# Bounds and asymptotic minimal growth for Gorenstein Hilbert functions

Juan Migliore [a,1], Uwe Nagel [b,2,3], Fabrizio Zanello [c,*,3]

[a] *Department of Mathematics, University of Notre Dame, Notre Dame, IN 46556, USA*
[b] *Department of Mathematics, University of Kentucky, 715 Patterson Office Tower, Lexington, KY 40506-0027, USA*
[c] *Department of Mathematical Sciences, Michigan Technological University, Houghton, MI 49931-1295, USA*

### A R T I C L E   I N F O

### A B S T R A C T

We determine new bounds on the entries of Gorenstein Hilbert functions, both in any fixed codimension and asymptotically.

Our first main theorem is a lower bound for the degree $i + 1$ entry of a Gorenstein $h$-vector, in terms of its entry in degree $i$. This result carries interesting applications concerning unimodality: indeed, an important consequence is that, given $r$ and $i$, all Gorenstein $h$-vectors of codimension $r$ and socle degree $e \geqslant e_0 = e_0(r, i)$ (this function being explicitly computed) are unimodal up to degree $i + 1$. This immediately gives a new proof of a theorem of Stanley that all Gorenstein $h$-vectors in codimension three are unimodal.

Our second main theorem is an asymptotic formula for the least value that the $i$th entry of a Gorenstein $h$-vector may assume, in terms of codimension, $r$, and socle degree, $e$. This theorem broadly generalizes a recent result of ours, where we proved a conjecture of Stanley predicting that asymptotic value in the specific case $e = 4$ and $i = 2$, as well as a result of Kleinschmidt which concerned the logarithmic asymptotic behavior in degree $i = \lfloor \frac{e}{2} \rfloor$.

© 2008 Elsevier Inc. All rights reserved.

* Corresponding author.
    *E-mail addresses:* juan.c.migliore.1@nd.edu (J. Migliore), uwenagel@ms.uky.edu (U. Nagel), zanello@math.kth.se (F. Zanello).

## 1. Introduction

It has been observed by Bass [1] that Gorenstein algebras are ubiquitous throughout mathematics. Despite this, it is often distressingly difficult to find ones with specific desired properties (e.g. in liaison theory, to find "small" Gorenstein subschemes containing a given subscheme of projective space). A first step is to have some understanding of which Hilbert functions can occur. These are completely classified in codimension $r \leqslant 3$ [17] (see also the third author's [21]; Macaulay first proved the result in the simpler case $r = 2$ in [13]), but a complete description is unknown if the codimension is at least 4, in spite of a remarkable amount of work performed by several authors (see, e.g., [2,4,5, 11,14,15,17]).

A first step is to have some idea of the extremal general "shape" of the Hilbert function as the codimension gets arbitrarily large. The upper bound is, of course, the case of compressed Gorenstein algebras (see Emsalem and Iarrobino's [7], which was the seminal work on compressed algebras, and also [8,10,19,20], which further developed and extended this theory), so the interesting question is to ask for a lower bound. This has first been done by Stanley [18]. In particular, he considered the special case where the socle degree is 4, and gave a precise conjecture for the asymptotic growth of the least value, $f(r)$, of the Hilbert function in degree 2, in terms of the codimension, $r$. Specifically, he conjectured that

$$\lim_{r \to \infty} \frac{f(r)}{r^{2/3}} = 6^{2/3}.$$

This conjecture appeared in the first edition of [18], in 1983. Bounds were given by Stanley and by Kleinschmidt [12], but the precise limit was only proved (verifying Stanley's conjecture) in 2006 by the current authors.

The purpose of this paper is to give a very broad generalization of this result. There are at least two initial questions that one can ask concerning the general shape of the Hilbert function of a Gorenstein algebra, and those will be answered in this paper. First, if we know the entry of a Hilbert function in any given degree, what is a "good" lower bound for the value it can assume in the next degree?

Our answer to this question, Theorem 2.4, carries very interesting applications concerning uni-modality: indeed, an important consequence of our result is that, given $r$ and $i$, all Gorenstein $h$-vectors of codimension $r$ and socle degree $e \geqslant e_0 = e_0(r, i)$ (this constant being explicitly computed) are unimodal up to degree $i + 1$.

In codimension $r \leqslant 3$, this result is powerful enough to supply a new proof of a celebrated theorem of Stanley that all Gorenstein $h$-vectors are unimodal.

Second, one can ask for asymptotic bounds given only in terms of the codimension and the socle degree, as in Stanley's situation. In Theorem 3.6, we will supply the least asymptotic value that the $i$th entry of a Gorenstein $h$-vector may assume, in terms of codimension $r$ and socle degree $e$. This generalizes the recent result of ours mentioned above, where we solved a conjecture of Stanley pre-dicting that asymptotic value in the specific case $e = 4$ and $i = 2$, as well as a result of Kleinschmidt (see [12], Theorem 1) for $i = \lfloor \frac{e}{2} \rfloor$.

Our asymptotic result follows by combining our lower bounds and a construction of suitable Gorenstein algebras. We illustrate this with a specific example.

**Example.** Consider the Gorenstein $h$-vectors of the form

$$(1, 125, h_2, h_3, \dots, 125, h_8 = 1).$$

The proof of Theorem 3.6 guarantees the existence of a Gorenstein $h$-vector $(1, 125, 95, 77, 71, \dots)$. On the other hand, with this value of $h_1$, Theorem 2.4 provides $h_2 \geqslant 95$, which is thus sharp. Taking $h_2 = 95$, it provides $h_3 \geqslant 77$, again sharp thanks to our explicit construction. Then taking $h_3 = 77$, we obtain $h_4 \geqslant 70$, but now our example does not achieve the bound; indeed, we do not know if this bound is sharp or not. See also Example 3.7 for a broad generalization of this example.

On the other hand, it is not surprising that our bound in Theorem 2.4 is not always sharp, since a sharp bound would probably make it easy to decide if non-unimodal Gorenstein $h$-vectors of codimension four do exist. See also Example 2.10. What we do find surprising is that Theorem 2.4 is strong enough to give a sharp asymptotic bound (Theorem 3.6), as described above.

## 2. A general lower bound and its applications

Throughout this paper, $k$ will denote an infinite field, and $R = k[x_1, \ldots, x_r]$ a graded polynomial ring in $r$ variables. Each standard graded $k$-algebra $A$ can be written as $A = R/I$, where $I \subset R$ is a homogeneous ideal.

We begin by recalling results of Macaulay, Green, and Stanley that we will need in this paper.

**Definition 2.1.** Let $n$ and $i$ be positive integers. The *$i$-binomial expansion of $n$* is

$$n_{(i)} = \binom{n_i}{i} + \binom{n_{i-1}}{i-1} + \cdots + \binom{n_j}{j},$$

where $n_i > n_{i-1} > \cdots > n_j \geqslant j \geqslant 1$. We remark that such an expansion always exists and it is unique (see, e.g., [6] Lemma 4.2.6).

Following [3], we define, for any integers $a$ and $b$,

$$(n_{(i)})^b_a = \binom{n_i + b}{i + a} + \binom{n_{i-1} + b}{i - 1 + a} + \cdots + \binom{n_j + b}{j + a},$$

where we set $\binom{m}{q} = 0$ whenever $m < q$ or $q < 0$.

**Theorem 2.2.** *Let $L \in A$ be a general linear form. Denote by $h_d$ the degree $d$ entry of the Hilbert function of $A$ and by $h'_d$ the degree $d$ entry of the Hilbert function of $A/LA$. Then:*

  (i) *(Macaulay) $h_{d+1} \leqslant ((h_d)_{(d)})^1_1$.*
  (ii) *(Green) $h'_d \leqslant ((h_d)_{(d)})^{-1}_0$.*

**Proof.** (i) See [6], Theorem 4.2.10.
  (ii) See [9], Theorem 1.  □

The following simple observation is not new (see for instance [18], bottom of p. 67):

**Lemma 2.3** *(Stanley). Let $A$ be an artinian Gorenstein algebra, and let $L \in A$ be any linear form. Then the Hilbert function of $A$ can be written as*

$$h := (1, h_1, \ldots, h_e) = (1, b_1 + c_1, \ldots, b_e + c_e = 1),$$

*where*

$$b = (b_1 = 1, b_2, \ldots, b_{e-1}, b_e = 1)$$

*is the $h$-vector of $A/(0 : L)$ (with the indices shifted by 1), which is a Gorenstein algebra, and*

$$c = (c_0 = 1, c_1, \ldots, c_{e-1}, c_e = 0)$$

*is the $h$-vector of $A/LA$.*

Perhaps the most important, and definitely the most consequential, result of this section is a lower bound for the value of a Gorenstein Hilbert function in terms of the value in the previous degree. This result generalizes [14], Theorem 4, which treated the case $i = 1$. Specifically, we have that:

**Theorem 2.4.** *Suppose that $h = (1, h_1 = r, h_2, \ldots, h_{e-2}, h_{e-1} = r, h_e = 1)$ is the h-vector of an artinian Gorenstein algebra over $R = k[x_1, \ldots, x_r]$. Assume that $i$ is an integer satisfying $1 \leqslant i \leqslant \frac{e}{2} - 1$. Then*

$$h_{i+1} \geqslant \big((h_i)_{(e-i)}\big)^{-1}_{-1} + \big((h_i)_{(e-i)}\big)^{-(e-2i)}_{-(e-2i-1)}.$$

**Proof.** As in Stanley's Lemma 2.3, let us write $h = (1, h_1, \ldots, h_e)$ as $b + c = (1, b_1 + c_1, \ldots, b_e + c_e = 1)$, where we have picked the form $L$ to be general inside $R$. Notice that $b$ is a Gorenstein $h$-vector of socle degree $e - 1$. Therefore, by symmetry and our choice of the indices, $b_j = b_{e+1-j}$ for all $j$.

Hence, by Green's Theorem 2.2(ii), we have that

$$c_{e-i} \leqslant \big((h_{e-i})_{(e-i)}\big)^{-1}_0 = \big((h_i)_{(e-i)}\big)^{-1}_0.$$

Thus (using the Pascal's Triangle inequality)

$$b_{i+1} = b_{e-i} \geqslant h_i - \big((h_i)_{(e-i)}\big)^{-1}_0 = \big((h_i)_{(e-i)}\big)^{-1}_{-1},$$

which implies

$$c_{i+1} \leqslant h_{i+1} - \big((h_i)_{(e-i)}\big)^{-1}_{-1}.$$

By iterating Macaulay's Theorem 2.2(i), we obtain another upper bound on $c_{e-i}$, namely:

$$c_{e-i} \leqslant \Big(\big(h_{i+1} - \big((h_i)_{(i)}\big)^{-1}_{-1}\big)_{(i+1)}\Big)^{e-2i-1}_{e-2i-1}.$$

Now, since by Green's theorem, $c_{e-i} \leqslant \big((h_i)_{(e-i)}\big)^{-1}_0$, we write $c_{e-i} = \big((h_i)_{(e-i)}\big)^{-1}_0 - a$, for some integer $a \geqslant 0$. Therefore,

$$c_{i+1} = h_{i+1} - b_{i+1} = h_{i+1} - b_{e-i} = h_{i+1} - h_i + \big((h_i)_{(e-i)}\big)^{-1}_0 - a = h_{i+1} - \big((h_i)_{(e-i)}\big)^{-1}_{-1} - a,$$

from which we get, again by iterating Macaulay's theorem:

$$\Big(\big(h_{i+1} - \big((h_i)_{(e-i)}\big)^{-1}_{-1} - a\big)_{(i+1)}\Big)^{e-2i-1}_{e-2i-1} = \big((c_{i+1})_{(i+1)}\big)^{e-2i-1}_{e-2i-1} \geqslant c_{e-i} = \big((h_i)_{(e-i)}\big)^{-1}_0 - a.$$

Therefore, since $(m_{(d)})^1_1$ is a strictly increasing function of $m$, we have

$$\Big(\big(h_{i+1} - \big((h_i)_{(e-i)}\big)^{-1}_{-1}\big)_{(i+1)}\Big)^{e-2i-1}_{e-2i-1} \geqslant \Big(\big(h_{i+1} - \big((h_i)_{(e-i)}\big)^{-1}_{-1} - a\big)_{(i+1)}\Big)^{e-2i-1}_{e-2i-1} + a$$

$$\geqslant \big((h_i)_{(e-i)}\big)^{-1}_0.$$

In particular, we have proved that

$$\Big(\big(h_{i+1} - \big((h_i)_{(e-i)}\big)^{-1}_{-1}\big)_{(i+1)}\Big)^{e-2i-1}_{e-2i-1} \geqslant \big((h_i)_{(e-i)}\big)^{-1}_0.$$

From the last inequality, we again use monotonicity. Notice first that $\big((h_i)_{(e-i)}\big)^{-1}_0$ already presents itself in its $(e - i)$ binomial expansion (possibly after eliminating those binomial coefficients that are

equal to 0). Similarly, the left-hand side of the last inequality is also already written as an $(e-i)$ binomial expansion, since $(i+1)+(e-2i-1)=e-i$. Hence we easily get

$$h_{i+1} - \left((h_i)_{(e-i)}\right)^{-1}_{-1} \geqslant \left((h_i)_{(e-i)}\right)^{-1-(e-2i-1)}_{-(e-2i-1)} = \left((h_i)_{(e-i)}\right)^{-(e-2i)}_{-(e-2i-1)},$$

as we wanted to show.  □

Let us now present some interesting applications of the above theorem.

**Example 2.5.** In [2], D. Bernstein and A. Iarrobino gave the first known example of a non-unimodal Gorenstein Hilbert function in codimension 5, namely

$$1 \ \ 5 \ \ 12 \ \ 22 \ \ 35 \ \ 51 \ \ 70 \ \ 91 \ \ 90 \ \ 91 \ \ 70 \ \ 51 \ \ 35 \ \ 22 \ \ 12 \ \ 5 \ \ 1.$$

One can check, using Theorem 2.4, that with a value of 91 in degree 7, the 90 in degree 8 is optimal. Notice that, for $i \leqslant 6$, the value of $h_{i+1}$ is not optimal with respect to our bound, but in any case Theorem 2.4 guarantees that the Hilbert function is unimodal in those degrees; only in degrees 7 and 8 it might be possible to violate unimodality.

This motivates Proposition 2.6 below.

**Proposition 2.6.** *Let $h = (1, h_1, \ldots, h_{e-1}, h_e)$ be a Gorenstein h-vector, and fix an index $i \leqslant \frac{e}{2} - 1$. If*

$$h_i < \binom{e-i+2}{2} - \binom{e-2i-1}{2} = \frac{1}{2}(i+3)(2e-3i),$$

*then $h_{i+1} \geqslant h_i$.*

**Proof.** Since $\binom{e-i+2}{2} = \binom{e-i+2}{e-i}$, the $(e-i)$-binomial expansion of $h_i$ clearly begins with $\binom{e-i+1}{e-i} + \cdots$ or $\binom{e-i}{e-i} + \cdots$. In particular, since $h_i < \binom{e-i+2}{2} - \binom{e-2i-1}{2}$, there exists an integer $\ell$, $e-2i-1 \leqslant \ell \leqslant e-i+1$, such that

$$h_i = \binom{e-i+1}{e-i} + \cdots + \binom{\ell+1}{\ell} + (*) = (e-i+1) + \cdots + (\ell+1) + (*),$$

where $(*)$ is a sum of at most $\ell - 1$ binomial coefficients of the form $\binom{j}{j}$. (Of course, $h_i := (*)$ if $\ell = e - i + 1$.) Theorem 2.4 then gives

$$h_{i+1} \geqslant \left((e-i) + \cdots + \ell + (*)\right) + (e-i-\ell+1)$$

$$= (e-i+1) + (e-i) + \cdots + (\ell+1) + (*) = h_i,$$

as desired.  □

A simple, but very important, application of the previous result is the following:

**Corollary 2.7.** *Fix r and i. Then all Gorenstein h-vectors of codimension r and socle degree*

$$e > \frac{(i+1)(i+2)\cdots(i+r-1)}{(i+3)(r-1)!} + \frac{3}{2}i$$

*are unimodal up to degree $i + 1$.*

**Proof.** One can show that, for any $r \geqslant 2$,

$$\frac{(i+1)(i+2)\cdots(i+r-1)}{(i+3)(r-1)!} + \frac{3}{2}i \geqslant \frac{i(i+1)\cdots(i+r-2)}{(i+2)(r-1)!} + \frac{3}{2}(i-1).$$

Hence it suffices to prove unimodality only in degree $i+1$.

Since $\frac{1}{2}(i+3)(2e-3i)$ is an increasing function of $e$, and $h_i$ clearly cannot exceed $\binom{i+r-1}{i}$, by Proposition 2.6 it is enough to show that $\frac{1}{2}(i+3)(2e-3i) > \binom{i+r-1}{i}$, and a standard computation shows that this is equivalent to the inequality on $e$ of the statement. $\quad\square$

In particular, our result is strong enough to reprove the well-known theorem of Stanley that all codimension $r \leqslant 3$ Gorenstein $h$-vectors are unimodal (see also [21]):

**Theorem 2.8.** *(See [17].) All Gorenstein $h$-vectors of codimension $r \leqslant 3$ are unimodal.*

**Proof.** Notice that, a fortiori, it suffices to show that the inequality of the statement of Corollary 2.7 is satisfied for $i = \lfloor \frac{e}{2} \rfloor - 1$ and $r = 3$. Therefore, we want to prove that

$$e > \frac{\lfloor \frac{e}{2} \rfloor (\lfloor \frac{e}{2} \rfloor + 1)}{2(\lfloor \frac{e}{2} \rfloor + 2)} + \frac{3}{2}\left(\left\lfloor \frac{e}{2} \right\rfloor - 1\right).$$

But the right-hand side is equal to

$$2\left\lfloor \frac{e}{2} \right\rfloor - \frac{2\lfloor \frac{e}{2} \rfloor + 3}{\lfloor \frac{e}{2} \rfloor + 2},$$

and the desired inequality immediately follows, since $e \geqslant 2\lfloor \frac{e}{2} \rfloor$. $\quad\square$

For $r = 4$ the estimate we obtain is still a very interesting one. Namely, from Corollary 2.7, we immediately have:

**Corollary 2.9.** *All Gorenstein $h$-vectors of codimension 4 and socle degree $e > \frac{1}{6}(i^2 + 12i + 2)$ are unimodal up to degree $i + 1$.*

This complements the main result of [15], which focused on the initial degree of $I$ rather than on the socle degree of $R/I$. There it was shown that, whenever $r = 4$ and $h_4 \leqslant 33$, then the possible $h$-vectors for Gorenstein algebras are *precisely* the SI-sequences.

We conclude this section with an example showing that the bound given in Theorem 2.4 is not always sharp. However, in the next section we will prove that this bound is asymptotically sharp.

**Example 2.10.** Consider Gorenstein $h$-vectors of the form

$$(1, 4, 10, 20, h_4, h_5, h_6 = h_4, 20, 10, 4, 1).$$

Assume $h_4 = 33$. Then Theorem 2.4 gives $h_5 \geqslant 30$, whereas Theorem 3.1 in [15] says that $h_5 \geqslant h_4 = 33$. In fact, using the methods of [15], Theorem 3.1, one can show that all the above Gorenstein $h$-vectors are unimodal.

Notice that the methods in [15] work nicely for algebras with low initial degree whose codimension is at most four. The methods developed in this paper work in general. This is the big advantage of the current paper.

## 3. Asymptotic minimal growth

The following definition generalizes one introduced in [18] and extended in [14].

**Definition 3.1.** Fix integers $e$ and $i$. Then $f_{e,i}(r)$ is the least possible value in degree $i$ of the Hilbert function of a Gorenstein algebra with socle degree $e$ and codimension $r$.

**Lemma 3.2.** *(See [3], Lemma 3.3.) Let $A, d$ be positive integers. Then*

(1) *Assume that $d > 1$ and $s := (A_{(d)})^{-1}_{-1}$. Then $s$ is the smallest integer such that $A \leqslant (s_{(d-1)})^{+1}_{+1}$.*
(2) *Assume that $d > i$. Then*

$$(A_{(d)})^{-i-1}_{-i-1} = \left(\left((A_{(d)})^{-i}_{-i}\right)_{(d-i)}\right)^{-1}_{-1}.$$

We note the following two immediate consequences of Lemma 3.2.

**Corollary 3.3.** *With the notation of Lemma* 3.2 *we have*

(1) $\left(\left((A_{(d)})^{-1}_{-1}\right)_{(d-1)}\right)^{-1}_{-1} = (A_{(d)})^{-2}_{-2}.$
(2) $\left(\ldots\left(\left(\left((A_{(d)})^{-1}_{-1}\right)_{(d-1)}\right)^{-1}_{-1}\right)\ldots\right)_{(d-i)})^{-1}_{-1} = (A_{(d)})^{-(i+1)}_{-(i+1)}.$

We need two more preliminary results before proving our main theorem. Remember that, given two functions $f$ and $g$, we say that $f(m) \in O(g(m))$ if, for $m$ large, there exists a positive constant $C$ such that $|f(m)| \leqslant C \cdot g(m)$.

**Lemma 3.4.** *Given $e \geqslant 1$, every positive integer $r$ can be written in the form*

$$r = m + \binom{m+e-3}{e-1} + \binom{a_{e-2}}{e-2} + \binom{a_{e-3}}{e-3} + \cdots + \binom{a_1}{1},$$

*where $m$ is the largest integer such that $m + \binom{m+e-3}{e-1} \leqslant r$, $a_{e-2} \geqslant a_{e-1} \geqslant \cdots \geqslant a_1 \geqslant 0$ (the inequalities being strict if the $a_i$'s are positive), and each $a_i \in O(m)$.*

**Proof.** The $a_i$'s are simply obtained from the $(e-2)$-binomial expansion of $r - m - \binom{m+e-3}{e-1}$ (we can consider them to be all 0's if $r = m - \binom{m+e-3}{e-1}$). □

The following result is due to Stanley, even if its idea was already contained in a paper of Reiten [16].

**Lemma 3.5.** *Given a level algebra with h-vector $(1, h_1, \ldots, h_j)$, there exists a Gorenstein algebra (called its trivial extension) having h-vector $H = (1, H_1, \ldots, H_j, H_{j+1})$, where, for each $i = 1, 2, \ldots, j$, we have*

$$H_i = h_i + h_{j+1-i}.$$

**Proof.** See [17], Example 4.3. □

The following is the main result of this paper. Notice that once we have fixed the socle degree $e$, by symmetry it is enough to determine the behavior of the Hilbert function in degrees $i \leqslant \frac{e}{2}$ as $r \to \infty$. Notice also that the following result generalizes Stanley's conjecture when $i = 2$ and $e = 4$, which we proved in [14]. Also, it greatly generalizes a theorem of Kleinschmidt (see [12], Theorem 1), which

supplied a logarithmic estimate for the middle entry, namely:

$$\log f_{e,\lfloor \frac{e}{2} \rfloor}(r) \sim_r \frac{\lfloor \frac{e+1}{2} \rfloor}{e-1} \log r.$$

(Recall that two arithmetic functions $f$ and $g$ are asymptotic, i.e., $f(r) \sim_r g(r)$, when $\lim_{r \to +\infty} \frac{f(r)}{g(r)} = 1$. One often simply writes $f(r) \sim_r g(r)$ in place of $f(r) \sim_{r \to +\infty} g(r)$, since $+\infty$ is the only accumulation point for the natural numbers with respect to the discrete topology they naturally inherit from the reals.)

A surprising fact is that the asymptotic formula we will show for $f_{e,i}(r)$ suddenly increases by a factor of 2 exactly in the middle (i.e., when $i = \frac{e}{2}$; therefore this pathology occurs only when the socle degree $e$ is even).

**Theorem 3.6.** *Fix $e$ and $i$. Then*

$$\lim_{r \to \infty} \frac{f_{e,i}(r)}{r^{\frac{e-i}{e-1}}} = \begin{cases} \frac{((e-1)!)^{\frac{e-i}{e-1}}}{(e-i)!} & \text{if } i < \frac{e}{2}, \\[2mm] 2 \cdot \frac{((e-1)!)^{\frac{e/2}{e-1}}}{(e/2)!} & \text{if } i = \frac{e}{2}, \end{cases}$$

*where $f_{e,i}(r)$, as in the above definition, denotes the least possible value that the Hilbert function of a Gorenstein algebra of codimension $r$ and socle degree $e$ may assume in degree $i$. (Notice that, if $i = \frac{e}{2}$, the left-hand side of the displayed equation has denominator equal to $r^{\frac{e/2}{e-1}}$.)*

**Proof.** Let $F(r) := f_{e,i}(r)/r^{\frac{e-i}{e-1}}$. We have to show that the limit exists and is equal to the asserted value. This was done for $e = 4$ and $i = 2$ in [14], so we will assume that $e \geqslant 5$. We will exhibit functions $G$ and $H$ such that, for all $r$, $G(r) \leqslant F(r) \leqslant H(r)$ and both $G$ and $H$ converge to the limit asserted in the theorem. We begin by producing $G(r)$.

We first assume that $i < \frac{e}{2}$. Observe that by Theorem 2.4 (or by Theorem 4 of [14]) and the fact that $h_1 = r$, we have

$$h_2 \geqslant (r_{(e-1)})^{-1}_{-1} + (r_{(e-1)})^{-(e-2)}_{-(e-3)} \geqslant (r_{(e-1)})^{-1}_{-1}. \tag{1}$$

Consider the $(e-2)$-binomial expansion of $h_2$:

$$(h_2)_{(e-2)} = \binom{\alpha_{e-2}}{e-2} + \binom{\alpha_{e-3}}{e-3} + \cdots + \binom{\alpha_1}{1}.$$

Then again by Theorem 2.4 we have for $h_3$ that

$$
\begin{aligned}
h_3 &\geqslant \big((h_2)_{(e-2)}\big)^{-1}_{-1} + \big((h_2)_{(e-2)}\big)^{-(e-4)}_{-(e-5)} \\
&\geqslant \big((h_2)_{(e-2)}\big)^{-1}_{-1} \\
&\geqslant \big(((r_{(e-1)})^{-1}_{-1})_{(e-2)}\big)^{-1}_{-1} \quad \text{(by (1))} \\
&= (r_{(e-1)})^{-2}_{-2} \quad \text{(by Corollary 3.3).}
\end{aligned}
$$

Proceeding inductively in the same way, we obtain for $i < \frac{e}{2}$, using Corollary 3.3,

$$f_{e,i}(r) \geqslant (r_{(e-1)})^{-(i-1)}_{-(i-1)}. \tag{2}$$

Consider the $(e-1)$-binomial expansion of $r$:

$$r_{(e-1)} = \binom{k}{e-1} + \binom{k_{e-2}}{e-2} + \cdots + \binom{k_1}{1}.$$

Note that $k$ is obtained as a function of $r$. Thus, invoking (2), we obtain

$$f_{e,i}(r) \geqslant \binom{k-i+1}{e-i}.$$

Since $k$ is a function of $r$, and $e$ and $i$ are fixed in advance, $\binom{k-i+1}{e-i}$ is also a function of $r$, which we denote by $G_1(r)$.

Since asymptotically $r \sim_r k^{e-1}/(e-1)!$, we have

$$k \sim_r r^{\frac{1}{e-1}} \cdot \left((e-1)!\right)^{\frac{1}{e-1}},$$

and so

$$G_1(r) \sim_r \frac{k^{e-i}}{(e-i)!} \sim_r \frac{r^{\frac{e-i}{e-1}} \cdot \left((e-1)!\right)^{\frac{e-i}{e-1}}}{(e-i)!}.$$

Denoting $G(r) := G_1(r)/r^{\frac{e-i}{e-1}}$, we see that $G(r) \leqslant F(r)$ and $G(r)$ has the desired limit when $i < \frac{e}{2}$.

The argument is similar when $i = \frac{e}{2}$, with essentially one difference. We now have, using Theorem 2.4, that

$$h_{\frac{e}{2}} \geqslant \left((h_{\frac{e}{2}-1)(e-\frac{e}{2}+1)}\right)_{-1}^{-1} + \left((h_{\frac{e}{2}-1)(e-\frac{e}{2}+1)}\right)_{-1}^{-2} = \left((h_{\frac{e}{2}-1)(\frac{e}{2}+1)}\right)_{-1}^{-1} + \left((h_{\frac{e}{2}-1)(\frac{e}{2}+1)}\right)_{-1}^{-2}.$$

Arguing as before, we now obtain

$$f_{e,\frac{e}{2}} \geqslant (r_{(e-1)})_{-\frac{e}{2}+1}^{-\frac{e}{2}+1} + (r_{(e-1)})_{-\frac{e}{2}+1}^{-\frac{e}{2}}.$$

Since asymptotically both terms carry equal weight, we proceed as before with a factor of two, as asserted.

We now want to show the upper bound, by exhibiting a function $H(r) \geqslant F(r)$ which converges to the limit of the statement.

Let us write $r$ as in Lemma 3.4, and consider the integer $r - m = \binom{m+e-3}{e-1} + \binom{a_{e-2}}{e-2} + \binom{a_{e-3}}{e-3} + \cdots + \binom{a_1}{1}$. First suppose that $r > m + \binom{m+e-3}{e-1}$, i.e. that $a_{e-2} \geqslant e-2$.

We construct an $h$-vector $h$ of socle degree $e$ and type $h_{e-1} = r - m$ as follows. For all indices $i$, let

$$h_i = \left((h_{e-1)(e-1)}\right)_{-(e-1-i)}^{-(e-1-i)} = \binom{m-2+i}{i} + \binom{a_{e-2}-e+i+1}{i-1} + \cdots + \binom{a_{e-i-1}-e+i+1}{0}.$$

In particular,

$$h_1 = \binom{m-1}{1} + \binom{a_{e-2}-e+2}{0} = (m-1)+1 = m.$$

It is easy to see, by the fact that all $a_i$'s are $O(m)$, that $h_i \sim_m \frac{m^i}{i!}$.

Furthermore, by Lemma 3.2 and Corollary 3.3, we have that $h_i$ is the minimum possible value of $h$ in degree $i$, given $h_{e-1}$. It is easy to show that this construction guarantees that $h$ be level, since the lex-segment ideal corresponding to $h$ is a level ideal (see, e.g., [3] or [22]).

Hence, by trivial extension (Lemma 3.5), we can construct a Gorenstein $h$-vector $(1, H_1, \ldots, H_e)$ of socle degree $e$, where $H_i = h_i + h_{e-i}$.

In particular, $H_1 = h_1 + h_{e-1} = m + (r - m) = r$. Also, for all indices $i \leqslant \frac{e}{2}$, we have

$$H_i \sim_m \frac{m^i}{i!} + \frac{m^{e-i}}{(e-i)!},$$

which is asymptotic to $\frac{m^{e-i}}{(e-i)!}$ if $i < \frac{e}{2}$, and to $2\frac{m^{e/2}}{(e/2)!}$ if $i = \frac{e}{2}$.

Since $m$ is a function of $r$, $H_i$ is also a function of $r$. Also, notice that, asymptotically, $r \sim_r \frac{m^{e-1}}{(e-1)!}$, whence $m \sim_r ((e-1)!)^{\frac{1}{e-1}} r^{\frac{1}{e-1}}$.

Thus, since by definition, $f_{e,i}(r) \leqslant H_i(r)$, we have

$$\frac{f_{e,i}(r)}{r^{\frac{e-i}{e-1}}} \leqslant \frac{H_i(r)}{r^{\frac{e-i}{e-1}}},$$

and it is easy to check that the right-hand side converges to the desired value for all $i \leqslant \frac{e}{2}$.

It remains to prove the upper bound when $r$ is of the form $r = m + \binom{m+e-3}{e-1}$.

We proceed exactly as before, by starting with a level $h$-vector of type $h_{e-1} = r - m = \binom{m+e-3}{e-1}$, and obtaining, by trivial extensions, a Gorenstein $h$-vector $(1, H_1, \ldots, H_e)$, where $H_i = \binom{m+i-2}{i} + \binom{m+e-i-2}{e-i}$ if $0 \leqslant i \leqslant e - 1$. The only difference is that now $H_1 = \binom{m-1}{1} + \binom{m+e-3}{e-1} = (m-1) + (r - m) = r - 1$.

But it is easy to show that if $(1, H_1, H_2, \ldots, H_{e-1}, 1)$ is a Gorenstein $h$-vector, then also $(1, H_1 + 1, H_2 + 1, \ldots, H_{e-1} + 1, 1)$ is always a Gorenstein $h$-vector (for instance using Macaulay's inverse systems; see, e.g., the proof of Proposition 8 in [14]).

Hence, we have constructed a Gorenstein $h$-vector of codimension $r$ also when $r = m + \binom{m+e-3}{e-1}$, and, employing the same argument as above, we obtain that asymptotically its entries again satisfy the estimate of the statement, since adding 1 clearly does not change their asymptotic value.

The proof of the theorem is complete. $\quad\square$

We illustrate the quality of our bounds by an example in which we focus on degrees two and three.

**Example 3.7.** Consider the degrees 2 and 3 entries of a Gorenstein $h$-vector $(1, h_1, h_2, h_3, \ldots, h_e)$, where $r = h_1 = \binom{m+e-3}{e-1} + m$ for some integer $m$ satisfying $1 \leqslant m \leqslant e - 2$. Assume that $e \geqslant 6$. Note that the construction given in the last part of the proof of Theorem 3.6 gives a Gorenstein algebra with $h$-vector

$$\left(1, \binom{m+e-3}{e-1} + m, \binom{m+e-4}{e-2} + \binom{m}{2} + 1, \binom{m+e-5}{e-3} + \binom{m+1}{3} + 1, \ldots\right). \qquad (3)$$

One quickly checks that

$$h_1 = r = \binom{m+e-3}{e-1} + m = \binom{m+e-3}{e-1} + \binom{e-2}{e-2} + \cdots + \binom{e-m-1}{e-m-1}$$

so applying Theorem 2.4 to get a bound for $h_2$, we obtain

$$h_2 \geqslant \left[ \binom{m+e-4}{e-2} + \binom{e-3}{e-3} + \cdots + \binom{e-m-2}{e-m-2} \right] + \left[ \binom{m+e-3-e+2}{e-1-e+3} + 0 \right]$$
$$= \binom{m+e-4}{e-2} + m + \binom{m-1}{2}.$$

Since $\binom{m-1}{2} + m = \binom{m}{2} + 1$, we see that for this class of examples the bound for $h_2$ given in Theorem 2.4 is sharp!

Similarly, let us consider the bound that we obtain for $h_3$. We have already computed in (3) the value of $h_3$ obtained in the construction of Theorem 3.6. To apply Theorem 2.4, we need to write the $(e-2)$-binomial expansion of $h_2$. To that end, suppose that $a \geqslant 1$ and $k \leqslant e-2$ are integers satisfying

$$(e-2) + (e-3) + \cdots + (e-k) + a = \binom{m}{2} + 1.$$

Notice that, since $m \leqslant e-2$ and $e \geqslant 6$, such integers $a$ and $k$ always exist. Hence

$$h_2 \geqslant \binom{m+e-4}{e-2} + \binom{e-2}{e-3} + \binom{e-3}{e-4} + \cdots + \binom{e-k}{e-k-1} + a$$

where here we are thinking of $a$ as a sum of binomial coefficients of the form $\binom{e-j}{e-j}$. Then Theorem 2.4 gives

$$h_3 \geqslant \left[ \binom{m+e-5}{e-3} + \binom{e-3}{e-4} + \cdots + \binom{e-k-1}{e-k-2} + a \right] + \left[ \binom{m+e-4-(e-4)}{(e-2)-(e-5)} + (k-1) \right]$$
$$= \left[ \binom{m+e-5}{e-3} + (e-3) + \cdots + (e-k-1) + a \right] + \left[ \binom{m}{3} + (k-1) \right]$$
$$= \binom{m+e-5}{e-3} + \binom{m}{2} + 1 + \binom{m}{3}$$
$$= \binom{m+e-5}{e-3} + \binom{m+1}{3} + 1.$$

Hence the bound of Theorem 2.4 is attained. Choosing $e = 8$ and $m = 5$ we obtain the example given in the introduction.

## Acknowledgment

## References

[1] H. Bass, On the ubiquity of Gorenstein rings, Math. Z. 82 (1963) 8–28.
[2] D. Bernstein, A. Iarrobino, A non-unimodal graded Gorenstein Artin algebra in codimension five, Comm. Algebra 20 (8) (1992) 2323–2336.
[3] A.M. Bigatti, A.V. Geramita, Level algebras, lex segments and minimal Hilbert functions, Comm. Algebra 31 (2003) 1427–1451.
[4] M. Boij, Graded Gorenstein Artin algebras whose Hilbert functions have a large number of valleys, Comm. Algebra 23 (1) (1995) 97–103.
[5] M. Boij, D. Laksov, Nonunimodality of graded Gorenstein Artin algebras, Proc. Amer. Math. Soc. 120 (1994) 1083–1092.
[6] W. Bruns, J. Herzog, Cohen-Macaulay Rings, revised ed., Cambridge Stud. Adv. Math., vol. 39, Cambridge Univ. Press, Cambridge, UK, 1998.
[7] J. Emsalem, A. Iarrobino, Some zero-dimensional generic singularities; finite algebras having small tangent space, Compos. Math. 36 (1978) 145–188.

 [8] R. Fröberg, D. Laksov, Compressed algebras, in: Conference on Complete Intersections in Acireale, in: Lecture Notes in Math., vol. 1092, Springer, 1984, pp. 121–151.
 [9] M. Green, Restrictions of linear series to hyperplanes, and some results of Macaulay and Gotzmann, in: Algebraic Curves and Projective Geometry, Trento, 1988, in: Lecture Notes in Math., vol. 1389, Springer, Berlin, 1989, pp. 76–86.
[10] A. Iarrobino, Compressed algebras: Artin algebras having given socle degrees and maximal length, Trans. Amer. Math. Soc. 285 (1984) 337–378.
[11] A. Iarrobino, H. Srinivasan, Some Gorenstein Artin algebras of embedding dimension four, I: Components of $PGOR(H)$ for $H = (1, 4, 7, \ldots, 1)$, J. Pure Appl. Algebra 201 (2005) 62–96.
[12] P. Kleinschmidt, Über Hilbert-Funktionen graduierter Gorenstein-Algebren, Arch. Math. (Basel) 43 (1984) 501–506.
[13] F.H.S. Macaulay, The Algebraic Theory of Modular Systems, Cambridge Univ. Press, Cambridge, UK, 1916.
[14] J. Migliore, U. Nagel, F. Zanello, On the degree two entry of a Gorenstein $h$-vector and a conjecture of Stanley, Proc. Amer. Math. Soc. 136 (8) (2008) 2755–2762.
[15] J. Migliore, U. Nagel, F. Zanello, A characterization of Gorenstein Hilbert functions in codimension four with small initial degree, Math. Res. Lett. 15 (2) (2008) 331–349.
[16] I. Reiten, The converse to a theorem of Sharp on Gorenstein modules, Proc. Amer. Math. Soc. 32 (1972) 417–420.
[17] R. Stanley, Hilbert functions of graded algebras, Adv. Math. 28 (1978) 57–83.
[18] R. Stanley, Combinatorics and Commutative Algebra, second ed., Progr. Math., vol. 41, Birkhäuser, Boston, MA, 1996.
[19] F. Zanello, Extending the idea of compressed algebra to arbitrary socle-vectors, J. Algebra 270 (1) (2003) 181–198.
[20] F. Zanello, Extending the idea of compressed algebra to arbitrary socle-vectors, II: Cases of non-existence, J. Algebra 275 (2) (2004) 730–748.
[21] F. Zanello, Stanley's theorem on codimension 3 Gorenstein $h$-vectors, Proc. Amer. Math. Soc. 134 (1) (2006) 5–8.
[22] F. Zanello, When is there a unique socle-vector associated to a given $h$-vector? Comm. Algebra 34 (5) (2006) 1847–1860.