adic valuation of  $A_{l,m}$  to that of  $A_{\lfloor l/2 \rfloor, m}$ .

**Proposition 4.6.** Let

$$\lambda_l := \frac{1 - (-1)^l}{2}, \quad M_0 := \left\lfloor \frac{m + \lambda_l}{2} \right\rfloor. \quad (4.5)$$

Then

$$v_2(A_{l,m}) = 2l - \lfloor l/2 \rfloor + \lambda_l v_2(M_0 - \lfloor l/2 \rfloor) + v_2(A_{\lfloor l/2 \rfloor, M_0}). \quad (4.6)$$

**Proof.** We present the details for  $v_2(A_{2l,2m})$ . Theorem 2.1 gives

$$\begin{aligned}
 v_2(A_{2l,2m}) &= v_2((2m - 2l + 1)_{4l}) + 2l \\
 &= v_2((2m - 2l + 1)(2m - 2l + 2) \cdots (2m + 2l - 1)(2m + 2l)) + 2l \\
 &= v_2(2^{2l}(m - l + 1)(m - l + 2) \cdots (m + l)) + 2l \\
 &= 4l + v_2((m - l + 1)_{2l}) \\
 &= 3l + v_2(A_{l,m}).
 \end{aligned}$$

A similar calculation shows that

$$v_2(A_{2l+1,2m}) = 3l + 2 + v_2(A_{l,m}) + v_2(m - l). \quad (4.7)$$

The general case then follows from Theorem 3.1.  $\square$

**Corollary 4.7.** *The 2-adic valuation of  $A_{l,m}$  satisfies*

$$v_2(A_{l,m}) = 2l + v_2(l!) + \sum_{k \geq 0} \lambda_{\lfloor l/2^k \rfloor} v_2(M_k - \lfloor l/2^{k+1} \rfloor) \quad (4.8)$$

where

$$M_k = \left\lfloor \frac{m + \lambda_l + 2\lambda_{\lfloor l/2 \rfloor} + \cdots + 2^k \lambda_{\lfloor l/2^k \rfloor}}{2^{1+k}} \right\rfloor = \left\lfloor \frac{m + \sum_{n=0}^k 2^n \lambda_{\lfloor l/2^n \rfloor}}{2^{1+k}} \right\rfloor. \quad (4.9)$$

**Proof.** This is a repeated application of Proposition 4.6. The first term results from

$$\sum_{k \geq 0} \left( 2 \left\lfloor \frac{l}{2^k} \right\rfloor - \left\lfloor \frac{l}{2^{k+1}} \right\rfloor \right) = 2l + \sum_{k \geq 1} \left\lfloor \frac{l}{2^k} \right\rfloor = 2l + v_2(l!). \quad \square$$

## 5. Verification of the algorithm and the reduction sequence

In this section we show that the algorithm presented in Section 4 terminates after a finite numbers of cycles. Moreover, we prove that  $\Omega(l)$ , the reduction sequence of  $l$ , is identical to the composition sequence of  $l$ .

**Notation.** The constant sequences will be denoted by  $(t) = \{t, t, t, \dots\}$ .

**Definition 5.1.** A sequence  $(a) = \{a_1, a_2, a_3, \dots\}$  is a *translate* of  $(b) = \{b_1, b_2, b_3, \dots\}$  if  $(a) = (b) + (t)$ , for some constant sequence  $(t)$ . Addition of sequences is performed term by term.

We first consider the base case  $l = 1$ .

**Lemma 5.2.** *The initial case  $l = 1$  satisfies*

$$W(1) = F(T(X(1)) - c) = (2), \quad (5.1)$$

where  $(c)$  is given in (4.3).

**Proof.** Since  $v_2(A_{1,m}) = v_2(m(m+1)) + 1$  and  $v_2(2m-1) = 0$ , we have

$$T(X(1)) = \{v_2((2m-1)(2m)) + 1 : m \geq 1\} = \{v_2(m) + 2 : m \geq 1\} = c + (2).$$

Then the assertion follows from  $F((t)) = (t)$  for a constant  $(t)$ .  $\square$

**Theorem 5.3.** *The algorithm terminates after finitely many iterations. Furthermore, in each cycle,  $W(l)$  is a translate of  $X(j)$ , for some  $j < l$ .*

**Proof.** Start by rewriting the terms in  $X(l)$  as

$$v_2\left(\frac{(m-1+2l)!}{(m-1)!}\right) + l = v_2((m-1+2l)(m-2+2l) \cdots (m+1)m) + l, \quad m \geq 1.$$

Then, the operator  $T$  acts on these to yield (for  $m \geq 1$ )

$$\begin{aligned} v_2((2m-2+2l)(2m-3+2l)\cdots(2m)(2m-1)) + l &= v_2((m-1+l)\cdots(m)) + 2l \\ &= v_2\left(\frac{(m-1+l)!}{(m-1)!}\right) + 2l. \end{aligned} \quad (5.2)$$

**Case I:**  $l$  is even. From (5.2), we can easily obtain the relation

$$T(X(l)) = \left\{ v_2\left(\frac{(m-1+l)!}{(m-1)!}\right) + l/2 + t : m \geq 1 \right\} = X(l/2) + (t), \quad t = 3l/2.$$

**Case II:**  $l$  is odd. Upon subtracting the sequence  $c = \{v_2(m) : m \geq 1\}$  from (5.2) we get that

$$v_2\left(\frac{(m+l-1)!}{m!}\right) + 2l = v_2\left(\frac{(m+l-1)!}{m!}\right) + \frac{l-1}{2} + \frac{3(l-1)}{2} + 2,$$

for  $m \geq 1$ . Then, apply the operator  $F$  to the last sequence and find

$$W(l) = \left\{ v_2\left(\frac{(m-2+l)!}{(m-1)!}\right) + \frac{l-1}{2} + t : m \geq 1 \right\} = X\left(\frac{l-1}{2}\right) + (t), \quad t = (3l+1)/2.$$

Here, we have utilized the property that  $v_2(r!) = v_2((r-1)!)$ , when  $r \geq 1$  is odd. This justifies that the first term augmented in the sequence, as a result of the action of  $F$ , coincides with the next term (these are values at  $m=1$  and  $m=2$ , respectively).

We can now conclude that in either of the two cases (or a combination thereof), the index  $l$  shrinks dyadically. Thus the reduction algorithm must end in a finite step into a translate of  $X(1)$ . Since Lemma 5.2 handles  $X(1)$ , the proof is completed.  $\square$

**Corollary 5.4.** For general  $k \in \mathbb{N}$ , the sequence  $X_k(l)$  is  $2^{n_k}$ -simple for some  $n_k \in \mathbb{N}$ .

**Theorem 5.5.** Let  $\{k_1, \dots, k_n : 0 \leq k_1 < k_2 < \dots < k_n\}$ , be the unique collection of distinct nonnegative integers such that

$$l = \sum_{i=1}^n 2^{k_i}. \quad (5.3)$$

Then the reduction sequence  $\Omega(l)$  of  $l$  is  $\{k_1 + 1, k_2 - k_1, \dots, k_n - k_{n-1}\}$ .

**Proof.** The argument of the proof is to check that the rules of formation for  $\Omega_1(l)$  also hold for the reduction sequence  $\Omega(l)$ . The proof is divided according to the parity of  $l$ . The case  $l$  odd starts with  $l=1$ , where the block length is 2. From Theorem 2.1 we obtain a constant sequence after iterating the algorithm once. Thus the algorithm terminates and the reduction sequence for  $l=1$  is  $\Omega(1) = \{1\}$ .

Now consider the general even case:  $X(2l)$ . Theorem 5.3 shows that applying  $T$  to this sequence yields a translate of  $X(l)$ . This does not affect the reduction sequence  $\Omega(l)$ , but the doubling of block length increases the first term of  $\Omega(l)$  by 1. Therefore

$$\Omega(2l) = \{k_1 + 2, k_2 - k_1, \dots, k_n - k_{n-1}\}. \quad (5.4)$$

This is precisely what happens to the binary digits of  $l$ : if

$$l = \sum_{i=1}^n 2^{k_i}, \quad \text{then } 2l = \sum_{i=1}^n 2^{k_i+1}.$$

This concludes the argument for even indices.

For the general odd case,  $X(2l+1)$ , we apply  $T$ , subtract  $c$  and then apply  $F$ . Again, by Theorem 5.3, this gives us a translate of  $X(l)$ . We conclude that, if the reduction sequence of  $l$  is

$$\{k_1 + 1, k_2 - k_1, \dots, k_n - k_{n-1}\}, \quad (5.5)$$

then that of  $2l + 1$  is

$$\{1, k_1 + 1, k_2 - k_1, \dots, k_n - k_{n-1}\}. \quad (5.6)$$

This is precisely the behavior of  $\Omega_1$ . The proof is complete.  $\square$

**Corollary 5.6.** *The reduction sequence  $\Omega(l)$  associated to an integer  $l$  is the sequence of compositions of  $l$ , that is,*

$$\Omega(l) = \Omega_1(l). \quad (5.7)$$

**Corollary 5.7.** *The reduced constant is  $2l + v_2(l!) = v_2(A_{l,l})$ .*

**Proof.** In Corollary 4.7, subtract the last term as per the reduction algorithm.  $\square$

**Corollary 5.8.** *The set  $\Omega(l)$  has cardinality*

$$s_2(l) = \text{the number of ones in the binary expansion of } l. \quad (5.8)$$

**Note.** The function  $s_2(l)$  defined in (5.8) has recently appeared in a different divisibility problem. Lengyel [7] conjectured, and De Wannemacker [10] proved, that the 2-adic valuation of the Stirling numbers of the second kind  $S(n, k)$  is given by

$$v_2(S(2^n, k)) = s_2(k) - 1. \quad (5.9)$$

The reader will find in [1] a general study of the 2-adic valuation of Stirling numbers.

## 6. A symmetry conjecture on the graphs of $v_2(A_{l,m})$

The graphs of the function  $v_2(A_{l,m})$ , where we take every other  $2^{1+v_2(l)}$ -element to reduce the repeating blocks to a single value, are shown in the next figures. We conjecture that these graphs have a symmetry property generated by what we call an *initial segment* from which the rest is determined by adding a *central piece* followed by a *folding rule*. We conclude with sample pictures of this phenomenon.

**Example 6.1.** For  $l = 1$ , the first few values of the reduced table are

$$\{2, 3, 2, 4, 2, 3, 2, 5, 2, 3, \dots\}.$$

The ingredients are:

- *initial segment*:  $\{2, 3, 2\}$ ,
- *central piece*: the value at the center of the initial segment, namely 3,
- *rules of formation*: start with the initial segment and add 1 to the central piece and reflect.

This produces the sequence

$$\begin{aligned} \{2, 3, 2\} &\rightarrow \{2, 3, 2, 4\} \rightarrow \{2, 3, 2, 4, 2, 3, 2\} \rightarrow \{2, 3, 2, 4, 2, 3, 2, 5\} \\ &\rightarrow \{2, 3, 2, 4, 2, 3, 2, 5, 2, 3, 2, 4, 2, 3, 2\}. \end{aligned}$$

The details are shown in Fig. 1.

**Remark.** We have found no way to predict the initial segment nor the central piece. Fig. 2 shows the beginning of the case  $l = 9$ . From here one could be tempted to anticipate that this graph extends as in the case  $l = 1$ . This is not correct however, as can be seen in Fig. 3. In fact, the initial segment is depicted in Fig. 3 and its extension is shown in Fig. 4.

The initial pattern can be quite elaborate. Fig. 5 illustrates the case  $l = 53$  and Fig. 6 shows it for  $l = 59$ . A complete description of these initial segments is open to further exploration.

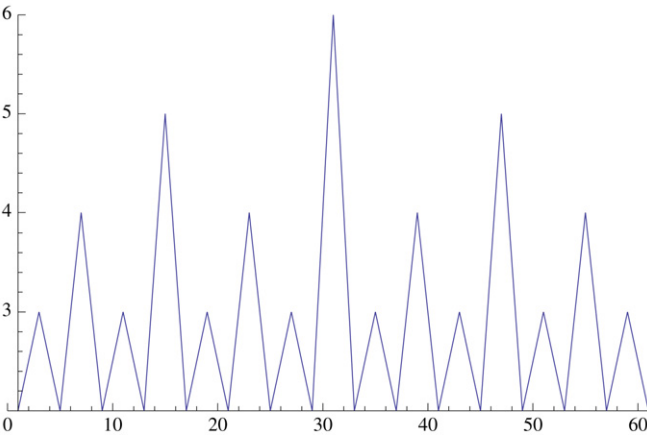


Fig. 1. The 2-adic valuation of  $A_{1,m}$ .

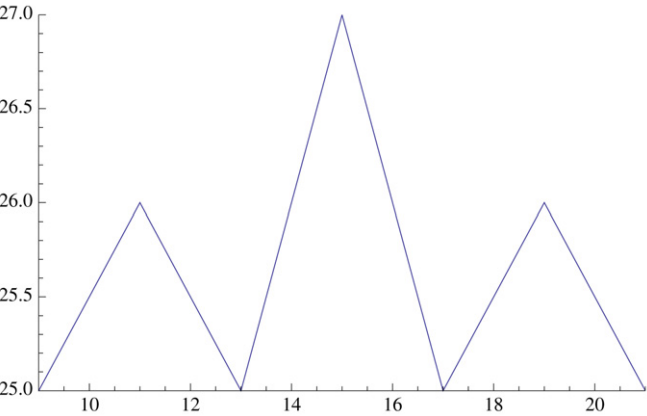


Fig. 2. The beginning for  $l = 9$ .

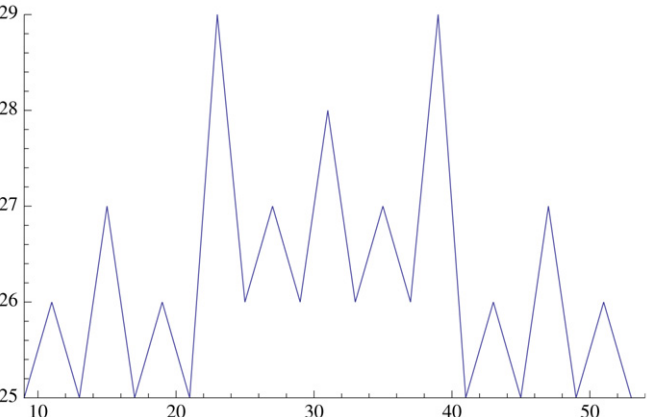


Fig. 3. The continuation of  $l = 9$ .

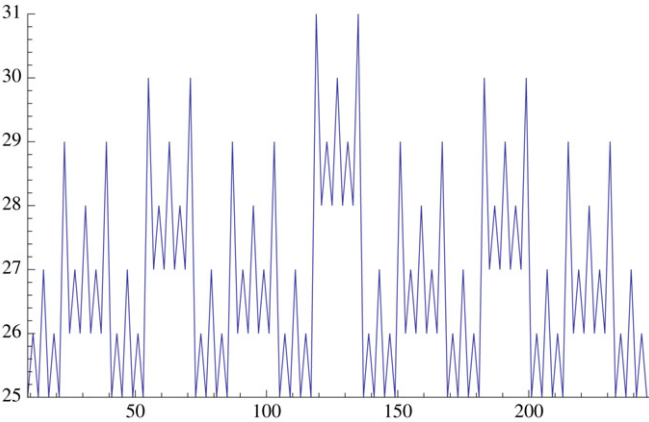


Fig. 4. The pattern for  $l = 9$  persists.

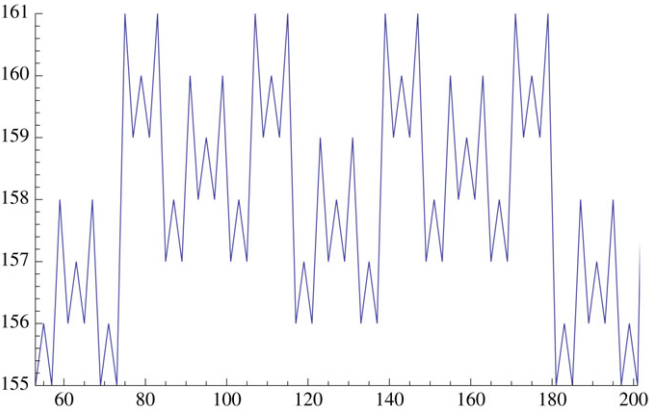


Fig. 5. The initial pattern for  $l = 53$ .

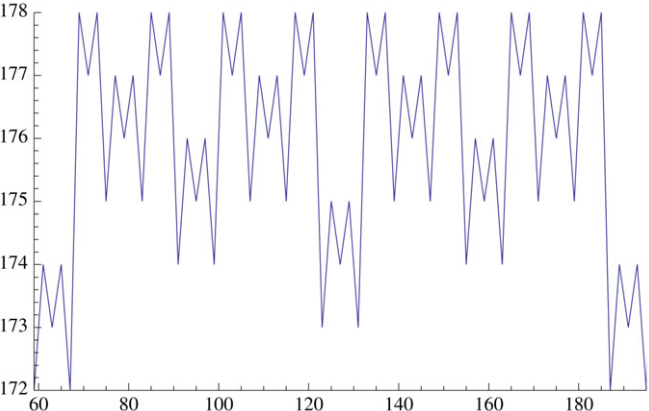


Fig. 6. The initial pattern for  $l = 59$ .

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