Approximations for stochastic differential equations with reflecting convex boundaries

Roger Pettersson

Department of Mathematical Statistics, Lund University, Box 118, S-221 00 Lund, Sweden

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Abstract

We consider convergence of a recursive projection scheme for a stochastic differential equation reflecting on the boundary of a convex domain $G$. If $G$ satisfies Condition (B) in Tanaka (1979), we obtain mean square convergence, pointwise, with the rate $O((\delta \log 1/\delta)^{1/2})$, and if $G$ is a convex polyhedron we obtain mean square convergence, uniformly on compacts, with the rate $O(\delta \log 1/\delta)$. An application is given for stochastic differential equations with hysteretic components.

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1. Introduction

Reflecting stochastic differential equations (RSDEs) are often written in the form

$$d\xi = b \, dt + \sigma \, dB + d\varphi,$$

where $b$ is a drift term, $\sigma$ a dispersion matrix, $B$ a Brownian motion and $\varphi$ reflects $\xi$ to the interior of a given set $G$ if $\xi$ is at the boundary of $G$.

A natural question is: what does a numerical solution for a RSDE look like, and in what sense does it converge? For differential equations without reflecting boundaries there are a variety of different approximations (cf. Kloeden and Platen, 1992). The method proposed in this paper is a recursive projection scheme which is essentially Euler's method forced to remain in the constraining set $G$.

Recently, several authors have also considered numerical methods for RSDEs. Asmussen et al. (1994) considered convergence for different numerical methods if $G = [0, \infty)$ and Lépine (1993) constructed a numerical method if $G$ is a half space in several dimensions. In these, an explicit expression of a solution to the so-called

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Skorohod problem is known. Liu (1993) constructed, for convex $G$, a penalization scheme and another scheme which converges weakly. For a brief comparison between the penalization scheme and the projection scheme see Section 4. Słomiński (1994) considered, for various types of sets $G$, two types of numerical schemes of which one was discussed by Asmussen et al. (1994) and Lépine (1993) and the other one coincides with that discussed in this paper. Also Chitashvili and Lazrieva (1981) and Kinkladze (1983) considered this method, in which $G$ was an interval in the real line and $[0, \infty) \times \mathbb{R}^{d-1}$, respectively.

In this paper, we let $G$ be a convex set satisfying Condition (B) in Tanaka (1979). We consider in particular a convex polyhedron for $G$ (a finite intersection of closed half spaces) since this is the case in several important applications, such as mechanical systems with hysteresis (see Section 3). Further, we shall use a strong result for the so-called Skorohod problem on such sets (Lemma 5.1), obtained by Dupuis and Ishii (1991).

For convex $G$ satisfying Condition (B) we show convergence of the projection scheme in the sense of mean square convergence, pointwise, and obtain the rate $O((\delta \log 1/\delta)^{1/2})$, where $\delta$ is the mesh of the stepsize. A related result was given by Słomiński (1994), who obtained the slower rate $O(\delta^{1/2-\varepsilon})$ for all $\varepsilon > 0$, but with mean square, uniformly on compact sets. We also note that in a certain product situation we obtain the rate $O(\sqrt{\delta})$ in mean square convergence, pointwise. If $G$ is a convex polyhedron we consider convergence in mean square, uniformly on compacts, and obtain the rate $O(\delta \log 1/\delta)$. This rate is slightly faster than indicated by Słomiński (1994) who obtained the rate $O(\delta^{1-\varepsilon})$ for all $\varepsilon > 0$.

In the next section we give a short introduction to the Skorohod problem and RSDEs. In Section 3 we demonstrate a direct application to polylinear hysteresis. In Section 4 we formulate the projection scheme and give convergence results if $G$ is a convex set satisfying Condition (B). In the fifth section a stronger convergence result is shown if $G$ is a convex polyhedron. We also include an Appendix proving an expected version of a modulus of continuity result for the Brownian motion.

2. Skorohod problem and RSDE

Throughout this section let $G$ be an arbitrary closed and convex set in $\mathbb{R}^d$. For $x \in \mathbb{R}^d$, let $\Pi_G(x)$ be the projection of $x$ on $G$, i.e. the point in $G$ which is closest to $x$. If $x \in \partial G$ let $\mathcal{N}(x)$ be the set of inwards directed normals for $G$ at $x$.

$$\mathcal{N}(x) = \{ \gamma \in \mathbb{R}^d : |\gamma| = 1 \text{ and } \langle x - y, \gamma \rangle \leq 0 \forall y \in G \}.$$  \hspace{1cm} (2)

For given $T > 0$ let $D([0,T],\mathbb{R}^d)$ be the set of right continuous functions from $[0,T]$ into $\mathbb{R}^d$ with left limits (the set of càdlàgs) and $C([0,T],G)$ the set of continuous functions from $[0,T]$ into $G$. For a function $\eta$ in $D([0,T],\mathbb{R}^d)$ let $|\eta|(t)$ be the variation of $\eta$ on the interval $[0,t]$. Denote by $\langle \cdot, \cdot \rangle$ the usual inner product in $\mathbb{R}^d$. 


Definition 2.1 (Skorohod problem). Given $w \in D([0,T], \mathbb{R}^d)$ with $w(0) \in G$, the triple $(\xi, w, \varphi)$ is said to be associated on $G$ or solve the Skorohod problem if

(i) $\xi(t) = w(t) + \varphi(t)$, $\xi(0) = w(0)$,

(ii) $\xi(t) \in G$,

(iii) $|\varphi(t)| < \infty$,

(iv) $|\varphi(t)| = \int_{[0,t]} I_{\{\xi(s) \in \partial G\}} d|\varphi|(s)$,

(v) $\varphi(t) = \int_{[0,t]} \gamma(s) d|\varphi|(s)$, where $\gamma(s) \in N(\xi(s))$ (d|\varphi| -a.e.)

for $t \in [0, T]$.

Example 2.2. Given a right continuous step function $w$ with $w(0) \in G$ and jumps at $t_1 < \cdots < t_n$, where $t_1 > t_0 = 0$ and $t_n \leq T$. Then a well known result (see e.g. Saisho, 1987, Remark 1.4) is that $(\xi, w, \varphi)$ is an associated triple if $\xi(t) = w(0)$ for $0 \leq t < t_1$,

$$\xi(t) = \Pi_G(\xi(t_{k-1}) + \Delta w(t_k)), \quad t_k \leq t < t_{k+1}$$

for $k \geq 1$ where $\Delta w(t_k) = w(t_k) - w(t_{k-1})$ and $\varphi(\cdot) := \xi(\cdot) - w(\cdot)$.

Let $(\Omega, \mathcal{F}, P)$ be a probability space with filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions. Let $\{B(t)\}_{t \geq 0}$ be an $\mathcal{F}_t$-adapted $m$-dimensional Brownian motion. Suppose $G$ is given and let $x_0$ be an arbitrary fixed point in $G$.

Definition 2.3. Given maps $b: [0, T] \times \mathbb{R}^d \mapsto \mathbb{R}^d$ and $\sigma: [0, T] \times \mathbb{R}^d \mapsto \mathbb{R}^d \times \mathbb{R}^m$, a solution to the RSDE

$$d\xi(t) = b(t, \xi(t)) dt + \sigma(t, \xi(t)) dB(t), \quad \xi(0) = x_0,$$

is any triple $(\xi, w, \varphi)$ which is (a.s.) associated on $G$ with

$$w(t) = x_0 + \int_0^t b(s, \xi(s)) ds + \int_0^t \sigma(s, \xi(s)) dB(s),$$

interpreted in the sense of Itô.

For convenience, we identify a solution to (3) by its first component $\xi$. We shall assume $G$ is satisfying the following condition (Tanaka, 1979, Condition (B)).

Condition (B). $G$ is a closed convex domain with the property that there exist $\varepsilon > 0$ and $\delta > 0$ such that $\forall x \in \partial G$ one can find a point $y_x$ and an open ball $\{y \in \mathbb{R}^d: |y - y_x| < \varepsilon\}$ contained in $G$ where $|y_x - x| \leq \delta$.

It is well known that Condition (B) is satisfied for any closed and convex set $G$ with nonempty interior if $G$ is bounded or $d \leq 2$ (see Tanaka, 1979, p 170); this is also true if $G$ is a convex polyhedron. The following theorem contains two important results from Tanaka (1979) Theorem 2.1(i) and Theorem 4.1, respectively.

Theorem 2.4. Assume $G$ satisfies Condition (B).

(a) Then for any continuous function $w$ there exists a unique triple $(\xi, w, \varphi)$ which solves the Skorohod problem on $G$. 
(b) If the drift term $b(t,x)$ and dispersion matrix $\sigma(t,x)$ are Borel measurable in $(t,x)$ and are Lipschitz continuous and satisfies a linear growth condition in $x$, uniformly in $t$, then there exists a pathwise unique solution $\zeta$ to the RSDE (3) reflecting on the boundary of $G$. Furthermore, the solution $\zeta$ is (a.s.) continuous.

3. Polylmear hysteresis and RSDE

In this section we give an application of RSDEs that is useful for seismic reliability analysis. The constraining set $G$ is $\mathbb{R}^2 \times [-1,1]$.

When a structure is exposed to stress, the structure may be deformed. For small stress the material shows an approximately elastic behaviour, but when the stress is larger, a nonelastic phenomenon, plastic or imperfectly plastic behaviour may occur: the deformations then will be larger and permanent. This phenomenon is an example of hysteresis. In this section we assume the stress is in the form of random excitation. We study in particular polylmear hysteresis subjected to Gaussian white noise. Using a Markovianization technique (see e.g. Krée and Soize, 1982) it is possible to extend our approach to a substantially wider class of Gaussian processes. Polylmear and especially bilinear hysteresis are the simplest and most widely used hysteretic models.

Example 3.1 (Bilinear hysteresis and earthquake). Let $-\dot{B}$ be an acceleration in a given direction generated by an earthquake and assume that is modelled by a Gaussian white noise. Assume this acceleration influences a structure to move $x$ units in this direction. Let $-F$ be the restoring force produced by the deformation of the structure. If we take the mass equal to one, the equation of motion can be written as

$$\ddot{x} + 2c\dot{x} + F = \dot{B}, \quad x(0) = \dot{x}(0) = F(0) = 0,$$

where the constant $c > 0$ characterizes the structural damping. Usually, the restoring force has hysteretic properties. Here we shall use a bilinear hysteresis model. In this case it is common (cf. Suzuki and Minai, 1988) to write $F$ in the following form:

$$F = \alpha x + (1 - \alpha)z,$$

where $\alpha$ is a fixed constant in $[0,1]$ and $z$ an absolutely continuous function with derivative $\dot{z}$ given by

$$\dot{z} = \begin{cases} \dot{x}I_{\{\dot{x} < 0\}}, & z = 1, \\ \dot{x}, & -1 < z < 1, \\ \dot{x}I_{\{\dot{x} > 0\}}, & z = -1. \end{cases}$$

Hence, the units have been chosen so that the elasticity constant $dF/dx$ equals one if $\alpha x - (1 - \alpha) < F < \alpha x + (1 - \alpha)$ (i.e. $|z| < 1$) and the structure is subjected to a plastic deformation if $F = \alpha x + (1 - \alpha)$ (i.e. $z = 1$) and $\dot{x} > 0$ or $F = \alpha x - (1 - \alpha)$ (i.e. $z = -1$) and $\dot{x} < 0$. It is convenient to rewrite Eq. (4) with conditions (5) and (6) as a first degree system. Put $\xi = [x,\dot{x},z]'$. 
Fig. 1. (a) simulation of $x^h$ and $F^h$ (with $z = 0.1$) given by the projection scheme (11); (b) corresponding simulation of $t$ and $|\phi^g(t)|$.

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -\alpha & -2c & -(1 - \alpha) \\ 0 & 1 & 0 \end{bmatrix},$$

$b = A\xi$ and $\sigma = [0, 1, 0]'$ where prime denotes transpose; then we have

$$\dot{\xi} = b(\xi) + \sigma \dot{B} + \phi, \quad \xi(0) = 0,$$

(7)

where $\phi = [0, 0, \phi_3]'$ with

$$\phi_3 = \begin{cases} -\dot{x}_I(x \geq 0), & z = 1, \\ 0, & |z| < 1, \\ -\dot{x}_I(x < 0), & z = -1. \end{cases}$$

(8)

It is not evident that there exists a solution to (7) and (8), but if we interpret (7) with condition (8) as an RSDE of the form (3) on $G = \mathbb{R}^2 \times [-1, 1]$, where $x_0 = 0$, both existence and uniqueness of a solution is ascertained.

Another interesting approach, which we do not consider further here, is to identify (7) and (8) as a multivalued stochastic differential equation (cf. Krée, 1982).

Often the plastic damage is of main interest. This can be measured by the number of passages of $F$ between the plastic regions or by the variation of $\phi$ (see Fig. 1).

The main problem leading to this note was to find approximate solutions for (7) and (8). Results for this are presented in Section 5.

For the preceding example but with polylinear hysteresis, $F$ is often represented in the following form:

$$F = \alpha_0 x + \sum_{i=1}^{n} \alpha_i z_i,$$
where $\sum_{i=1}^{n} z_i = 1, z_i \geq 0$ and

$$\dot{z}_i = \begin{cases} \dot{x}_i < 0, & z_i = \delta_i \\ \dot{x}_i, & -\delta_i < z < \delta_i \\ \dot{x}_i > 0, & z = -\delta_i, \end{cases} \quad (9)$$

where $1 = \delta_1 < \cdots < \delta_n$ (cf. Suzuki and Minai, 1988). This can be interpreted as an $(n + 2)$-dimensional RSDE with $\zeta = [x, \dot{x}, z_1, \ldots, z_n]'$, $b = A\zeta$, where

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ -\alpha_0 & -2c & -\alpha_1 & \cdots & -\alpha_n \\ 0 & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 1 & 0 & \cdots & 0 \end{bmatrix},$$

$\sigma = [0, 1, 0, \ldots, 0]'$ and $G = \mathbb{R}^2 \times [-1, 1] \times [-\delta_2, \delta_2] \times \cdots \times [-\delta_n, \delta_n]$. Obviously, $G$ is a convex polyhedron.

4. Numerical approximation for RSDEs

In this section we consider a numerical method given by an iterative projection scheme. We assume, that $G$ satisfies Condition (B). For convenience, we also assume that the drift vector $b$ and dispersion matrix $\sigma$ are maps $\mathbb{R}^d \mapsto \mathbb{R}^d$ and $\mathbb{R}^d \mapsto \mathbb{R}^d \times \mathbb{R}^m$, respectively, i.e. not depending on $t$, and such that

$$|b(x) - b(y)| \vee |\sigma(x) - \sigma(y)| \leq L|x - y| \quad (10)$$

for $x, y \in \mathbb{R}^d$ where $|b| = \left(\sum_{i=1}^{d} b_i^2\right)^{1/2}$ and $|\sigma| = \left(\sum_{i,j=1}^{d,m} \sigma_{ij}^2\right)^{1/2}$. We observe that this implies, by Theorem 2.4(b), existence and pathwise uniqueness of a solution to (3).

We consider the following numerical method. Let $x_0$ be the initial point for the RSDE (3) and $\{0 = t_1 < t_2 < \cdots < t_{c_6} = T\}$ be a partition of $[0, T]$ with mesh $\delta = \max\{\Delta t_k; 1 \leq k \leq c_5\}$, where $\Delta t_k = t_k - t_{k-1}$. Define $\zeta^\delta(t) := x_0$ for $0 \leq t < t_1$ and for $t_k \leq t < t_{k+1}$, where $k \geq 1$,

$$\zeta^\delta(t) := \Pi_G\{\zeta^\delta(t_{k-1}) + b(\zeta^\delta(t_{k-1})) \Delta t_k + \sigma(\zeta^\delta(t_{k-1})) \Delta B(t_k)\}, \quad (11)$$

where $\Delta B(t_k) = B(t_k) - B(t_{k-1})$. If $G = \mathbb{R}^d$ then (11) at the points $t_1, t_2, \ldots$ coincides with the usual Euler's method for stochastic differential equations. Note that as in Example 2.2, $(\zeta^\delta, w^\delta, \varphi^\delta)$ is an associated triple, with $w^\delta(t) = x_0$ for $0 \leq t < t_1$ and

$$w^\delta(t) = w^\delta(t_{k-1}) + b(\zeta^\delta(t_{k-1})) \Delta t_k + \sigma(\zeta^\delta(t_{k-1})) \Delta B(t_k) \quad \text{for} \quad t_k \leq t < t_{k+1}, \quad \text{where} \quad k \geq 1 \quad \text{and} \quad \varphi^\delta(\cdot) = \zeta^\delta(\cdot) - w^\delta(\cdot).$$

The main tool in this section is the following lemma from which we obtain convergence of $\zeta^\delta$ to $\zeta$ as $\delta \downarrow 0$. It also reveals some of the connections between a solution of a reflecting stochastic differential equation and related projection scheme.
Lemma 4.1. Let \( \zeta^0 \) be given by (11) and assume \( \xi \) is a solution to (3). Then

\[
|\zeta^0(t_n) - \xi(t_n)|^2 \leq \left| \sum_{k=1}^{n} \sigma(\zeta^0(t_{k-1})) \triangle B(t_k) - \int_0^{t_n} \sigma(\xi(s)) dB(s) \right|^2
\]

\[
+ 2 \sum_{k=1}^{n} \left( \zeta^0(t_k) - \xi(t_k), b(\zeta^0(t_{k-1})) \right) \triangle t_k - 2 \int_0^{t_n} \left( \zeta^0(s) - \xi(s), b(\xi(s)) \right) ds
\]

\[
+ 2 \sum_{k=1}^{n} \left( \sum_{j=k+1}^{n} \sigma(\zeta^0(t_{j-1})) \triangle B(t_j) - \int_{t_k}^{t_n} \sigma(\xi(s)) dB(s), b(\zeta^0(t_{k-1})) \right) \triangle t_k
\]

\[
+ 2 \sum_{k=1}^{n} \left( \sum_{j=k+1}^{n} \sigma(\zeta^0(t_{j-1})) \triangle B(t_j) - \int_{t_k}^{t_n} \sigma(\xi(s)) dB(s), \triangle \phi^0(t_k) \right)
\]

\[
- 2 \int_0^{t_n} \left( \sum_{k:s < t_k \leq t_n} \sigma(\zeta^0(t_{k-1})) \triangle B(t_k) - \int_{t}^{t_n} \sigma(\zeta(s)) dB(u), b(\zeta(s)) \right) ds
\]

\[
- 2 \int_0^{t_n} \left( \sum_{k:s < t_k \leq t_n} \sigma(\zeta^0(t_{k-1})) \triangle B(t_k) - \int_{t}^{t_n} \sigma(\zeta(u)) dB(u), d\phi(s) \right).
\]

Furthermore,

\[
|\zeta^0(t_k) - x_0|^2 \leq |\zeta^0(t_{k-1}) - x_0|^2 + |\sigma(\zeta^0(t_{k-1})) \triangle B(t_k)|^2
\]

\[
+ 2 \left( \zeta^0(t_{k-1}) - x_0, \sigma(\zeta^0(t_{k-1})) \right) \triangle B(t_k)
\]

\[
+ 2 \left( \zeta^0(t_k) - x_0, b(\zeta^0(t_{k-1})) \right) \triangle t_k.
\]

\[
|\zeta^0(t_k) - \zeta^0(t_{k-1})|^2 \leq |\sigma(\zeta^0(t_{k-1})) \triangle B(t_k)|^2
\]

\[
+ 2 \left( \zeta^0(t_k) - \zeta^0(t_{k-1}), b(\zeta^0(t_{k-1})) \right) \triangle t_k
\]

and

\[
|\zeta(t) - \zeta(s)|^2 \leq \left| \int_s^t \sigma(\zeta(u)) dB(u) \right|^2
\]

\[
+ 2 \int_s^t \left( \zeta(u) - \zeta(s), b(\zeta(u)) \right) du
\]

\[
+ 2 \int_s^t \left( \int_u^t \sigma(\zeta(v)) dB(v), b(\zeta(u)) du + d\phi(u) \right)
\]

for \( 0 \leq s \leq t. \)

Proof. The inequalities (12) and (15) follows by noting that \((\zeta^0, \omega^0, \phi^0)\) and \((\zeta, \omega, \phi)\) are associated triples and using Remark 2.2 in Tanaka (1979) carefully.

The inequalities (13) and (14) are easily verified by modifying the arguments for Tanaka (1979), Remark 2.2.

Inequality (13) implies, with elementary computations, that if \( b \) and \( \sigma \) satisfy the Lipschitz condition (10), then \( \max \{ E|\zeta^0(t_k) - x_0|^2 : 1 \leq k \leq n \} \) is uniformly bounded.
for all small $\delta$. Similarly, it follows (with $s = 0$) that $\sup \{E[|z(t) - x_0|^2]: 0 \leq t \leq T\}$ is finite from (15). This is sufficient for our purposes. However, by Słomiński (1994, pp. 206-207), we in fact have

$$\sup_{0 < \delta \leq T} E \max_{1 \leq k \leq c_0} |z^\delta(t_k) - x_0|^2 < \infty$$

and

$$E \sup_{0 \leq t \leq T} |z(t) - x_0|^2$$

We also need continuity results for $z^\delta$ and $z$.

Lemma 4.2. Assume $b$ and $\sigma$ satisfy the Lipschitz condition (10). Let $z^\delta$ be given by (11) and assume $z$ is a solution to (3). Then there exists a constant $C = C_T$ such that

$$\sup_{0 < \delta \leq T} E|z^\delta(t_k) - z^\delta(t_{k-1})|^2 \leq C \triangle t_k.$$  (18)

and

$$E|z(t) - z(s)|^2 \leq C(t - s)$$  (19)

for $0 \leq s \leq t \leq T$.

Proof. We show (18) first. By taking expectations of both the sides of inequality (14), we obtain

$$E|z^\delta(t_k) - z^\delta(t_{k-1})|^2 \leq E(\sigma(z^\delta(t_{k-1})))^2 \triangle t_k$$

$$+ 2E(z^\delta(t_k) - z^\delta(t_{k-1}), b(z^\delta(t_{k-1}))) \triangle t_k.$$  (18)

(18) follows from the Lipschitz conditions of $b$ and $\sigma$ and (16).

The continuity result (19) is shown similarly as (18) by using (15) instead of (14) and (17) instead of (16). $\square$

The main theorem, the convergence rate of $z^\delta$ to $z$, can now be stated. Note that we assume $\sigma$ is bounded.

Theorem 4.3. Assume $G$ satisfies Condition (B) and $b$ and $\sigma$ satisfy the Lipschitz continuity condition (10) and $|\sigma(x)| \leq L$ for $x \in \mathbb{R}^d$. Let $z$ and $z^\delta$ be given by (3) and (11), respectively. Then

$$\sup_{0 \leq t \leq T} E|z^\delta(t) - z(t)|^2 = O\left(\left(\frac{\delta \log \frac{1}{\delta}}{\delta}\right)^{1/2}\right)$$  (20)

for small $\delta$.

In the case, when $G = \mathbb{R}^d$, it is well known that $E|z^\delta(t) - z(t)|^2 = O(\delta)$ (see e.g. Kloeden and Platen, 1992, Theorem 10.2.2).
Under the conditions in Theorem 4.3, Słomiński (1994) showed that
\[ E \sup_{0 \leq r \leq T} |\zeta^\delta(t) - \bar{\zeta}(t)|^2 = O(\delta^{1/2-\epsilon}) \]
for all \( \epsilon > 0 \), which is slower than (20), but the mean square convergence by Słomiński (1994) is uniformly on \([0, T]\).

Liu (1993) suggested the following penalization scheme:
\[
\dot{\zeta}^\delta(t_k) = \dot{\zeta}^\delta(t_{k-1}) + b(\zeta^\delta(t_{k-1})) \Delta t_k + \sigma(\zeta^\delta(t_{k-1})) \Delta B(t_k) \\
- \frac{1}{\lambda} \beta(\zeta^\delta(t_{k-1})) \Delta t_k,
\]
where \( \beta(x) = x - \Pi_G(x) \). If \( \lambda = \sqrt{\delta} \) and \( \Delta t_k = \delta \) then Liu obtained, for bounded \( G \), that \( E|\zeta^\delta(T) - \bar{\zeta}(T)|^2 = O(\delta^{1/2-\epsilon}) \) for all \( \epsilon > 0 \). Observe that the projection scheme (11) can be rewritten as a modified version of (21) with \( \lambda = \Delta t_k \) and \( \beta \) evaluated at \( \zeta^\delta(t_{k-1}) + b(\zeta^\delta(t_{k-1})) \Delta t_k + \sigma(\zeta^\delta(t_{k-1})) \Delta B(t_k) \) instead of at \( \zeta^\delta(t_{k-1}) \).

The convergence rate (20) depends on the following modulus of continuity result.

**Lemma 4.4.** Let \( \{B(t)\} \) be an \( m \)-dimensional Brownian motion and \( \{0 = t_0 < t_1 < \cdots < t_{s_{\delta}} = T\} \) a partition of \([0, T]\) with mesh \( \delta \). Then
\[ E \max_{1 \leq k \leq s_{\delta}} \sup_{t \in [t_{k-1}, t_k]} |B(t) - B(t_{k-1})|^2 = O(\delta \log 1/\delta) \]
for small \( \delta \).

We prove Lemma 4.4 in Appendix.

**Proof of Theorem 4.3.** Throughout this proof let \( C \) be a universal constant. By (18) and (19), and since \( \zeta^\delta \) is constant in \([t_{k-1}, t_k)\), we only need to show the theorem at the grid points \( t_k \). Inequality (12) gives, by the Itô isomorphism and the independent increments of the Brownian motion,
\[ E|\zeta^\delta(t_n) - \bar{\zeta}(t_n)|^2 \leq \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} E|\sigma(\zeta^\delta(t_{k-1})) - \sigma(\zeta(s))|^2 ds \\
+ 2 \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} E \langle \zeta^\delta(t_k) - \zeta(t_k), b(\zeta^\delta(t_{k-1})) \rangle - E \langle \zeta^\delta(s) - \zeta(s), b(\zeta(s)) \rangle ds \\
- 2E \int_{0}^{t_n} \langle \sigma(\zeta^\delta(s))(B(s) - B(s^\delta)), b(\zeta(s)) \rangle ds \\
- 2E \int_{0}^{t_n} \langle \sigma(\zeta^\delta(s))(B(s) - B(s^\delta)), d\varphi(s) \rangle,
\]
where \( s^\delta = \max\{t_k: t_k \leq s\} \). The most critical term in (23) is the last one. It is dominated by
\[ 2L \left( E \max_{0 \leq t \leq T} \sup_{s \in [t_{k-1}, t_k]} |B(s) - B(t_{k-1})|^2 \right)^{1/2} \left( E(\varphi(T))^2 \right)^{1/2}, \]
where the first factor, by Lemma 4.4, is $O((\delta \log 1/\delta)^{1/2})$ for small $\delta$ and the second one, by Słomiński (1994, p. 210), is finite (from Condition (B)).

We now consider the other terms in (23). The first term is, by the Lipschitz assumption of $\sigma$, less or equal to

$$2L^2 \sum_{k=1}^{n} E|\xi^\delta(t_{k-1}) - \zeta(t_{k-1})|^2 \Delta t_k + 2L^2 \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} E|\xi(t_{k-1}) - \zeta(s)|^2 ds,$$

where the second sum, by (19), is $O(\delta)$.

Consider the second term in (23). By the Lipschitz continuity of $b$, and the fact that $\xi^\delta$ is constant at $[t_{k-1}, t_k)$, we have for $t_{k-1} \leq s < t_k$,

$$E\langle \xi^\delta(t_k) - \zeta(t_k), b(\xi^\delta(t_{k-1})) \rangle - E\langle \xi^\delta(s) - \zeta(s), b(\xi(s)) \rangle$$

$$\leq \frac{1}{2} E|\xi^\delta(t_k) - \zeta(t_k)|^2 + \frac{1}{2} E|\xi^\delta(t_{k-1}) - \zeta(s)|^2$$

$$+ (E|\xi^\delta(t_k) - \zeta(t_k)|^2 E|b(\xi(s))|^2)^{1/2} + (E|\xi(t_k) - \zeta(s)|^2 E|b(\xi(s))|^2)^{1/2},$$

where the last two terms are $O(\sqrt{\delta})$ by (18), (19), the Lipschitz condition of $b$ and (17). Further,

$$E|\xi^\delta(t_k) - \zeta(t_k)|^2$$

$$\leq 3E|\xi^\delta(t_k) - \zeta(t_{k-1})|^2 + 3E|\xi^\delta(t_{k-1}) - \zeta(t_{k-1})|^2 + 3E|\xi(t_{k-1}) - \zeta(t_k)|^2$$

$$\leq 3E|\xi^\delta(t_{k-1}) - \zeta(t_{k-1})|^2 + C\delta$$

and

$$E|\xi^\delta(t_{k-1}) - \zeta(s)|^2 \leq 2E|\xi^\delta(t_{k-1}) - \zeta(t_{k-1})|^2 + 2E|\xi(t_{k-1}) - \zeta(s)|^2$$

$$\leq 2E|\xi^\delta(t_{k-1}) - \zeta(t_{k-1})|^2 + C\delta.$$

Hence, the second term in (23) is bounded by

$$C \sum_{k=1}^{n} E|\xi^\delta(t_{k-1}) - \zeta(t_{k-1})|^2 \Delta t_k + C(\delta + \sqrt{\delta}).$$

The third term in (23) is, by the Cauchy–Schwarz Inequality, easily seen to be $O(\sqrt{\delta})$. Thus,

$$E|\xi^\delta(t_\alpha) - \zeta(t_\alpha)|^2 \leq C \sum_{k=1}^{n} E|\xi^\delta(t_{k-1}) - \zeta(t_{k-1})|^2 \Delta t_k + C\left(\delta + \sqrt{\delta} + \sqrt{\delta \log \frac{1}{\delta}}\right),$$

where the last term trivially is $O((\delta \log 1/\delta)^{1/2})$ (for small $\delta$). The proof is completed by using arguments as for the proof of Bellman–Gronwall’s Inequality (or see Słomiński, 1994, Lemma A.1).

\[\square\]

**Remark 4.5.** A stronger convergence rate than (20) can be obtained in the following 'product situation'. For some integer $p$, $1 < p < d$, let $G = \mathbb{R}^p \times G'$, for a closed and convex set $G'$ in $\mathbb{R}^{d-p}$ and $\sigma_{ij} = 0$ for $i = p+1, \ldots, d$ and $j = 1, \ldots, m$. In this
case, the last term in (23) vanishes and, by simplifying the proof of Theorem 4.3, it is easily seen that

$$\sup_{0 \leq t \leq T} E|\xi^0(t) - \xi(t)|^2 = O(\sqrt{\delta}).$$

Condition (B) is not required in $G$ here. Example 3.1 illustrated a product situation.

5. Convergence when $G$ is a convex polyhedron

In this section we prove convergence of $\xi^0$, given by the projection scheme (11), to the solution $\xi$ of the RSDE (3), in the sense of mean square convergence, uniformly on compacts. We assume $G$ is a convex polyhedron with nonempty interior. Clearly, condition (B) is satisfied in this case. In order to have shorter notation let $\|\eta\|_T = \sup\{|\eta(t)|: 0 \leq t \leq T\}$ for $\eta$ càdlàg. The following important result from Dupuis and Ishii (1991, Theorem 2.2) will be used.

**Lemma 5.1.** Assume $G$ is a convex polyhedron with nonempty interior. If $(\xi_1, w_1, \varphi_1)$ and $(\xi_2, w_2, \varphi_2)$ are associated on $G$ where $w_1$ and $w_2$ are càdlàg, then

$$\|\xi_1 - \xi_2\|_T \leq K \|w_1 - w_2\|_T$$

for some positive constant $K$ (independent of $T$).

**Remark 5.2.** If $G$ is convex but not a polyhedron with nonempty interior, then there is no $K$ such that (24) is valid for all couples of associated triples $(\xi_i, w_i, \varphi_i) (i = 1, 2)$ (cf. Dupuis and Ishii, 1991, Proposition 4.1).

**Remark 5.3.** If $G$ is a convex polyhedron with nonempty interior and $b$ and $\sigma$ satisfy the Lipschitz condition (10) then, since $(\xi^0, w^0, \varphi^0)$ and $(x_0, x_0, 0)$ are associated triples, it is easy to give an alternative proof for the boundedness statements (16) and (17) by using Lemma 5.1.

**Theorem 5.4.** Assume $G$ is a convex polyhedron with nonempty interior and assume $b$ and $\sigma$ satisfy the Lipschitz condition (10) and $\sigma$ is bounded. Let $\xi$ and $\xi^0$ be given by (3) and (11), respectively. Then

$$E\|\xi^0 - \xi\|^2_T = O(\delta \log 1/\delta)$$

for small $\delta$.

Observe that for stochastic differential equations without reflection, under Lipschitz-and linear growth assumptions of the drift term and the dispersion matrix, $E\|\xi^0 - \xi\|^2_T = O(\delta)$ (cf. Kloeden and Platen, 1992, Remark 10.2.3).

The rate in Theorem 5.4 is slightly faster than the rate $O(\delta^{1-\varepsilon})$, for any $\varepsilon > 0$, which was recently obtained by Słomiński (1994). Chitashvili and Lazrieva (1981)
also suggested the numerical method given by (11) in the special case when the polyhedron $G$ is an interval on the real line. However, the convergence by Chitashvili and Lazrieva (1981) was slower than the convergence given by Słomiński (1994).

**Proof of Theorem 5.4.** Again, let $C$ be a universal constant and $\delta$ small. We may assume $|\sigma(x)| \leq L$ for $x \in \mathbb{R}^d$. By Theorem 2.4(a) we can introduce an associated ‘help’ triple $(\xi^\delta, \tilde{\xi}^\delta, \tilde{\phi}^\delta)$ with

$$
\tilde{\phi}^\delta(t) := x_0 + \int_0^t b(\xi^\delta(s)) \, ds + \int_0^t \sigma(\xi^\delta(s)) \, dB(s).
$$

Note that $\tilde{\phi}^\delta(t_k) = \phi^\delta(t_k), k = 0, \ldots, c_\delta$. In order to show that $E\|\tilde{\phi}^\delta - \xi\|_T^2$ is $O(\delta \log 1/\delta)$ for small $\delta$ we use upper estimates of $E\|\tilde{\phi}^\delta - \xi\|_T^2$ and $E\|\tilde{\phi}^\delta - \xi\|_T^2$. By Lemma 5.1 and the boundedness condition of $\sigma$,

$$
E\|\tilde{\phi}^\delta - \xi\|_T^2 \leq K^2 E\|\phi^\delta - \tilde{\phi}^\delta\|_T^2
$$

$$
\leq 2K^2 E \max_{1 \leq k \leq c_\delta, t \in [t_{k-1}, t_k]} |b(\xi^\delta(t_k - 1))(t - t_{k-1})|^2
$$

$$
+ 2K^2 E \max_{1 \leq k \leq c_\delta, t \in [t_{k-1}, t_k]} |\sigma(\xi^\delta(t_k - 1))(B(t) - B(t_{k-1}))|^2
$$

$$
\leq 2K^2 \left\{ E \max_{1 \leq k \leq c_\delta} |b(\xi^\delta(t_k - 1))|^2 \delta^2 + L^2 E \max_{1 \leq k \leq c_\delta, t \in [t_{k-1}, t_k]} |B(t) - B(t_{k-1})|^2 \right\},
$$

where the first term, by the Lipschitz continuity condition of $b$ and (16) is $O(\delta)$, and the second one, by Lemma 4.4, is $O(\delta \log 1/\delta)$. Hence

$$
E\|\tilde{\phi}^\delta - \xi\|_T^2 \leq C\delta \log 1/\delta
$$

for small $\delta$. Further, for $t \in [0, T],

$$
E\|\phi^\delta - \xi\|_T^2 \leq K^2 E\|\phi^\delta - \tilde{\phi}^\delta\|_T^2
$$

$$
\leq 2K^2 \sup_{s \in [0, t]} \left| \int_s^t b(\xi^\delta(u)) - b(\xi(u)) \, du \right|^2
$$

$$
+ 2K^2 \sup_{s \in [0, t]} \left| \int_s^t \sigma(\xi^\delta(u)) - \sigma(\xi(u)) \, dB(u) \right|^2
$$

$$
\leq 2K^2 L^2 \left\{ t \int_0^t E\|\phi^\delta - \xi\|_u^2 \, du + 4 \int_0^t E\|\phi^\delta - \xi\|_u^2 \, du \right\} \leq C \int_0^t E\|\phi^\delta - \xi\|_u^2 \, du.
$$

Thus,

$$
E\|\phi^\delta - \xi\|_T^2 \leq 2E\|\phi^\delta - \tilde{\phi}^\delta\|_T^2 + 2E\|\phi^\delta - \xi\|_T^2 \leq C\delta \log 1/\delta + C \int_0^t E\|\phi^\delta - \xi\|_u^2 \, du
$$

for small $\delta$. Since the integrand by (16) and (17) is uniformly bounded for $u$ in $[0, T]$ and $0 < \delta \leq T$, we deduce from the Bellman–Gronwall Inequality that $E\|\phi^\delta - \xi\|_T^2 \leq C\delta \log 1/\delta$. □
Remark 5.5. Theorem 4.3, Remark 4.5 and Theorem 5.4 can easily be extended to the case when \( b \) and \( \sigma \) depends on \( t \) with \( b(\xi^\delta(t_k-1)) \) and \( \sigma(\xi^\delta(t_k-1)) \) in the scheme (11) substituted by \( b(t_{k-1},\xi^\delta(t_{k-1})) \) and \( \sigma(t_{k-1},\xi^\delta(t_{k-1})) \), respectively. In this case, condition (10) should be replaced by
\[
|b(t,x) - b(t,y)| \vee |\sigma(t,x) - \sigma(t,y)| \leq L|x - y|
\]
and
\[
|b(t,x)| \leq L(1 + |x|), \quad |\sigma(t,x)| \leq L
\]
for \( x, y \in \mathbb{R}^d \) and \( 0 \leq t \leq T \). We also need a condition of, for example, the type
\[
|b(t,x) - b(s,x)| \vee |\sigma(t,x) - \sigma(s,x)| \leq L(1 + |x|)((t-s)\log 1/(t-s))^\alpha
\]
for \( x \in \mathbb{R}^d \), \( 0 \leq s < t \leq T \) and \( t - s \) small, where \( \alpha = \frac{1}{4} \) for Theorem 4.3 and \( \alpha = \frac{1}{2} \) for Theorem 5.4.

Appendix

Proof of Lemma 4.4. It is sufficient to show the lemma for \( m = 1 \). The left-hand side in (22) is less or equal to \( E \sup \{|B(s_1) - B(s_2)|^2 : (s_1, s_2) \in [0, T]^2 \text{ and } |s_1 - s_2| \leq \delta \} \). We use a result presented in Pollard (1992).

Let \((S, \rho)\) be a pseudometric space \( S \) with pseudometric \( \rho \) and let \( \{X(s) : s \in S\} \) a real Gaussian process with \( \rho \)-continuous sample paths, for which \( E|X(s) - X(t)|^2 \leq \rho(s,t)^2 \) for all \( s, t \) in \( S \). Assume \((S, \rho)\) is totally bounded. For given \( \varepsilon \) let \( M(\varepsilon) \) be the largest integer \( n \) such that there are points \( s_1, \ldots, s_n \) in \( S \) with \( \rho(s_i, s_j) > \varepsilon \) for \( i \neq j \). Then there exists \( C \) such that, for any \( s^0 \) in \( S \),
\[
\sqrt{E \sup_{s \in S} X(s)^2} \leq \sqrt{EX(s^0)^2} + C \int_0^{\sup_{(\rho(s,s^0),s \in S)}} \sqrt{\log M(\varepsilon)} \, d\varepsilon. \tag{A.1}
\]
In our case, for given \( \delta \) we let \( S = \{(s = (s_1, s_2) \in [0, T]^2 : |s_1 - s_2| \leq \delta \} \) and \( \rho \) be the pseudometric defined by
\[
\rho(s, t) := \sqrt{E\{(X(s) - X(t))^2\}}
\]
for \( s = (s_1, s_2), t = (t_1, t_2) \) in \( S \) where \( X((s_1, s_2)) = B(s_1) - B(s_2) \). Thus,
\[
\rho((s_1, s_2), (t_1, t_2)) = \sqrt{\lambda([s_1 \wedge s_2, s_1 \vee s_2] \triangle [t_1 \wedge t_2, t_1 \vee t_2])},
\]
where \( \lambda \) is the Lebesgue measure on \( \mathbb{R} \) and \( \triangle \) is denoting the symmetric set difference. For \( s = (s_1, s_2) \) in \( S \) with \( s^0 = (0,0) \), \( \rho(s,s^0) = |s_1 - s_2|^{1/2} \leq \sqrt{\delta}. \) Hence,
\[
\sup_{s \in S} \rho(s,s^0) = \sqrt{\delta}. \tag{A.2}
\]
We now turn to the bound of \( M(\varepsilon) \). Let \( N(\varepsilon) \) be the smallest number of closed \( \rho \)-balls with radius \( \varepsilon \) required to cover \( S \). Clearly \( \rho((s_1, s_2), (t_1, t_2)) \leq (|s_1 - t_1| + |s_2 - t_2|)^{1/2} \).
Hence for fixed \((t_0, t_0)\), the set \(\{(s_1, s_2) : \rho((s_1, s_2), (t_0, t_0)) \leq \varepsilon\}\) contains the square \(\{s_1 : |s_1 - t_0| \leq \varepsilon^2/2\} \times \{s_2 : |s_2 - t_0| \leq \varepsilon^2/2\}\), which implies that

\[
N(\varepsilon) \leq \left(2 + \frac{2\delta}{(\varepsilon^2/2)}\right)(1 + \frac{T}{(\varepsilon^2/2)}),
\]

(A.3)

where \([\cdot]\) denotes the integer part. Finally, from (A.2), (A.3) and since \(M(\varepsilon) \leq N(\varepsilon/2)\), it follows that the integral in (A.1) is \(O((\delta \log 1/\delta)^{1/2})\) which shows the lemma.

Observe that the result in Lemma 4.4 cannot be improved for an equidistant partition since by the relation \(T \leq c_6 \delta\), we have

\[
\liminf_{\delta \to 0} E \max_{1 \leq k \leq c_6} |\Delta B(t_k)|^2 \geq \liminf_{\delta \to 0} E \max_{1 \leq k \leq c_6} \frac{|\Delta B(t_k)|^2}{\Delta t_k 2 \log(c_6/T)},
\]

which by Leadbetter et al. (1985, Theorem 1.5.3) is not less than one.

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