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On the essential spectrum of magnetic Schrödinger operators in exterior domains

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Abstract. We establish equality between the essential spectrum of the Schrödinger operator with magnetic field in the exterior of a compact arbitrary dimensional domain and that of the operator defined in all the space, and discuss applications of this equality.

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1. Introduction

Magnetic Schrödinger operators in domains with boundaries appear in several areas of physics, one can mention the Ginzburg–Landau theory of superconductors, the theory of Bose–Einstein condensates, and the study of edge states in Quantum mechanics. We refer the reader to [1,2,7] for details and additional references on the subject. From the point of view of spectral theory, the presence of boundaries has an effect similar to that of perturbing the magnetic Schrödinger operator by an electric potential. If we focus at present on two dimensional domains and constant magnetic fields, we observe in both cases (exterior domain and electric potential), that the essential spectrum consists of the Landau levels and the discrete spectrum form clusters of eigenvalues around the Landau levels. Several papers are devoted to the study of different aspects of these clusters of eigenvalues in domains with or without boundaries. In case of domains with

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boundaries, one can cite [3–5,8,9] for results in the semi-classical context, [10,11] and the references therein for the study of accumulation of eigenvalues.

Consider a *compact* and *connected* domain $K \subset \mathbb{R}^d$ whose boundary consists of a finite number of *smooth* closed curves. Denote by $\Omega = \mathbb{R}^d \setminus K$. Given a function $\gamma \in L^\infty(\partial\Omega)$ and a vector potential $A \in C^1(\mathbb{R}^d;\mathbb{R}^d)$, we define the Schrödinger operator $L^\gamma_{\Omega,\mathbf{B}}$ with domain $D\left(L^\gamma_{\Omega,\mathbf{B}}\right)$ as follows,

$$D\left(L^\gamma_{\Omega,\mathbf{B}}\right) = \{u \in L^2(\Omega) : (\nabla - i\mathbf{A})^j u \in L^2(\Omega), j = 1,2; \nu \cdot (\nabla - i\mathbf{A})u + \gamma u = 0 \text{ on } \partial\Omega\}, \tag{1.1}$$

$$\forall u \in D\left(L^\gamma_{\Omega,\mathbf{B}}\right), \quad L^\gamma_{\Omega,\mathbf{B}} u = -(\nabla - i\mathbf{A})^2 u. \tag{1.2}$$

The vector ν is the unit *outward* normal vector of the boundary $\partial\Omega$. The magnetic field $\mathbf{B}$ is identified by an antisymmetric matrix $(b_{k,j})_{1\leqslant k,j\leqslant d}$ whose entries are defined by the components (a_j) of $\mathbf{A}$ as follows, $b_{k,j} = \partial_{x_j} a_k - \partial_{x_k} a_j$. Associated with the operator $L^\gamma_{\Omega,\mathbf{B}}$ is the quadratic form,

$$q^\gamma_{\Omega,\mathbf{B}}(u) = \int_\Omega |(\nabla - i\mathbf{A})u|^2 \mathrm{d}x + \int_{\partial\Omega} \gamma |u|^2 \mathrm{d}S, \quad u \in H^1_{\mathbf{A}}(\Omega), \tag{1.3}$$

where the space $H^1_{\mathbf{A}}(\Omega) = \{u \in L^2(\Omega) : (\nabla - i\mathbf{A})u \in L^2(\Omega)\}$ is the form domain of $q^\gamma_{\Omega,\mathbf{B}}$.

Since the function $\gamma \in L^\infty(\partial\Omega)$, the operator $L^\gamma_{\Omega,\mathbf{B}}$ is semi-bounded from below and its associated quadratic form is closed, Friedrichs' theorem tells us that $L^\gamma_{\Omega,\mathbf{B}}$ is self-adjoint in $L^2(\Omega)$.

We introduce the magnetic Schrödinger operator $L_{\mathbf{B}}$ in $L^2(\mathbb{R}^d)$ with magnetic field $\mathbf{B}$ as follows. The domain of the operator is $D(L_{\mathbf{B}}) = \{u \in L^2(\mathbb{R}^d) : (\nabla - i\mathbf{A})^j u \in L^2(\mathbb{R}^d), j = 1,2\}$, and the action of the operator on its domain is as follows,

$$L_{\mathbf{B}} u = -(\nabla - i\mathbf{A})^2 u, \quad (\text{in } L^2(\mathbb{R}^d)). \tag{1.4}$$

In this note, we establish the following result and discuss consequences of it.

Theorem 1.1. *The essential spectrum of the operator $L^\gamma_{\Omega,\mathbf{B}}$ is the same as that of the operator $L_{\mathbf{B}}$.*

Earlier versions of Theorem 1.1 are already proven for two-dimensional domains [10,11] under different boundary conditions and for constant magnetic fields only. Theorem 1.1 remains true for the magnetic Schrödinger operator with Dirichlet boundary condition (that is when replacing the Robin condition in (1.1) by the condition $u = 0$ on $\partial\Omega$). The proof is exactly the same as the one we present here.

2. Proof of Theorem 1.1

We denote by Γ the common boundary of Ω and K and define the following operator on Γ,

$$\partial_\Gamma u = \partial_N u + \gamma\, u = \nu \cdot (\nabla - ib\mathbf{A})u + \gamma\, u, \tag{2.1}$$

where ν is the unit *outward* normal vector to the boundary of Ω.

We have introduced the operator $L^{\gamma}_{\Omega,\mathbf{B}}$ with quadratic form $q^{\gamma}_{\Omega,\mathbf{B}}$ from (1.3). We will also use the corresponding operator in K, namely $L^{-\gamma}_{K,\mathbf{B}}$. Since the quadratic forms $q^{\gamma}_{\Omega,\mathbf{B}}$ and $q^{-\gamma}_{K,\mathbf{B}}$ are semi-bounded (see (1.3)), we get up to a shift by a positive constant that they are strictly positive. Thus we assume, the hypothesis:

(H1) The operators $L_{\mathbf{B}}$, $L^{\gamma}_{\Omega,\mathbf{B}}$ and $L^{-\gamma}_{K,\mathbf{B}}$ are invertible.

Since Ω and K are complementary, the Hilbert space $L^2(\mathbb{R}^d)$ is decomposed as the direct sum $L^2(\Omega)\oplus\ L^2(K)$ in the sense that any function $u\in L^2(\mathbb{R}^d)$ can be represented as $u_\Omega\oplus u_K$ where u_Ω and u_K are the restrictions of u to Ω and K, respectively. Notice that, for all $u=u_\Omega\oplus u_K\in L^2(\mathbb{R}^d)$ such that $u_\Omega\in D(L_{\Omega,\mathbf{B}})$ and $u_K\in D(L_{K,\mathbf{B}})$, then $\partial_\Gamma u_\Omega=\partial_\Gamma u_K=0$, where ∂_Γ is the trace operator from (2.1).

We can extend the operator $L^{\gamma}_{\Omega,\mathbf{B}}$ in $L^2(\Omega)$ to an operator $\widetilde{L}$ in $L^2(\mathbb{R}^d)$. Actually, let $\widetilde{L}=L^{\gamma}_{\Omega,\mathbf{B}}\oplus L^{-\gamma}_{K,\mathbf{B}}$ in $D\left(L^{\gamma}_{\Omega,\mathbf{B}}\right)\oplus D\left(L^{-\gamma}_{K,\mathbf{B}}\right)\subset L^2(\mathbb{R}^d)$. More precisely, $\widetilde{L}$ is the self-adjoint extension associated with the quadratic form

$$\tilde{q}(u)=q^{\gamma}_{\Omega,\mathbf{B}}(u_\Omega)+q^{-\gamma}_{K,\mathbf{B}}(u_K),\quad u=u_\Omega\oplus u_K\in L^2(\mathbb{R}^d). \tag{2.2}$$

By the hypothesis (H1), we may speak of the resolvent $\widetilde{R}=\widetilde{L}^{-1}$ of $\widetilde{L}$. Since $\sigma(\widetilde{L})=\sigma\left(L^{\gamma}_{\Omega,\mathbf{B}}\right)\cup\sigma\left(L^{-\gamma}_{K,\mathbf{B}}\right)$ and $L^{-\gamma}_{K,\mathbf{B}}$ has a compact resolvent, then we get the following lemma.

Lemma 2.1. *With $\widetilde{L}$, $\widetilde{R}$ and $L^{\gamma}_{\Omega,\mathbf{B}}$ defined as above, it holds true that:*

(1) $\sigma_{\mathrm{ess}}\left(L^{\gamma}_{\Omega,\mathbf{B}}\right)=\sigma_{\mathrm{ess}}(\widetilde{L})$.

(2) $\lambda\in\sigma_{\mathrm{ess}}(\widetilde{R})\setminus\{0\}$ if and only if $\lambda\neq0$ and $\lambda^{-1}\in\sigma_{\mathrm{ess}}\left(L^{\gamma}_{\Omega,\mathbf{B}}\right)$.

In the next lemma, we observe that the operator $L^{\gamma}_{\Omega,\mathbf{B}}$ can be viewed as a compact perturbation of the magnetic Schrödinger operator $L_{\mathbf{B}}$ in $L^2(\mathbb{R}^d)$ introduced in (1.4).

Lemma 2.2. *The operator $V=\widetilde{L}^{-1}-L_{\mathbf{B}}^{-1}$ is compact. Moreover, for all $f,g\in L^2(\mathbb{R}^d)$, it holds that*

$$\langle f,Vg\rangle_{L^2(\mathbb{R}^d)}=\int_\Gamma\partial_\Gamma u\cdot\overline{(v_\Omega-v_K)}\,\mathrm{d}S, \tag{2.3}$$

where $u=L_{\mathbf{B}}^{-1}f$ and $v=\widetilde{L}^{-1}g$.

Proof. Since $f=L_{\mathbf{B}}u$ and $g=\widetilde{L}v=L^{\gamma}_{\Omega,\mathbf{B}}v_\Omega\oplus L^{-\gamma}_{K,\mathbf{B}}v_K$, it follows that,

$$\langle f,Vg\rangle_{L^2(\mathbb{R}^d)}=\int_\Omega L_{\mathbf{B}}u\cdot\overline{v_\Omega}\,\mathrm{d}x+\int_K L_{\mathbf{B}}u\cdot\overline{v_K}\,\mathrm{d}x-\int_\Omega u\cdot\overline{L^{\gamma}_{\Omega,\mathbf{B}}v_\Omega}\,\mathrm{d}x-\int_K u\cdot\overline{L^{-\gamma}_{K,\mathbf{B}}v_K}\,\mathrm{d}x.$$

The identity in (2.3) then follows by integration by parts and by using the boundary conditions $\partial_\Gamma v_\Omega=\partial_\Gamma v_K=0$.

As we will show below, compactness of the trace operators together with (2.3) give us compactness of the operator V. Let (g_n) be a sequence in $L^2(\mathbb{R}^d)$ that converges weakly to 0. We will prove that (Vg_n) converges strongly in $L^2(\mathbb{R}^d)$. We define,

$$u^{(n)} = L_{\mathbf{B}}^{-1} V g_n, \quad v^{(n)} = \widetilde{L}^{-1} g_n.$$

Let $U \subset \mathbb{R}^d$ be an open and bounded set that contains the common boundary Γ of Ω and K. We claim that there exists a positive constant C such that,

$$\forall n \in \mathbb{N}, \quad \|u^{(n)}\|_{H^2(\Omega)} + \left\|v_\Omega^{(n)}\right\|_{H^2(\Omega\cap U)} + \left\|v_K^{(n)}\right\|_{H^2(K\cap U)} \leqslant C. \tag{2.4}$$

Once the estimate in (2.4) is established, we get compactness of the operator V as follows. Since the embeddings of $H^2(U \cap \Omega)$ and $H^2(\Omega \cap K)$ in $L^2(\Gamma)$ are compact, we get that,

$$\left\|v_\Omega^{(n)}\right\|_{L^2(\Gamma)} + \left\|v_K^{(n)}\right\|_{L^2(\Gamma)} \to 0 \quad \text{as } n \to \infty.$$

Also, the trace theorem yields that $\|\partial_\Gamma u^{(n)}\|_{L^2(\Gamma)}$ is bounded. Now, we may use (2.3) with $f = Vg_n$, $g = g_n$ and deduce that,

$$\|Vg_n\|^2_{L^2(\mathbb{R}^d)} = \int_\Gamma \partial_\Gamma u^{(n)} \cdot \overline{\left(v_\Omega^{(n)} - v_K^{(n)}\right)} \mathrm{d}\, S \to 0 \quad \text{as } n \to \infty,$$

thereby establishing compactness of V. To finish the proof of Lemma 2.2, we need to prove the claim in (2.4). Since Vg_n is in $L^2(\mathbb{R}^d)$ we get by definition of $L_{\mathbf{B}}^{-1}$ that $L_{\mathbf{B}} u^{(n)} = Vg_n$. As a consequence, elliptic L^2-estimates yield boundedness of $u^{(n)}$ in $H^2(U)$. In a similar way we obtain boundedness of $v_\Omega^{(n)}$ and $v_K^{(n)}$ in H^2. Actually, it holds true that,

$$L_{\mathbf{B},\Omega} v_\Omega^{(n)} = g_n \quad \text{in} \Omega, \quad L_{\mathbf{B},K} v_K^{(n)} = g_n \quad \text{in } L^2(K),$$

together with the boundary conditions $\partial_\Gamma v_\Omega^{(n)} = 0$ and $\partial_\Gamma v_K^{(n)} = 0$. Boundedness of $v_\Omega^{(n)}$ and $v_K^{(n)}$ in H^2 then result from elliptic L^2-estimates (up to the boundary). □

Proof of Theorem 1.1. As corollary of Lemma 2.2 and Weyl's theorem, we get that $L_{\mathbf{B}}$ and $\widetilde{L}$ have the same essential spectrum. Consequently, Lemma 2.1 tells us that Theorem 1.1 is true. □

3. Applications of Theorem 1.1

The spectrum of the operator $L_{\mathbf{B}}$ is studied in several papers, see [6] and the references therein. Under the assumptions made in Corollary 3.1 below, it is proved in [6, Thm. 1.5] that the essential spectrum of $L_{\mathbf{B}}$ is exactly the union of spectra of all operators of the form $L_{\mathbf{B}_\infty}$, where $\mathbf{B}_\infty$ is a cluster value of the magnetic field $\mathbf{B}$ at ∞. The spectrum of $L_{\mathbf{B}_\infty}$ is either the interval $[\mathbf{B}_\infty, \infty)$ (if $\mathbf{B}_\infty = 0$ or d is odd) or the Landau levels otherwise. Since $L^\gamma_{\Omega,\mathbf{B}}$ has the same essential spectrum as $L_{\mathbf{B}}$, we get the result in Corollary 3.1 below.

Corollary 3.1. *Suppose that the magnetic field* $\mathbf{B} \in C^3$ *satisfies the following condition,*

$$\sum_{1\leqslant|\alpha|\leqslant 3}\ \sum_{1\leqslant i,j\leqslant n}|D^\alpha b_{i,j}(x)| = \mathcal{O}(|x|^{-\alpha}),\quad \text{as } |x|\to\infty, \tag{2.5}$$

where $b_{i,j}$ are the components of $\mathbf{B}$ *and α is a positive real number. The following statements are true.*

(1) *If* $d \geqslant 2$, *then* $\inf\sigma_{\mathrm{ess}}\left(L^\gamma_{\Omega,\mathbf{B}}\right) \geqslant \liminf_{|x|\to\infty}|B(x)|$.
(2) *If* $\liminf_{|x|\to\infty}|B(x)| = 0$, *then* $\sigma_{\mathrm{ess}}\left(L^\gamma_{\Omega,\mathbf{B}}\right) = [0,\infty)$.
(3) *If* $d = 2$, $\lim_{|x|\to\infty}|\mathbf{B}(x)| = b$ *and* $b > 0$, *then,* $\sigma_{\mathrm{ess}}\left(L^\gamma_{\Omega,\mathbf{B}}\right) = \{(2n-1)b : n\in\mathbb{N}\}$.
(4) *If* $d = 3$ *and* $\liminf_{|x|\to\infty}|\mathbf{B}(x)| = b$, *then,* $\sigma_{\mathrm{ess}}\left(L^\gamma_{\Omega,\mathbf{B}}\right) = [b,\infty)$.

The next corollary indicates situations where the spectrum of $L^\gamma_{\Omega,\mathbf{B}}$ is purely discrete.

Corollary 3.2. *Suppose that there exists a non-negative integer r such that* $\mathbf{B}\in C^{r+1}(\mathbb{R}^d)$. *Let $b_{k,j}$ be the components of* $\mathbf{B}$. *If there exists a positive constant C such that,*

$$\sum_{k,j}\ \sum_{\substack{\alpha\in\mathbb{N}^d\\ |\alpha|=r+1}}|D^\alpha b_{k,j}(x)| \leqslant C\left(\sum_{k,j}\ \sum_{\substack{\alpha\in\mathbb{N}^d\\ |\alpha|\leqslant r}}|D^\alpha b_{k,j}(x)|+1\right),$$

and $\sum_{k,j}\sum_{\substack{\alpha\in\mathbb{N}^d\\ |\alpha|\leqslant r}}|D^\alpha b_{k,j}(x)|\to\infty$ *as* $|x|\to\infty$, *then the operator* $L^\gamma_{\Omega,\mathbf{B}}$ *has compact resolvent.*

Under the conditions in Corollary 3.2, the operator $L_{\mathbf{B}}$ has compact resolvent [6, Corollaire 1.2]. As a consequence of Lemma 2.2, we get that the operator $L^\gamma_{\Omega,\mathbf{B}}$ has compact resolvent too, thereby proving Corollary 3.2.

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