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Elimination Theory in Codimension 2

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New formulae are given for Chow forms, discriminants and resultants arising from (not necessarily normal) toric varieties of codimension 2. The Newton polygon of the discriminant is determined exactly.

1. Introduction

Sparse elimination theory concerns the study of Chow forms and discriminants associated with toric varieties, that is, subvarieties of projective space which are parametrized by monomials (Sturmfels, 1993; Gel'fand *et al.*, 1994). This theory has its origin in the work of Gel'fand *et al.* on multivariate hypergeometric functions (Gel'fand *et al.*, 1989). The singularities of these functions occur on the projectively dual hypersurfaces to the torus orbit closures on the given toric variety X. The singular locus of the hypergeometric system is described by the full discriminant of X, which is a natural specialization of the Chow form.

Classical hypergeometric functions in one variable arise when X is a toric hypersurface, defined by one homogeneous binomial equation $x_1^{b_1} \cdots x_r^{b_r} = x_{r+1}^{b_r+1} \cdots x_n^{b_n}$. The Chow form of this hypersurface X is just its defining polynomial. The discriminant of X equals, up to an integer factor (Gel'fand *et al.*, 1994, Section 9.1),

$$D_X = b_{r+1}^{b_r+1} \cdots b_n^{b_n} \cdot x_1^{b_1} \cdots x_r^{b_r} - (-1)^{\deg(X)} b_1^{b_1} \cdots b_r^{b_r} \cdot x_{r+1}^{b_{r+1}} \cdots x_n^{b_n}, \tag{1.1}$$

and the full discriminant equals D_X times $\prod_{i=1}^n x_i^{\deg(X)-b_i}$. It is the purpose of this article to generalize these formulae to toric varieties of codimension 2.

We introduce our objects of study by means of an example. Let X be the toric 6-fold in projective 8-space given parametrically by the cubic monomials

$$(a : b : \cdots : i) = (u_1x^2 : u_2y^2 : u_3z^2 : u_0^2u_1 : u_0^2u_2 : u_0^2u_3 : u_0u_4x : u_0u_4y : u_0u_4z).$$

The prime ideal of the toric variety X is generated by the 2×2-minors of

$$\begin{pmatrix} a & b & c \\ dg^2 & eh^2 & fi^2 \end{pmatrix}. \tag{1.2}$$

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Thus X is arithmetically Cohen–Macaulay and has degree 13. The Chow form of X is gotten by eliminating the variable t from the 2×2-minors of

$$\begin{pmatrix} a_0 + ta_1 & b_0 + tb_1 & c_0 + tc_1 \\ (d_0 + td_1)(g_0 + tg_1)^2 & (e_0 + te_1)(h_0 + th_1)^2 & (f_0 + tf_1)(i_0 + ti_1)^2 \end{pmatrix}. \tag{1.3}$$

The Chow form is an irreducible polynomial of degree 26 in the 18 variables a_0, a_1, b_0, $b_1, \ldots, i_0, i_1$ having exactly 57 726 terms. It equals the determinant

$$\begin{pmatrix} 123 & 124 & 125 & 126 \\ 134 & 135 + 234 & 136 + 235 & 236 \\ 135 & 136 + 145 + 235 & 146 + 236 + 25 & 246 \\ 136 & 146 + 236 & 156 + 246 & 256 \end{pmatrix} \tag{1.4}$$

where ijk is the 3×3-minor with row indices i, j and k of the 6×3-matrix

$$\begin{pmatrix} a_0 & b_0 & c_0 \\ a_1 & b_1 & c_1 \\ d_0 g_0^2 & e_0 h_0^2 & f_0 i_0^2 \\ d_1 g_0^2 + 2d_0 g_0 g_1 & e_1 h_0^2 + 2e_0 h_0 h_1 & f_1 i_0^2 + 2f_0 i_0 i_1 \\ d_0 g_1^2 + 2d_1 g_0 g_1 & e_0 h_1^2 + 2e_1 h_0 h_1 & f_0 i_1^2 + 2f_1 i_0 i_1 \\ d_1 g_1^2 & e_1 h_1^2 & f_1 i_1^2 \end{pmatrix}. \tag{1.5}$$

Note that the Chow form can also be written as a polynomial of degree 13 in the brackets $[ab] = a_0 b_1 - a_1 b_0$, $[ac] = a_0 c_1 - a_1 c_0, \ldots, [hi] = h_0 i_1 - h_1 i_0$. We obtain the full discriminant of X from the Chow form by substituting

$$\begin{pmatrix} a_0 & a_1 \\ b_0 & b_1 \\ \vdots & \vdots \\ i_0 & i_1 \end{pmatrix} \mapsto \operatorname{diag}(a, b, c, d, e, f, g, h, i) \cdot B, \tag{1.6}$$

where B is the 9×2-matrix with row vectors $(1, 0)$, $(0, 1)$, $(-1, -1)$, $(-1, 0)$, $(0, -1)$, $(1, 1)$, $(-2, 0), (0, -2), (2, 2)$. The result of this substitution is the dual full discriminant $\tilde{E}_X$. It has exactly 12 terms and factors as follows:

$$\tilde{E}_X = 2^{14} \cdot (aeh^2 - bdg^2) \cdot (afi^2 - cdg^2) \cdot (bfi^2 - ceh^2) \cdot \tilde{D}_X, \tag{1.7}$$

where the last factor $\tilde{D}_X$ is the irreducible polynomial

$$\begin{aligned} &a^2 e^2 f^2 h^4 i^4 + b^2 d^2 f^2 g^4 i^4 + c^2 d^2 e^2 g^4 h^4 \\ &\qquad - 2abdef^2 g^2 h^2 i^4 - 2acdfe^2 g^2 h^4 i^2 - 2bcefd^2 g^4 h^2 i^2. \end{aligned}$$

Replacing each variable in $\tilde{D}_X$ by its reciprocal, that is, $a \mapsto 1/a$, $b \mapsto 1/b, \ldots$ and clearing denominators, we get the discriminant D_X, an irreducible polynomial of degree 10 which defines the hypersurface projectively dual to X.

In this paper we establish exact formulae for the Chow form (Theorems 2.1 and 2.7), the full discriminant (Proposition 3.2), and the discriminant (Theorem 4.2) associated with an arbitrary toric variety X of codimension 2 in a projective space. A combinatorial construction is given for the secondary polygon (Theorem 3.4) and the Newton polygon of the discriminant (Theorem 4.3). This construction shows that the dual variety $X^\vee$ is a hypersurface if and only if the secondary polygon is not centrally symmetric (Corollary 4.5). In Section 5 we study mixed resultants, that is, we apply our theory to codimension 2 toric varieties which arise from the Cayley trick (Gel'fand *et al.*, 1994, Section 3.2.D).

The toric 6-fold X in our example does arise from the Cayley trick. This can be seen from the defining parametrization $(u_1x^2 : \cdots : u_0u_4z)$. Hence the discriminant D_X is actually a resultant. Indeed, if we eliminate x, y, z from

$$a \cdot x^2 + d = b \cdot y^2 + e = c \cdot z^2 + f = g \cdot x + h \cdot y + i \cdot z = 0 \tag{1.8}$$

then the result is precisely the six-term discriminant D_X described earlier.

2. The Chow Form

Let $B = (b_{il})$ be an $n \times 2$-integer matrix of rank 2 with both column sums equal to zero. The *lattice ideal* I_B is the ideal in $k[x_1, \ldots, x_n], k$ any field, generated by the binomials $x^{u+} - x^{u-}$ where $u = u_+ - u_-$ runs over the two-dimensional lattice $L_B \subset \mathbb{Z}^n$ spanned by the columns of B. An explicit description of the minimal generators and the higher syzygies of I_B was given in Peeva and Sturmfels (1998). The ideal I_B is homogeneous with respect to the usual $\mathbb{Z}$-grading and hence defines a subscheme X_B of projective $(n-1)$-space $\mathbf{P}^{n-1}$. The lattice ideal I_B is prime if and only if $\mathbb{Z}^n/L_B$ is a free Abelian group, or equivalently, if and only if the row vectors of B generate the two-dimensional lattice $\mathbb{Z}^2$.

In this section we compute the Chow form and the Chow polygon of the projective scheme X_B. The *degree* of X_B, denoted $d_B = degree(X_B)$, is the number of intersection points with a generic 2-plane in $\mathbf{P}^{n-1}$. Let $Y = (y_{il})$ be an $n \times 2$-matrix of indeterminates. It represents a generic parametric line $(y_{11} + ty_{12}, \ldots, y_{n1} + ty_{n2})$ in $\mathbf{P}^{n-1}$. Following (Gel'fand *et al.*, 1994, Section 3.2.B), the *Chow form* $\tilde{\mathcal{C}}_B$ of the homogeneous lattice ideal I_B is the unique (up to sign) irreducible homogeneous polynomial in $\mathbb{Z}[y_{il}]$ which vanishes if and only if the corresponding line in $\mathbf{P}^{n-1}$ meets X_B. The degree of $\tilde{\mathcal{C}}_B$ equals $2 \cdot d_B$.

Classical invariant theory (cf. Gel'fand *et al.*, 1994, Proposition 3.1.6) tells us that the Chow form $\tilde{\mathcal{C}}_B$ can be written (non-uniquely) as a polynomial of degree d_B in the (*dual*) *Plücker coordinates* of a generic line, which we write as brackets

$$[ij] := y_{i1}y_{j2} - y_{i2}y_{j1} \qquad \text{for } 1 \leq i < j \leq n.$$

We further introduce a non-negative integer ν_{ij} for any $1 \leq i < j \leq n$ as follows: if the ith row vector b_i and the jth row vector b_j of $B = (b_{il})$ have the same sign in one of the two coordinates then set $\nu_{ij} = 0$; otherwise we set

$$\nu_{ij} := \min\{|b_{i1}b_{j2}|, |b_{i2}b_{j1}|\}. \tag{2.1}$$

Thus, $\nu_{ij} = 0$ unless b_i and b_j lie in the interior of opposite quadrants. Let

$$H_\ell(t) = \prod_{i:b_{i\ell}>0} (y_{i1} + y_{i2}t)^{b_{i\ell}} - \prod_{i:b_{i\ell}<0} (y_{i1} + y_{i2}t)^{b_{i\ell}}, \qquad \ell = 1, 2. \tag{2.2}$$

We regard H_1 and H_2 as polynomials in a single variable t with coefficients in $\mathbb{Z}[y_{i\ell}, i = 1, \ldots, n, \ell = 1, 2]$. Let β_ℓ denote the sum of the positive entries in the ℓth column of B, for $\ell = 1, 2$. Clearly, $degree(H_\ell) = \beta_\ell$, $\ell = 1, 2$.

Theorem 2.1. *The Chow form of the codimension* 2 *lattice ideal* I_B *equals*

$$\tilde{\mathcal{C}}_B = \frac{\mathrm{Res}_t(H_1, H_2)}{\prod_{1 \leq r < s \leq n} [r\ s]^{\nu_{rs}}},$$

where Res_t *denotes the Sylvester resultant of two univariate polynomials.*

PROOF. The binomials $\prod_{b_{ij}>0} x_i^{b_{ij}} - \prod_{b_{ij}<0} x_i^{-b_{ij}}$, $j = 1, 2$, defined by the two columns of B determine a complete intersection Y_B of degree $\beta_1\beta_2$ in $\mathbf{P}^{n-1}$ which coincides with X_B over $(k^*)^{n-1}$. The irreducible decomposition of Y_B consists of the components of X_B—of which there is only one if $\mathbb{Z}^n/L_B$ is free Abelian—together with subschemes supported on coordinate flats $x_r = x_s = 0$, whose Chow forms are the bracket monomials $[r\ s]$. The theorem will be proved if we show that the cycle $\{x_r = x_s = 0\}$ occurs with multiplicity ν_{rs} in the complete intersection.

Suppose first that $\nu_{rs} = 0$. We may assume that b_{r1}, $b_{s1} \geq 0$. Then, $\{x_r = x_s = 0\}$ is not contained in Y_B, and thus occurs with multiplicity 0. Suppose now that $\nu_{rs} > 0$. We may assume that b_{r1}, $b_{r2} > 0$ and b_{s1}, $b_{s2} < 0$. Then, $\{x_r = x_s = 0\}$ is contained in Y_B, and after localizing and changing variable names, we are lead to the following situation: let $a, b, c, d \in \mathbb{Z}_{>0}$, $ad \geq bc$ and $\alpha, \beta \neq 0$ in an extension field K of k, and consider the univariate resultant

$$r := \mathrm{Res}_t((x_0 + x_1t)^a - \alpha(y_0 + y_1t)^b, (x_0 + x_1t)^c - \beta(y_0 + y_1t)^d).$$

We want to show that $x_0y_1 - y_0x_1$ appears with exponent bc as a factor of r.

Indeed, when $x_1, y_1 \neq 0$, the condition $x_0y_1 - y_0x_1 = 0$ holds if and only if there exists t such that $x_0 + x_1t = y_0 + y_1t = 0$, and so $x_0y_1 - y_0x_1$ occurs in r with exponent μ equal to the intersection multiplicity at the origin of the artinian ideal $I = \langle x^a - \alpha y^b, x^c - \beta y^d\rangle$ in $K[x, y]$. We claim $\mu = bc$.

When $ad > bc$, the given equations are a Gröbner basis with leading terms x^a and βy^d, for the term order defined by weight$(x) = b + d$ and weight$(y) = a + c$. Hence $\dim_K K[x, y]/I = ad$, that is, there are ad roots in the affine plane counting multiplicity. Of those, $ad - bc$ lie in the torus, i.e. have both coordinates non-zero. No root of I has precisely one zero coordinate. Therefore the multiplicity of I at the origin is the difference $\mu = ad - (ad - bc) = bc$. In the case $ad = bc$, the polynomials $x^a - \alpha y^b$, $x^c - \beta y^d$ are quasi-homogeneous. By a weighted version of Bezout's theorem, they have $ad = bc$ common roots, but as i is artinian the only possible root is the origin, with this multiplicity. □

COROLLARY 2.2. *The degree of a homogeneous lattice ideal I_B of codimension 2 can be computed from the defining $n \times 2$-matrix B by the following formula*

$$\mathrm{degree}(X_B) = \beta_1\beta_2 - \sum_{1\leq r<s\leq n} \nu_{rs}.$$

The polynomial ring $\mathbb{Z}[y_{i\ell}]$ has a natural $\mathbb{Z}^n$-grading defined by $\deg(y_{i\ell}) = e_i$, the ith unit vector. The *Chow polytope* CP_B is, by definition (Gel'fand *et al.*, 1994, Section 6.3), the convex hull in $\mathbb{R}^n$ of the degrees of all monomials appearing in the expansion of $\tilde{\mathcal{C}}_B$. Its faces correspond to toric deformations of the algebraic cycle X_B.

We assume that the row vectors $b_1, b_2, \ldots, b_m$ of the matrix B are ordered counter-clockwise in cyclic order, and that $b_{m+1}, \ldots, b_n = 0$. It may happen that b_{i+1} is a positive multiple of b_i. Let P_B denote the unique (up to translation) lattice polygon whose boundary consists of the directed edges $b_1, b_2, \ldots, b_m$. For each vector $b_i = (b_{i1}, b_{i2})$, the linear functional

$$u = (u_1, u_2) \mapsto \det(b_i, u) = b_{i1}u_2 - b_{i2}u_1$$

attains its *minimum* value over P_B at the edge parallel to b_i for $i = 1, \ldots, m$ and is identically zero for $i = m + 1, \ldots, n$. Let μ_i denote the *maximum* value of the linear functional $u \mapsto \det(b_i, u)$ as u ranges the polygon P_B. For $i = 1, \ldots, m$, this maximum is attained at a unique vertex of P_B unless $b_j = \lambda b_i$ for some j and $\lambda < 0$. For every lattice point v in P_B, the quantity

$$v^{(i)} := \mu_i - \det(b_i, v) \tag{2.3}$$

is a non-negative integer, invariant under translation of P_B. The vector $(v^{(1)}, v^{(2)}, \ldots, v^{(n)})$ expresses the point v in P_B in intrinsic coordinates.

Theorem 2.3. *The Chow polygon CP_B of a codimension 2 lattice ideal I_B is the image of the polygon P_B under the affine isomorphism $v \mapsto (v^{(1)}, \ldots, v^{(n)})$.*

The proof of this theorem will be given in the next section, after Gale duality and duality of Plücker coordinates have been introduced. See Theorem 3.4 for the same theorem in dual formulation. Theorems 2.3 and 3.4 will then be derived from the constructions in Sections 7.1.D and 8.3.B of Gel'fand et al. (1994).

Example 2.4. For the example in the Introduction we take $b_1 = (1, 0)$, $b_2 = (1, 1)$, $b_3 = (2, 2)$, $b_4 = (0, 1)$, $b_5 = (-1, 0)$, $b_6 = (-2, 0)$, $b_7 = (-1, -1)$, $b_8 = (0, -1)$, $b_9 = (0, -2)$, and P_B the hexagon with vertices $(0, 0)$, $(1, 0)$, $(4, 3)$, $(4, 4)$, $(1, 4)$, $(0, 3)$. The edges of P_B are labelled by the variables as follows: $a, \{f, i\}, b, \{d, g\}, c, \{e, h\}$, and we have $\mu = (4, 3, 6, 0, 0, 0, 1, 4, 8)$. The 12 points on the boundary of P_B correspond to the 12 monomials in the expansion of $\tilde{E}_X$. For instance, the vertex $v = (0, 0)$ has intrinsic coordinates $(v^{(1)}, \ldots, v^{(9)}) = (4, 3, 6, 0, 0, 0, 1, 4, 8)$ and corresponds to $a^4ce^4f^3h^8i^6$.

For any $v \in P_B$, the coordinate sum $\sum_{i=1}^n v^{(i)}$ coincides with $\sum_{i=1}^n \mu_i$, and this equals the degree of the Chow form $\mathcal{C}_B$ as a polynomial in the $y_{i\ell}$. From this we get an alternative formula for the degree of our lattice ideal.

Corollary 2.5. *The degree of the variety X_B equals $d_B = \frac{1}{2} \cdot \sum_{i=1}^n \mu_i$.*

Counting lattice points in the polygon P_B gives an upper bound for the number of monomials appearing in the full discriminant D_X (see Section 3):

Remark 2.6. The number of lattice points in the polygon P_B equals

$$1 + \tfrac{1}{2}\left(\sum_{i=1}^{n} gcd(b_{i1}, b_{i2}) + \sum_{1 \le i < j \le n} (b_{i2}b_{j1} - b_{i1}b_{j2})\right).$$

Proof. This is a reformulation of *Pick's formula* which states that the area of a lattice polygon equals the number of lattice points in that polygon minus half the number of lattice points in its boundary, minus one. □

If the lattice ideal I_B is a complete intersection then the denominator in Theorem 2.1 is 1 and we get a determinantal formula for the Chow form, namely, $\tilde{\mathcal{C}}_B$ equals the univariate resultant in the numerator, which can be computed as the determinant of a Sylvester or Bézoutian matrix.

It would be desirable to have a division-free determinantal formula for the Chow form $\tilde{\mathcal{C}}_B$ of any codimension 2 lattice ideal. At the current time we know such formulae only for special classes of matrices B. We present a formula for a class which includes the example in the Introduction. Recall from Peeva and Sturmfels (1998) that the lattice ideal I_B is

Cohen–Macaulay if and only if I_B is generated by the 2×2-minors of a 2×3-matrix of monomials in $x_1, \dots, x_n$:

$$\begin{pmatrix} m_1 & m_2 & m_3 \\ m_4 & m_5 & m_6 \end{pmatrix}.$$

Let d_i denote the total degree of the monomial m_i. In order for the lattice ideal I_B to be homogeneous it is necessary and sufficient that

$$d_1 + d_5 = d_2 + d_4 \qquad \text{and} \qquad d_1 + d_6 = d_3 + d_4.$$

For the following discussion we make an even more restrictive assumption:

$$d_1 = d_2 = d_3 \geq d_4 = d_5 = d_6. \tag{2.4}$$

We introduce four new indeterminates s, t, u, v. Let $m_i[t]$ denote the image of the monomial m_i under the substitution $x_i \mapsto y_{i1} + y_{i2}t$ for $i = 1, 2, \dots, n$. We define the *Bézout polynomial* to be the following expression:

$$\frac{1}{(s-u)(t-v)} \cdot \det \begin{pmatrix} m_1[t] + m_4[t] \cdot s & m_1[t] + m_4[t] \cdot u & m_1[v] + m_4[v] \cdot u \\ m_2[t] + m_5[t] \cdot s & m_2[t] + m_5[t] \cdot u & m_2[v] + m_5[v] \cdot u \\ m_3[t] + m_6[t] \cdot s & m_3[t] + m_6[t] \cdot u & m_3[v] + m_6[v] \cdot u \end{pmatrix}.$$

Set $\delta := d_1 + d_4$. The Bézout polynomial can be written uniquely in the form

$$(1, v, v^2, \dots, v^{d_1-1}, u, uv, uv^2, \dots, uv^{d_4-1}) \cdot \mathbf{B} \cdot \begin{pmatrix} 1 \\ t \\ \vdots \\ t^{\delta-1} \end{pmatrix},$$

where $\mathbf{B} = \mathbf{B}(y_{ij})$ is a $\delta \times \delta$-matrix with entries in $k[y_{11}, y_{12}, \dots, y_{n2}]$.

THEOREM 2.7. *If I_B is a Cohen–Macaulay lattice ideal of codimension* 2 *satisfying* (2.4) *then its Chow form $\tilde{\mathcal{C}}_B$ equals the determinant of* $\mathbf{B}(y_{ij})$.

PROOF. Consider the rational normal scroll of type (d_1, d_4), a toric surface of degree δ in a projective space of dimension $\delta + 1$. Its Chow form has an exact determinantal formula in terms of a Bézout matrix. A nice proof of this fact follows from recent results of Eisenbud and Schreyer (preprint), since the rational normal scroll is given by the 2×2-minors of a matrix of variables. This Chow form is the unmixed, sparse resultant for three polynomials with support

$$\{1, t, t^2, \dots, t^{d_1}, s, st, st^2, \dots, st^{d_4}\}.$$

The three polynomials $m_i[t] + m_{i+3}[t] \cdot s$ have exactly this support. Our formula is gotten by specializing the Bézout matrix for the scroll. □

EXAMPLE 2.8. The ideal in (1.2) satisfies the hypotheses of Theorem 2.7, with $\delta = 4$. The matrix (1.4) is precisely the matrix $\mathbf{B}(y_{ij})$ in this case. □

3. The Full Discriminant

There are two different ways of presenting a toric variety of codimension 2: by an $n \times 2$-matrix B as in Peeva and Sturmfels (1998), or by an $(n \times 2) \times n$-matrix A as in

Gel'fand *et al.* (1994, Section 5.1). The two matrices are *Gale dual*, which means that the image of B equals the kernel of A. Up to this point in the paper, we have only used the B-representation. We now make a switch and introduce the A-representation.

Let $A = (a_1, \ldots, a_n)$ be an $(n-2) \times n$-integer matrix of rank $n-2$, and suppose there exists a vector $w \in \mathbb{Q}^{n-2}$ such that $w \cdot a_i = 1$ for $i = 1, 2, \ldots, n$. We can choose an integral $n \times 2$ matrix B whose columns are a $\mathbb{Z}$-basis of $\ker_{\mathbf{z}}(A)$. The matrix B has rank 2 and $A \cdot B = 0$. It is unique modulo right multiplication by $GL(2, \mathbb{Z})$. Let $I_A = I_B$ denote the corresponding toric ideal in $k[x_1, \ldots, x_n]$ and $X = X_A = X_B$ the corresponding toric variety in $\mathbf{P}^{n-1}$.

Here it is important to note that not all integer matrices B arise as the Gale dual of some matrix A as earlier. For this it is necessary and sufficient that $\mathbb{Z}^n / im_{\mathbf{z}}(B)$ is torsion-free, or equivalently, that the ideal I_B is prime. On the other side, by possibly replacing $\mathbb{Z}^{n-2}$ by the lattice generated by the column vectors of A, we assume w.l.o.g that the columns of A generate $\mathbb{Z}^{n-2}$, or equivalently, that the maximal minors of A are relatively prime.

The *A-discriminant* D_A is an irreducible polynomial in $\mathbb{Z}[x_1, \ldots, x_n]$ which vanishes under a specialization if the corresponding Laurent polynomial

$$f = \sum_{i=1}^{n} x_i \cdot t_1^{a_{i1}} t_2^{a_{i2}} \ldots t_{n-2}^{a_{i,n-2}} \qquad \text{where } x_1, \ldots, x_n \in \mathbb{C}^*$$

has a multiple root $t = (t_1, \ldots, t_{n-2})$ in $(\mathbb{C}^*)^{n-2}$, i.e. f and all its partial derivatives vanish at t. Equivalently, the hypersurface $\{D_A = 0\}$ is projectively dual to the toric variety X, when the dual variety $X^\vee$ is a hypersurface, and $D_A = 1$ otherwise; see Gel'fand *et al.* (1994, Sections 1.1 and 9.1).

In the next section we give a formula for the A-discriminant D_A and its degree. In this section, we study a larger polynomial E_A which contains D_A as a factor. It is called the *principal A-determinant* in Gel'fand *et al.* (1994) but we prefer the term *full discriminant.* Actually, our full discriminant agrees with expression (1.1) in Gel'fand *et al.* (1994, 10.1.A), but there is a slight inaccuracy in Gel'fand *et al.* (1994, Theorem 10.1.2) since E_A does not generally have content 1. An extra integer factor is needed. This integer factor would be 2^{14} for the example (1.7) in the Introduction.

Before stating the definition of E_A, we first review the duality between primal and dual Plücker coordinates, and see how it ties in with Gale duality. For $1 \le i < j \le n$, let $B(i,j)$ be the submatrix of B consisting of the ith and jth rows, and let $A\langle i,j \rangle$ denote the submatrix of A obtained by omitting the ith and jth columns. Here signs are adjusted so that $\det A\langle i,j \rangle = \det B(i,j)$. In Section 2 we used an $n \times 2$ matrix $Y = (y_{i\ell})$ of indeterminates. The *dual Plücker coordinates* of a line in $\mathbf{P}^{n-1}$ are

$$[ij] := \det Y(i,j) = y_{i1} y_{j2} - y_{i2} y_{j1} \qquad \text{for } 1 \le i < j \le n. \tag{3.1}$$

Here we consider an $(n-2) \times n$-matrix $Z = (z_{ij})$ of indeterminates. The *primal Plücker coordinates* of our line are the $(n-2) \times (n-2)$-subdeterminants

$$\langle ij \rangle = \det Z\langle i,j \rangle \qquad \text{(with the sign adjusted as usual).}$$

The dual Chow form $\tilde{\mathcal{C}}_B$ is a polynomial of degree d_B in the brackets (3.1). Replacing $[ij] \mapsto \langle ij \rangle$ in $\tilde{\mathcal{C}}_B$ gives a homogeneous polynomial of degree $(n-2)d_B$ in the variables z_{ij}. It is denoted $\mathcal{C}_A$ and called the *primal Chow form.* Note that $\mathcal{C}_A$ coincides with the *A-resultant* defined in Gel'fand *et al.* (1994, Section 8.2.A).

Definition 3.1. The *full discriminant* E_A is the image of the primal Chow form $\mathcal{C}_A$ under the specialization $z_{ij} \mapsto a_{ij}x_j$ for $i = 1, \dots, n-2$, $j = 1, \dots, n$.

We next show how to compute the full discriminant directly from the dual Chow form $\tilde{\mathcal{C}}_B$ and hence from the formulae in Theorems 2.1 and 2.7.

Proposition 3.2. *The full discriminant E_A and the dual Chow form $\tilde{\mathcal{C}}_B(y_{i\ell})$ are related by the following formula:*

$$E_A(x_1, \dots, x_n) = (x_1 \dots x_n)^{d_B} \cdot \tilde{\mathcal{C}}_B(b_{i\ell}/x_i, i = 1, \dots, n, \ell = 1, 2). \tag{3.2}$$

The exponent d_B is the degree of the toric variety X and hence coincides with the normalized volume of the $(n-3)$-dimensional polytope $\mathrm{conv}(A)$. Gale dual formulae for this volume are given in Corollaries 2.2 and 2.5.

Proof. The specialization $z_{ij} \mapsto a_{ij}x_j$ in Definition 3.1 is equivalent to

$$\langle rs \rangle \rightarrow \det A\langle r, s\rangle \prod_{k \neq r,s} x_k \qquad \text{for } 1 \leq r < s \leq n \tag{3.3}$$

at the level of primal Plücker coordinates. The dual Chow form $\tilde{\mathcal{C}}_B$ is a $\mathbb{Z}$-linear combination of bracket terms $\prod [r\ s]$ of degree d_B. If we substitute $b_{i\ell}/x_i$ for $y_{i\ell}$ in the expansion of such a bracket term $\prod [r\ s]$ then we get

$$\begin{aligned} \prod [r\ s] &\longrightarrow \prod (\det B(r,s)/(x_r x_s)) = \prod (\det A\langle r,s\rangle/(x_r x_s)) \\ &= (x_1 \dots x_n)^{-d_B} \cdot \prod \left(\det A\langle r,s\rangle \prod_{k \neq r,s} x_k \right) \longleftarrow (x_1 \dots x_n)^{-d_B} \cdot \prod \langle rs \rangle. \end{aligned}$$

Hence the specialized dual Chow form on the right-hand side of (3.2) equals the specialization of the primal Chow form $\mathcal{C}_A$ under (3.3), as desired. □

It is known from Gel'fand *et al.* (1994, Theorem 10.1.2) that the full discriminant E_A is a product of irreducible factors $D_{A'}$ where A' ranges over facial discriminants. In particular, each monomial x_i corresponding to a vertex a_i of $\mathrm{conv}(A)$ appears to some positive power in the factorization of E_A. It is curious to note that the monomial factors disappear when we pass to dual coordinates. We define the *dual full discriminant* by specializing the dual Chow form:

$$\tilde{E}_B(x_1, \dots, x_n) = \tilde{\mathcal{C}}_B(b_{i\ell} \cdot x_i, i = 1, \dots, n, \ell = 1, 2). \tag{3.4}$$

Proposition 3.2 is equivalent to the reciprocity formula:

$$\tilde{E}_B(x_1, \dots, x_n) = (x_1 \dots x_n)^{d_B} \cdot E_A(1/x_1, \dots, 1/x_n). \tag{3.5}$$

Lemma 3.3. *The dual full discriminant $\tilde{E}_B$ has no monomial factors.*

Proof. Suppose that the variable x_i divides $\tilde{E}_B$. Then every bracket monomial appearing in the dual Chow form $\tilde{\mathcal{C}}_B$ contains the letter i. Equivalently, every bracket monomial in the primal Chow form $\mathcal{C}_A$ contains a bracket $\langle rs \rangle$ with $r = i$ or $s = i$. In view of Gel'fand *et al.* (1994, Theorem 8.3.3), this means that every regular triangulation of A contains a simplex for which a_i is not a vertex. But this is false, since a_i lies in every maximal simplex of the *reverse lexicographic triangulation* of A, for x_i smallest; see Sturmfels (1995, Proposition 8.6). □

The *secondary polygon* $\Sigma(A)$ of the configuration A coincides with the Newton polygon of the full discriminant E_A, by Gel'fand *et al.* (1994, Theorem 10.1.4). It is a two-dimensional convex polytope lying in $\mathbb{R}^n$. Let P_B be the polygon considered in Section 2. For $v \in P_B$, let $(v^{(1)}, \dots, v^{(n)})$ be the vector defined in (2.3).

THEOREM 3.4. *The secondary polytope $\Sigma(A)$ is the image of the polygon P_B under the affine isomorphism which sends v to $(d_B - v^{(1)}, \dots, d_B - v^{(n)})$.*

PROOF. It suffices to prove this Theorem for the case when all b_i are non-zero. Indeed, if $b_{m+1} = \cdots = b_n = 0$ then (Gel'fand *et al.*, 1994, Theorem 10.1.2) implies that

$$E_A(x_i, \dots, x_n) = (x_{m+1} \dots x_n)^{d_B} \cdot E_{A'}(x_1, \dots, x_m),$$

where A' is a Gale dual of the configuration $(b_1, \dots, b_m)$. Our assertion for $\Sigma(A')$ implies that for $\Sigma(A)$. We hence assume that $b_i \neq 0$ for all $i = 1, \dots, n$.

Each vertex $w = (w_1, \dots, w_n)$ of $\Sigma(A)$ corresponds uniquely to a regular triangulation Δ_w of A. This triangulation corresponds to a pair of adjacent linearly independent vectors b_k, b_{k+1}, the index k is determined by the property that $\sum_{i=1}^n w_i b_i$ lies in the cone spanned by b_k and b_{k+1}. (Indices are understood modulo n; recall that $\sum_i b_i = 0$.) More precisely, let C_k denote the set of index pairs (r, s) such that b_k and b_{k+1} lie in the cone spanned by b_r and b_s. Then, by Billera *et al.* (1990, Lemma 4.3), the pairs in C_k are the complements of the maximal cell in the triangulation Δ_w of A which is indexed by k. The normalized volume of such a maximal cell equals $|\det(b_r, b_s)|$.

By Gel'fand *et al.* (1994, Definition 7.1.6), the ith coordinate of w equals the sum of the normalized volumes of those simplices in Δ_w which contain the point a_i. Hence $w_i = \sum_{r,s} |\det(b_r, b_s)|$ where the sum is over all indices $r \neq i$, $s \neq i$ such that b_k and b_{k+1} lie in the cone spanned by b_r and b_s. Let v_w be the vertex of P_B between the edges parallel to b_k and b_{k+1}. We claim that $v_w \in \mathbb{Z}^2$ is mapped to $w \in \mathbb{Z}^n$ under the affine isomorphism given earlier.

We note that the maximum μ_i of the values $\det(b_i, v)$ is attained at the vertex $v \in P_B$ between the edges parallel to two independent vectors b_ℓ, $b_{\ell+1}$ such that $\det(b_i, b_\ell) \geq 0$ and $\det(b_i + b_{\ell+1}) < 0$ (indices modulo n). What we are claiming is the identity $w_i = d_B - \det(b_i, v) + \det(b_i, v_w)$.

Since the set C_k is Gale dual to our regular triangulation, we have $d_B = \mathrm{vol}(\mathrm{conv}(A))$ equals $\sum_{(r,s)\in C_k} |\det(b_r, b_s)|$. If we start drawing P_B from the origin, then, $v = \sum_{j=1}^{\ell} b_j$ and $v_w = \sum_{j=1}^{k} b_j$. Our assertion takes the following form:

$$\sum_{(r,s)\in C_k, r\neq i, s\neq i} |\det(b_r, b_s)| = \sum_{(r,s)\in C_k} |\det(b_r, b_s)| - \sum_{j=1}^{\ell} \det(b_i, b_j) + \sum_{j=1}^{k} \det(b_i, b_j).$$

After erasing equal terms on both sides, the following remains to be proved:

$$\sum_{j=1}^{\ell} \det(b_i, b_j) - \sum_{j=1}^{k} \det(b_i, b_j) = \sum_{j:(i,j)\in C_k} |\det(b_i, b_j).$$

The proof is straightforward by a case distinction involving the relative positions of the vectors b_i, b_k, b_{k+1} and b_ℓ in the plane. □

PROOF OF THEOREM 2.3. First assume that $b_1, \dots, b_n$ span the lattice $\mathbb{Z}^2$ and fix a corresponding matrix A. By Gel'fand *et al.* (1994, Theorem 10.1.4), the secondary polytope $\sigma(A)$ is the Newton polytope of E_A. By Theorem 2.3, the monomials appearing in E_A are $\prod_{i=1}^n x_i^{d_{B-v(i)}}$ where $v \in P_B$. In view of the reciprocity formula (3.5), the monomials appearing in $\tilde{E}_B$ are $(x_i \cdots x_n)^{d_B} \prod_{i=1}^n \frac{1}{x_i}^{d_{B-v(i)}} = \prod_{i=1}^n x_i^{v(i)}$ where $v \in P_B$. Hence P_B is the Newton polytope of $\tilde{E}_B$, and, in view of (3.4), it also equals the Chow polytope of X_B.

Suppose next the index of the sublattice spanned by $b_1, \dots, b_n$ in $\mathbb{Z}^2$ is $r > 1$. Then the scheme X_B is the equidimensional union of r torus translates of a fixed toric variety $X_{B'}$. Following Gel'fand *et al.* (1994, Section 4.1.A), the Chow form $\tilde{\mathcal{C}}_B$ factors into r irreducible polynomials, each of which is a torus translate of the irreducible Chow form $\tilde{\mathcal{C}}_{B'}$. Therefore the Chow polygon CP_B equals $r \cdot CP_{B'}$. The configuration B' is $GL(\mathbb{R}^2)$-equivalent to B, and it does possess a Gale dual A'. Our assertion holds for $CP_{B'}$ and it follows for CP_B by scaling. □

Let us now take a look at what happens to the formula in Theorem 2.1 under the specialization $y_{i\ell} \mapsto b_{i\ell} \cdot x_i$ in (3.4). A line through the origin in $\mathbb{R}^2$ is said to be *relevant* if it contains two vectors b_r, b_s in opposite directions. So, if the rows of B are in the general position, then there are no relevant lines. The example in the introduction has three relevant lines.

Consider the specializations of the two polynomial $H_\ell(t)$ in (2.2):

$$h_\ell(t) := \prod_{i:b_{i\ell}>0} (b_{i1} + b_{i2}t)^{b_{i\ell}} x_i^{b_{i\ell}} - \prod_{i:b_{i\ell}<0} (b_{i1} + b_{i1}t)^{b_{i\ell}} x_i^{-b_{i\ell}}, \qquad \ell = 1, 2. \tag{3.6}$$

REMARK 3.5. The polynomials h_1, h_2 have a common factor if and only if there exists a relevant line which is not a coordinate axis.

The presence of two vectors b_r, b_s in opposite directions in the interior of two quadrants then causes the resultant $\mathrm{Res}_t(h_1, h_2)$ to vanish. Also, $\det(B(r,s)) = 0$, while $\nu_{rs} \neq 0$. When there are two opposite vectors on a coordinate axis, both numbers are zero and $\det(B(r,s))^{\nu_{rs}} = 1$. We deduce:

PROPOSITION 3.6. *Assume there are no relevant lines for the configuration B except for the coordinate axes. Then the dual full discriminant equals*

$$\tilde{E}_B = \frac{\mathrm{Res}_t(h_1, h_2)}{\prod_{1 \le r < s \le n} \det(B(r,s))^{\nu_{rs}} \prod_{1 \le r < s \le n} (x_r \cdot x_s)^{\nu_{rs}}}.$$

In the next section we will show how to use Theorem 2.1 to compute discriminants even if the hypothesis of the earlier proposition is not satisfied.

4. The A-discriminant

Let $A \in \mathbb{Z}^{(n-2)\times n}$ and $B \in \mathbb{Z}^{n\times 2}$ be Gale dual matrices as before, and let X be the corresponding toric variety of codimension 2 in $\mathbf{P}^{n-1}$. The *A-discriminant* D_A is the defining irreducible polynomial of the dual variety X^v, unless $condim(X^v) > 1$ in which case $D_A = 1$. Gel'fand *et al.* (1994, Theorem 10.1.2) proved that D_A appears with exponent 1 in the factorization of the full discriminant E_A. In this section we compute D_A and all other factors of E_A in terms of the row vectors $b_i \in \mathbb{R}^2$ of B.

Throughout this section we shall assume that $b_i \neq 0$ for $i = 1, \ldots, n$. This means that X is not a cone over a coordinate point, or that $X^\vee$ does not lie in a coordinate hyperplane. All results in Section 4 require this hypothesis.

Each relevant line in the plane is identified with one of the two primitive vectors $v \in \mathbb{Z}^2$ on that line. We abbreviate $b_i^{(v)} := \det(b_i, v)$. With each such line v, we associate a codimension 1 discriminant as in (1.1).

$$D_v := \prod_{j:b_j^{(v)}<0} (b_j^{(v)})^{b_j^{(v)}} \prod_{i:b_i^{(v)}>0} x_i^{b_i^{(v)}} - \prod_{i:b_i^{(v)}>0} (b_i^{(v)})^{b_i^{(v)}} \prod_{j:b_j^{(v)}<0} x_j^{-b_j^{(v)}}. \tag{4.1}$$

Let $b_{i_1}, \ldots, b_{i_s}$ be all the row vectors of B which lie on the relevant line v. There is a unique integer vector $(\lambda_1, \ldots, \lambda_s)$ such that $b_{i_j} = \lambda_j \cdot v$ for $j = 1, \ldots, s$. We direct the primitive vector $v \in \mathbb{Z}^2$ so that the coordinate sum $\alpha_v := \lambda_1 + \cdots + \lambda_s$ is non-negative, and we define

$$\delta_v := \sum \{-\lambda_i : \lambda_i < 0\}. \tag{4.2}$$

Using this notation, Remark 3.5 can now be defined as follows:

Remark 4.1. If $v = (v_1, v_2)$ is a relevant line for B then $v_1 + v_2 t$ appears with exponent $\delta_v \cdot v_i$ in the factorization of the polynomial $h_i(t)$ in (3.6).

Denote by $p_1(t), p_2(t)$ the respective remaining factors, that is,

$$h_\ell(t) = p_\ell(t) \cdot \prod_{v \text{ relevant}} (v_1 + v_2 t)^{\delta_v \cdot v_\ell} \qquad \ell = 1, 2. \tag{4.3}$$

Now the resultant $r_B := \mathrm{Res}_t(p_1, p_2)$ is a non-zero polynomial in $x_1, \ldots, x_n$. It is customary to call r_B the *residual resultant* of h_1 and h_2. We shall prove the following formulae for the full discriminant and the A-discriminant.

Theorem 4.2. *There exist monomials $x^u, x^{u'}$ and integers ν, ν' such that*

$$D_A(x_1, \ldots, x_n) = (1/\nu) \cdot x^u \cdot r_B(1/x_1, \ldots, 1/x_n) \qquad \text{and}$$
$$E_A(x_1, \ldots, x_n) = \nu' \cdot x^{u'} \cdot D_A(x_1, \ldots, x_n) \cdot \prod_{v \text{ relevant}} D_v(x_1, \ldots, x_n)^{\delta_v}.$$

Proof. We shall first prove the following claim about the full discriminant:

$$r_B(1/x_1, \ldots, 1/x_n) \cdot \prod_{v \text{ relevant}} D_v(x_1, \ldots, x_n)^{\delta_v} \quad \text{divides} \quad E_A(x_1, \ldots, x_n)$$

in the Laurent polynomial ring $k[x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}]$.

Fix any relevant line v. Choose an isomorphism in $SL_2(\mathbb{Z})$ which maps v to $(0, 1)$, and apply this isomorphism to the rows of B. Also reorder the rows of B so that the multiples of v come first. After this transformation, the first column of B has the entries $0,\ldots,0,b_{s+1}^{(v)},b_{s+2}^{(v)},\ldots,b_n^{(v)}$.

For $\ell = 1,2$ and $i = 1,\ldots,s$ only, substitute $y_{i\ell} = b_{i\ell}/x_i$ into the Chow form $\mathcal{C}_B$. Let $\tilde{H}_\ell$ be the polynomials resulting from H_ℓ in (2.2) under the same substitution. Then $\tilde{H}_1 = H_1$, but $\tilde{H}_2$ is divisible by t^{δ_v}, and this is the highest possible power of t with this property (cf. Remark 4.1) Theorem 2.1 implies that the specialized Chow form factors, and one of its factors is

$$\mathrm{Res}_t(H_1, t^{\delta_v}) = (H_1(0))^{\delta_v}. \tag{4.4}$$

For all subsequent specializations, the Chow form factors accordingly. When we substitute $y_{i\ell} = b_{i\ell}/x_i$ for $i = s+1,\ldots,n, \ell = 1,2$, into $H_1(0)$ then we get the binomial D_v in (4.1). Clearly, the residual resultant r_B divides the full discriminant $\tilde{E}_B$. The earlier claim follows from this. Moreover, our argument shows that $D_v^{\delta_v}$ is the highest power of D_v which divides E_A, since the distinct factors in the numerator of the expression in Theorem 2.1 are mapped to distinct face discriminants D_v under the specialization described earlier.

Consider now the factorization formula given by Gel'fand *et al.* (1994, Theorem 10.1.2). The proper faces of the polytope $\mathrm{conv}(A)$ which are not simplices correspond to relevant lines v. This follows from the familiar description of faces of a polytope in terms of its Gale diagram (Grünbaum, 1967, Section 5.4). Hence the face discriminants of such faces are precisely the binomials D_v. In other words, the full discriminant E_A equals the A-discriminant D_A times the product of the expressions $D_v^{\delta_v}$ where v ranges over all relevant lines. We conclude from our claim that $r_B(1/x_1,\ldots,1/x_n)$ divides $D_A(x_1,\ldots,x_n)$ in the Laurent polynomial ring. Since D_A is irreducible, both of our assertions follow. □

We next compute the Newton polygon of the A-discriminant. Define

$$b_v := \alpha_v \cdot v = b_{i_1} + \cdots + b_{i_s}$$

with notation as in (4.2), for any relevant line v. It may happen that $b_v = 0$. We take all non-zero vectors b_v and all vectors b_i which do not lie in relevant lines, and we order them counterclockwise in cyclic order. Let Q_B denote the unique (up to translation) lattice polygon whose boundary consists of these directed edges. For any $i = 1,\ldots,n$ and any lattice point v in Q_B, we define

$$\nu_i := \min\{\det(b_i,u), u \in Q_B\} \qquad \text{and} \qquad \overline{v}^{(i)} := \det(b_i,v) - \nu_i. \tag{4.5}$$

Hence, $\overline{v}^{(i)} \in \mathbb{Z}_{\geq 0}$ is the normalized lattice distance from v to the boundary of Q_B, in the direction orthogonal to b_i.

Theorem 4.3. *The Newton polygon $N(D_A)$ of the A-discriminant D_A is the image of the polygon Q_B under the affine isomorphism $v \mapsto (\overline{v}^{(1)},\ldots,\overline{v}^{(n)})$.*

Proof. Suppose first that there are no relevant lines. Then, $Q_B = P_B$, and the secondary polygon $\Sigma(A)$ and the Newton polygon $N(D_A)$ are equal up to translation. More precisely, $\sum(A) = N(D_A) + \alpha$ where α_i is the exponent of x_i as a factor of E_A. We claim that

$$\alpha_i = d_B - \sum_{j:\det(b_i,b_j)>0} \det(b_i,b_j).$$

This can be seen as follows: the exponent of X_i in the factorization of E_A is the normalized volume of $\mathrm{conv}(A) \setminus \mathrm{conv}(A \setminus \{a_i\})$, which is the sharp lower bound for the total volume of all simplices with vertex a_i appearing in any regular triangulation. Now, the normalized volume of conv(A) equals d_B, while the normalized volume of conv$(A \setminus \{a_i\})$ equals $\sum_{j:\det(b_i,b_j)>0} \det(b_i, b_j)$. In light of Theorem 3.4, it suffices to show that

$$d_B - \mu_i + \det(b_i, v) = \alpha_i + \det(b_i, v) - \nu_i \qquad \text{for all } v \in Q_B, i = 1, \ldots, n.$$

After cancelling terms common to both sides, what remains to be shown is

$$\sum_{j:\det(b_i,b_j)>0} \det(b_i, b_j) = \mu_i - \nu_i.$$

This identity holds because both sides are equal to the normalized lattice width of the polygon $Q_B = P_B$ in the direction orthogonal to b_i.

We next assume that relevant lines exist. Then ν_i generally differs from $\nu_i' := \min\{\det(b_i, u), u \in P_B\}$. The secondary polytope $\Sigma(A)$ equals $N(D_A) + \alpha$ plus the Minkowski sum of the Newton segments of the binomials (4.1) where v runs over all relevant lines. Hence, if we draw P_B and Q_B from the same point,

$$P_B = Q_B + \sum_{v \text{ relevant}} \text{conv}\{0, v\}. \tag{4.6}$$

The minimum value of the linear functional $\det(b_i, *)$ over the line segment conv$\{0, v\}$ is $\det(b_i, v)$, when this value is negative and zero otherwise. Therefore (4.6) translates into the identity

$$\nu_i' = \nu_i + \sum_{v \text{ relevant}} \delta_v \cdot \min\{0, \det(b_i, v)\} \qquad \text{for } i = 1, \ldots, n.$$

The argument for the case of no relevant lines now completes the proof. □

We deduce the following formula for the degree of the A-discriminant:

Corollary 4.4.

$$\text{degree}(D_A) = -\sum_{i=1}^{n} \nu_i$$

We can also extract the following characterization from Theorem 4.3.

Corollary 4.5. *The A-discriminant D_A is equal to* 1 *if and only if the polygon P_B is centrally symmetric.*

Proof. The condition $D_A = 1$ is equivalent to Q_B being a point. This happens if and only if all vectors b_i lie in a relevant line, and $\alpha_v = 0$ for each relevant line v. This last condition is equivalent to P_B being centrally symmetric. □

The following variant to the formula of Theorem 4.2 works well in practice for computing the A-discriminant D_A. In the affine plane with coordinates (w_1, w_2), consider the following parametrically presented rational curve:

$$w_\ell = \frac{\prod_{b_{i\ell}>0}(b_{i1} + b_{i2}t)^{b_{i\ell}}}{\prod_{b_{i\ell}<0}(b_{i1} + b_{i2}t)^{-b_{i\ell}}}, \qquad \ell = 1, 2. \tag{4.7}$$

This is the *Horn uniformization* in Gel'fand *et al.* (1994, Section 9.3.C). Let $\Delta(w_1, w_2)$ be the irreducible polynomial defining this curve. This is a dehomogenization of the

A-discriminant, by Theorem 4.2 or by Gel'fand *et al.* (1994, Theorem 9.3.3. (a)). More precisely,

$$D_A(x_1,\dots,x_n) = (\text{a monomial}) \cdot \Delta\left(\prod_{i=1}^n x_i^{b_{i1}}, \prod_{i=1}^n x_i^{b_{i2}}\right). \tag{4.8}$$

The common factors in the numerator and denominator of (4.7) are precisely the relevant lines which are not a coordinate axis. In other words, cancelling common factors in (4.7) is equivalent to replacing $h_i(t)$ by $p_i(t)$ in (4.3).

We can get a description of the Newton polygon $N(\Delta)$ of $\Delta(w_1, w_2)$ by "dehomogenizing" the result in Theorem 4.3 as follows. Let $^\perp$ denote the linear rotation in the plane defined by $v^\perp := (v_2, -v_1)$.

Corollary 4.6. *Let $B^\perp = \{b_1^\perp, \dots, b_n^\perp\}$ and consider the polygon $Q_{B^\perp}$ translated so that it lies in the first quadrant and its boundary intersects both coordinate axes. Then $N(\Delta) = Q_{B^\perp}$.*

This result has been obtained independently by Sadykov (to appear), under the hypothesis that there are no relevant lines outside the coordinate axes.

Proof. Write $\Delta(w_1, w_2) = \sum_{\alpha \in N(\Delta)} \Delta_\alpha w^\alpha$. Then, by (4.8) $D_A(x)$ equals, up to a monomial, $\sum_{\alpha\in N(\Delta)} \Delta_\alpha \prod_{i=1}^n x_i^{\langle b_i, \alpha\rangle}$. On the other side, we deduce from Theorem 4.3 that $D_A(x)$ has the form

$$\prod_{i=1}^n x_i^{-\nu_i} \sum_{\beta\in Q_B} D_{A,\beta} \prod_{i=1}^n x_i^{\det(b_i,\beta)}.$$

Note that $\det(b_i, \beta)$ equals the inner product $\langle b_i, \beta^\perp\rangle$. Since $Q_{B^\perp}$ is precisely the image under the rotation of Q_B and Δ cannot have any monomial factors, the result follows. □

Example 4.7. We consider the toric 3-fold of degree 43 in $\mathbf{P}^5$ which appears as Example 5.10 in Peeva and Sturmfels (1998). It is defined by the 6×2 integer matrix

$$B = \begin{pmatrix} -1 & -3 \\ -5 & 1 \\ -1 & 4 \\ 2 & 3 \\ 3 & -2 \\ 2 & -3 \end{pmatrix}.$$

The lattice ideal I_B has seven minimal generators. There are no relevant lines. The polygon $P_B = Q_B$ is a hexagon. Using Remark 2.6 we find that Q_B contains 40 lattice points. They correspond to the 40 terms in the A-discriminant D_A. The six vertices of P_B correspond to the various leading terms in D_A. Using (4.8) in any computer algebra system we easily compute:

$$\begin{aligned} D_A = &- (7)^7(17)^{17}(19)^{19}x_1^{16}x_4^{11}x_5^{23}x_6^{22} \\ &- (2)^{34}(3)^{15}(5)^{15}(13)^{13}x_1^{20}x_2^{36}x_3^{11}x_6^{5} \\ &+ (2)^{10}(5)^{15}(11)^{11}(17)^{17}x_1^{23}x_2^{19}x_5^{13}x_6^{17} \\ &+ (2)^{64}(7)^{14}(13)^{13}x_3^{19}x_4^{28}x_5^{16}x_6^{9} \\ &+ (3)^{21}(7)^7(11)^{11}(13)^{13}x_2^{16}x_3^{26}x_4^{25}x_5^{5} \end{aligned}$$

$$
\begin{aligned}
&-(2)^{10}(5)^{15}(11)^{11}(17)^{17}x_1^9x_2^{29}x_3^{21}x_4^{13}\\
&+\text{interior terms.}
\end{aligned}
$$

We invite the reader to draw Q_B and verify Theorem 4.3 for this example. □

5. Resultants Having Newton Triangles

Mixed resultants form a subclass among all discriminants, by the Cayley trick of elimination theory (Gel'fand *et al.*, 1994, Section 9.1.A). This subclass is important for the theory of hypergeometric functions: conjecturally, it consists of the denominators of rational hypergeometric functions (Cattani *et al.*, 2001, Conjecture 1.4). In this section we examine the Cayley construction and mixed resultants in codimension 2.

Let $A_1, \ldots, A_{s+1}$ be vector configurations in $\mathbb{Z}^r$. Their *Cayley configuration* is defined as

$$A = \{e_1\} \times A_1 \cup \{e_2\} \times A_2 \cup \cdots \cup \{e_{s+1}\} \times A_{s+1} \subset \mathbb{Z}^{s+1} \times \mathbb{Z}^r, \tag{5.1}$$

where $e_1, \ldots, e_{s+1}$ is the standard basis of $\mathbb{Z}^{s+1}$.

If we assume that A is not a pyramid, each set A_i must contain at least two points. In the codimension 2 case that we are considering, this implies that $s \le r+1$. When $s = r+1$, each A_i has two elements and $D_A = 1$. We will be concerned in this section with the case $s = r$. Thus, A will be a $(2r+1) \times (2r+3)$ matrix and its Gale dual B is reducible, i.e. the configuration of its row vectors can be partitioned into $r+1$ subsets which have zero sum. We can reorder B to get a $(2r+3) \times 2$-matrix

$$B = (b_1, b_2, \ldots, b_r, c_1, c_2, -b_1, \ldots, -b_r, -c_1 - c_2)^T,$$

where the rows of the submatrix $\tilde{B} := (b_1, b_2, \ldots, b_r, c_1, c_2)^T$ span $\mathbb{Z}^2$. We assume that all b_i are non-zero and $\det(c_1, c_2) \neq 0$. By Corollary 4.5, $D_A \neq 1$.

If we reorder the Cayley matrix accordingly and perform row operations, we can replace it by a matrix which we also call A with the same discriminant (up to reordering the variables), which looks as follows:

$$A = \begin{pmatrix} \tilde{A} & 0 \\ I_{r+1}e_{r+1} & I_{r+1} \end{pmatrix}. \tag{5.2}$$

Here, I_{r+1} is the unit matrix of size $r+1$, $e_{r+1} = (0, 0, \ldots, 0, 1)^T$ and $\tilde{A}$ is an $r \times (r+2)$-matrix Gale dual to $\tilde{B}$ whose left $r \times r$-minor is diagonal

$$\tilde{A} = \begin{pmatrix} \gamma_1 & & & \alpha_1 & \beta \\ & \ddots & & \vdots & \vdots \\ & & \gamma_r & \alpha_r & \beta_r \end{pmatrix} \quad \text{where } \gamma_i \in \mathbb{Z}_{>0} \qquad \text{and} \qquad (\alpha_i, \beta_i) \in \mathbb{Z}^2 \setminus \{(0,0)\}.$$

The columns of A index the coefficients in a sparse system of $r+1$ equations:

$$
\begin{aligned}
f_0 &= z_1 \cdot t_1^{\alpha_1} \cdots t_r^{\alpha_r} + z_2 \cdot t_1^{\beta_1} \cdots t_r^{\beta_r} + z_3\\
f_i &= x_i \cdot t_i^{\gamma_i} + y_i \qquad \text{for } i = 1, \ldots, r.
\end{aligned}
$$

This system consists of r binomials and one Laurent trinomial, as in (1.8). The sparse resultant $\operatorname{Res}(f_0, f_1, \ldots, f_r)$ is the unique (up to sign) irreducible polynomial in $x_1, \ldots, x_r$, $y_1, \ldots, y_r, z_1, z_2, z_3$ which vanishes when the system has a common root in the torus $(\mathbb{C}^*)^r$. From Gel'fand *et al.* (1994, Section 9, Proposition 1.7) we get:

REMARK 5.1. The A-discriminant D_A equals the sparse resultant of $f_0, \ldots, f_r$.

We now apply the product formula for resultants (Pedersen and Sturmfels, 1993), which amounts to evaluating f_0 at the common zeros of $f_1, \ldots, f_r$. The number of zeros equals

$$\Gamma := \gamma_1 \gamma_2 \cdots \gamma_r = |\det(c_1, c_2)|.$$

Let η_i denote a primitive γ_ith root of unity. The product formula implies:

PROPOSITION 5.2. *Up to a Laurent monomial factor, the A-discriminant is*

$$D_A = \text{monomial} \cdot \prod_{i_1=1}^{\gamma_1} \cdots \prod_{i_r=1}^{\gamma_r} f_0(\eta_1^{i_1} z_1, \eta_2^{i_2} z_2, \ldots, \eta_r^{i_r} z_r)$$

where $z_i = (-\frac{y_i}{x_i})^{1/\gamma_i}$ *for* $i = 1, \ldots, r$.

Since f_0 is a trinomial, this formula gives an upper bound of $\binom{\Gamma+2}{2}$ for the number of terms appearing in the expansion of $D_A = \mathrm{Res}(f_0, \ldots, f_r)$. This bound is quadratic in Γ. In truth, this number grows linearly in Γ.

THEOREM 5.3. *The number of terms appearing in* D_A *is at most* $\frac{5}{4} \cdot \Gamma + \frac{7}{4}$.

This bound is tight if the vectors c_1 and c_2 span the lattice $\mathbb{Z}^2$. In this case, $\Gamma = \det(c_1, c_2) = 1$ and the resultant D_A has three terms. It is also tight for the example in the Introduction, where $\Gamma = 4$ and D_A has six terms

PROOF. According to Theorem 4.3, the Newton polygon of the discriminant D_A is essentially the lattice triangle $Q_B = \mathrm{conv}\{0, c_1, c_2\}$. The number of terms in D_A is at most the number of lattice points in Q_B. Using Pick's formula as in Remark 2.6, we find that the number $\#(Q_B \cap \mathbb{Z}^2)$ equals

$$1 + \tfrac{1}{2} \cdot (|\det(c_1, c_2)| + \gcd(c_{11}, c_{12}) + \gcd(c_{21}, c_{22}) + \gcd(c_{11} + c_{21}, c_{12} + c_{22})).$$

Using the inequality $a + b \leq ab + 1$, we find that the sum of any two of the three last summands is bounded above by $\Gamma + 1 = |\det(c_1, c_2)| + 1$. Therefore,

$$\#(Q_B \cap \mathbb{Z}^2) \leq 1 + \tfrac{1}{2} \cdot \big(\Gamma + \tfrac{3}{2} \cdot (\Gamma + 1)\big).$$

This is the desired inequality. □

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