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On the recognition of digital circles in linear time

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Abstract

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We consider the digitalization mapping dig : $\mathbb{R}^n \to \mathbb{Z}^n$ with

$$X = (x_1, \ldots, x_n) \to \begin{cases} P = (p_1, \ldots, p_n), & \text{if } \underset{1 \leq i \leq n}{\exists} (p_i - 0.5 < x_i \leq p_i + 0.5) \wedge \underset{j \neq i}{\forall} (p_j = x_j); \\ \emptyset & \text{otherwise.} \end{cases}$$

For a given object $s \subseteq \mathbb{R}^n$ one can obtain the so-called digitalization dig(s) of s. One problem of the image processing is the recognition of objects $s \subset \mathbb{R}^n$, where $\gamma = \text{dig}(s)$ is given. In case of dimension $n = 2$ we formulate necessary and sufficient conditions, that a given set $\gamma \subset Z^2$ is the digitalization of a Euclidean circle s.

Keywords. Image processing; digital circle; linear time algorithm; recognition of digital objects.

Introduction

We consider the so-called digitalization mapping dig : $\mathbb{R}^n \to \mathbb{Z}^n$ with

$$X = (x_1, \ldots, x_n) \to \begin{cases} P = (p_1, \ldots, p_n), & \text{if } \underset{1 \leq i \leq n}{\exists} (p_i - 0.5 < x_i \leq p_i + 0.5) \wedge \underset{j \neq i}{\forall} (p_j = x_j); \\ \emptyset & \text{otherwise} \end{cases}$$

For example, the dots of Fig. 1 showing the digitalization dig(s) of a given set $s \subseteq \mathbb{R}^2$.

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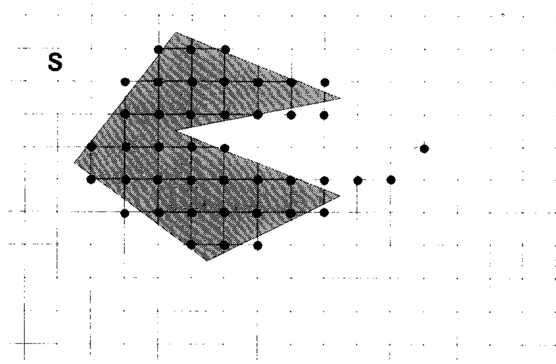

Fig. 1.

The conversion is a recognition problem of image processing: We consider a set *allobjects* of Euclidean objects, for example polygons in the plane. For a given set α of gridpoints one can ask (see Fig. 2): Is there a Euclidean object $object \in allobjects$ with the property $\alpha = \mathrm{dig}(object)$? In this paper we consider sets *allobjects* of circles and straight lines, respectively, in $\mathbb{R}^2$. The first aim is the transformation of straight lines onto circles and vice versa. From the view point of projective geometry both, Euclidean circles and straight lines, are no distinct objects. The question is, which properties of digitializing Euclidean straight lines are useful for digitalizing Euclidean circles?

In a first part we recall results of digitalizing Euclidean straight lines. In second one we consider digitalizing Euclidean circles. The main result of this part is a linear time algorithm for the recognition of digitalizing Euclidean circles.

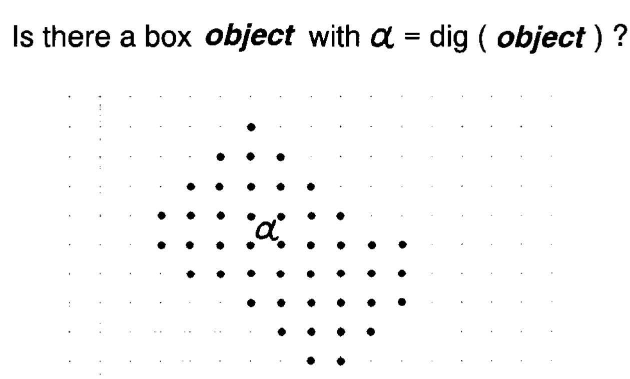

Fig. 2.

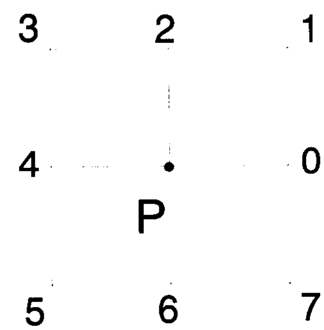

Fig. 3.

0. Some basic notions for planar sets

Neighbouring points. We call two lattice points P, Q *neighbours*, if their Euclidean distance is 1 of $\sqrt{2}$.

Freeman code. The encoding of the 2-dimensional neighbours of a point P is possible with the condenumbers shown in Fig. 3.

As Fig. 4 shows we obtain for a given sequence of lattice points a sequence of codenumbers

$$(C_i)_{i\in\mathbb{Z}} = \cdots 1\ 0\ 0\ 1\ 0\ 1\ 0\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 0\ 1 \cdots \quad .$$

Freeman sequence. Let be $k = 1$. We consider for a given sequence $(C_i)^{(k)}_{i\in\mathbb{Z}} := (C_i)_{i\in\mathbb{Z}}$ the conditions (C1), (C2) and (C3), respectively (see Freeman [1]):

(C1) There are at most two codenumbers. In case of two codenumbers they are neighbouring numbers (modulo 8).

(C2) If there are two codenumbers, then at least one of this codenumbers are singular, i.e., the neighbours in the sequence $(C_i)^{(k)}_{i\in\mathbb{Z}}$ are distinct of the singular number.

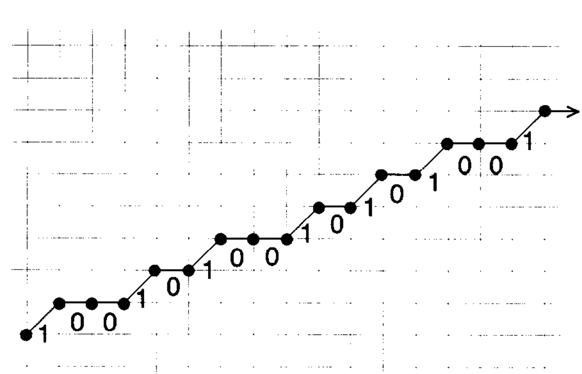

Fig. 4.

(C3) If the conditions (C1), (C2) are fullfilled, then we have two cases:

Case a: There is only one codenumber in the sequence $(C_i)_{i\in\mathbb{Z}}^{(k)}$. In this case we finish our considerations about sequences of codenumbers.

Case b: If not Case a let $k := k+1$. Observe a new codesequence $(C_i)_{i\in\mathbb{Z}}^{(k)}$ generating in the following way:

If both codenumbers are singular we define one as the nonsingular one, for example, the smallest.

We enumerate only the nonsingular codenumbers from one singular codenumber to the next one. With the sequence of the new codenumbers we continue at condition (C1). We call sequences with the properties (C1) up to (C3) for $k = 1, 2, \ldots$ *Freeman sequences.*

To give a concret example suppose

$$(C_i)_{i\in\mathbb{Z}}^{1} = \cdots 1\ 0\ 0\ 1\ 0\ 1\ 0\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 0\ 1\ 0\ 1\ 0\ 0\ 1\ 0\ 1\ 0\ 0\ 1\ 0\ 1\ 0$$
$$1\ 0\ 0\ 1\ 0\ 1\ 0\ 0\ 1\ 0\ 1\ 0\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 0 \cdots$$

with the singular codenumber 1.

We obtain with help of Freeman's algorithm (C1), (C2), (C3):

$$(C_i)_{i\in\mathbb{Z}}^{2} = \cdots 2\ 1\ 2\ 1\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 1\ 2 \cdots$$

with the singular codenumber 2. It follows

$$(C_i)_{i\in\mathbb{Z}}^{3} = \cdots 1\ 2\ 1\ 1\ 2\ 1\ 1\ 2 \cdots$$

with the singular codenumber 2, and, finally

$$(C_i)_{i\in\mathbb{Z}}^{4} = \cdots 2\ 2 \cdots$$

Th next notion is a topological one.

8-Curves. 8-*Curves* are sequences $\gamma \subseteq \mathbb{Z}^2$ of gridpoints with the property: For every point $P \in \gamma$ there is exactly two neighbours $Q, R \in \gamma$ in P, if γ is infinite.

In case of finite sequences γ there are exactly two points A, B of γ with exactly one neighbour in γ. For every point $P \in \gamma$, $P \notin \{A, B\}$ exists exactly two neighbours $Q, R \in \gamma$ of P.

1. Digital straight lines

Let a subset of gridpoints $\alpha \subseteq \mathbb{Z}^2$. We call α a *digital straight line,* if there is a Euclidean straight line g with $\alpha = \mathrm{dig}(g)$. Three results about properties of digital straight lines or line segments are given as follows.

(STR1) Rosenfeld [2] showed: The 8-curve γ is an digital straight line (or line segment), iff for every points $P, Q \in \gamma$ and every point $X \in \overline{PQ}$ of the Euclidean

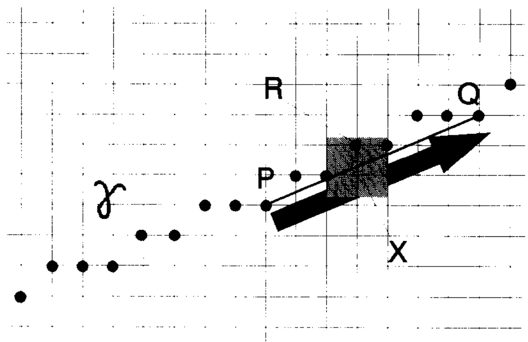

Fig. 5.

segment $\overline{PQ}$ there must exist at least one point $R \in \gamma$ with $\|R - X\|_\infty < 1$ (see Fig. 5).

(STR2) WU [3] proved: The conditions (C1), (C2), (C3) are necessary and sufficient conditions in case of codesequence $(C_i)_{i\in\mathbb{Z}}$ of digital straight lines γ.

(STR3) In [5] is shown: For every Euclidean straight line g with $\gamma = \mathrm{dig}(g)$ there is a stripe S parallel g with $\mathrm{dig}(S) = \gamma$ and the width of S is a function of the slope of g (see Fig. 6). In case of irrational slope m_1/m_2 with width of S is zero (for more details see [5]).

(STR4) Creutzburg [4] at all proved: the recognition of digital straight line segments with N points is possible in O(N).

2. Digital circles

We call $\xi \subseteq \mathbb{Z}^2$ a *digital circle*, if there is a Euclidean circle K with $\mathrm{dig}(k) = \xi$. The digital circle problem is given as follows.

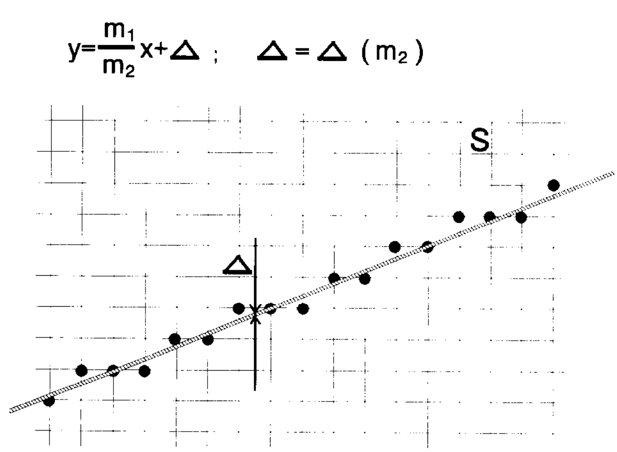

Fig. 6.

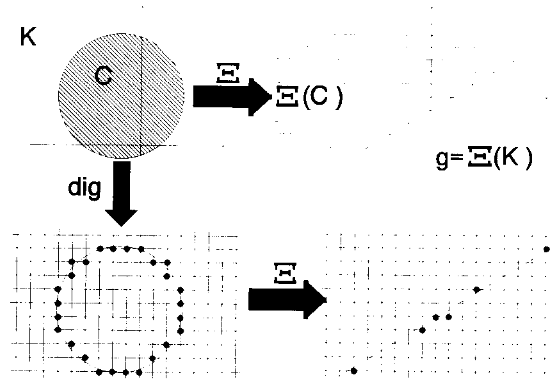

Fig. 7.

(DCP) Deciding, whether or not a given point set $\xi \subseteq \mathbb{Z}^2$ is a digital circle.

We denote in the following an arbitrary Euclidean circle with C and the boundary of C with K. In special cases we identify the Euclidean circle C with K.

Our aim is to find properties (CIR1) $\cdots$ (CIR4) about digital circles ξ analogously (STR1) $\cdots$ (STR4). For these reasons we firstly attempt to transform digital circles to digital straight lines and vice versa. The result is negative:

Let be Ξ an arbitrary topological transformation of the projective plane ε onto a projective plane π with the following property (T).

(T) C is a projective circle if and only if $\Xi(C)$ is a projective circle.

It holds (see Fig. 7).

Lemma 1. *If the image $\Xi(K)$ is a straight line g, then holds in general $\Xi(\mathrm{dig}(K))$ is no digital straight line segment.*

It is easy to show Lemma 1 and the proof is omitted here.

Let $K(\overline{A, B, C})$ denote the minimal arc of the projective circle containing the three points A, B, C (for the possible cases see Fig. 8).

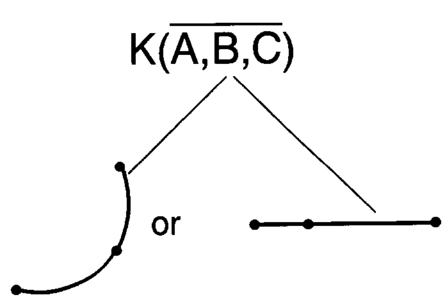

Fig. 8.

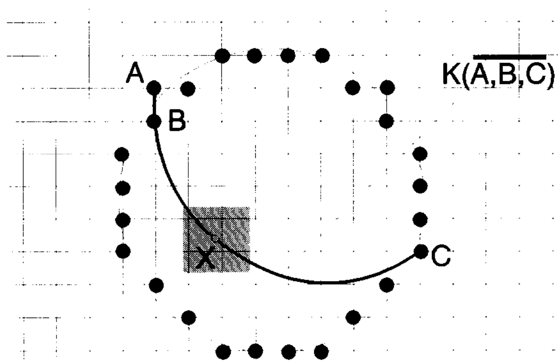

Fig. 9.

Then holds in general the following.

Lemma 2. *For every three points A, B, C of a given digital circle ξ and any point X of $K(\overline{A, B, C})$ there is not necessarily a point $R \in \xi$ with $\|R \times X\|_\infty < 1$ (see Fig.* 9).

For the proof of Lemma 2 we have to find an example as we have already done in Fig. 9. This is omitted here.

(CIR1) *It does not exist a similar result to Rosenfelds characterization* (STR1) *for digital circles.*

(CIR2) The author thinks there is no hope to find a general characterization of digital circles analogous Freeman's chaincode for straight lines (STR2).

Now we look for more positive results.

2.1. *The first result*

Let be given m convex and bounded polytops in the n-dimensional Euclidean space.

A remarkable theorem of convex geometry is the following.

Theorem of Helley: *If any $n + 1$ of the m polytops have at least one point in common, then all m polytops have one point in common.*

Let K denote the boundary of a Euclidean circle C and let ξ denote the digitalization dig(K) of K.

For the lattice point P denote cross(P) the grid cross around P with side length less than 1 without two endpoints. The set cross(P) contains the horizontal segment $h(P)$ and the vertical segment $v(P)$, respectively (see Fig. 10).

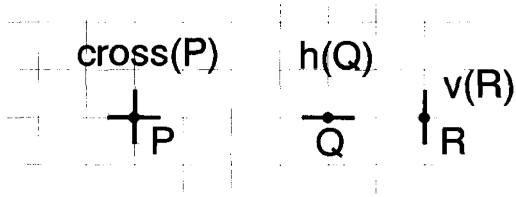

Fig. 10.

Clearly, $P \in \xi$, if and only if $K \cap \operatorname{cross}(P) \neq \emptyset$.

Let be ALLSEGMENTS $:= \bigcup_{P \in \xi} (h(P) \cup v(P))$.

Theorem 1. *ξ is a digital circle, if and only if the following two conditions are fullfilled*:

(1) *There is a set* SEGMENTS $\subseteq$ ALLSEGMENTS *with exactly one segment $h(P)$ or $v(P)$ included in* SEGMENTS, *for all $P \in \xi$.*

(2) *For any four elements $a, b, c, d \in$* segments *there is a Euclidean circle K' with $K' \cap a \neq \emptyset, \ldots, K' \cap d \neq \emptyset$.*

Proof. Firstly we transform the situation for the intersection of $v(P)$ or $h(P)$ with K', into the three-dimensional space. Then we use the theorem of Helly.

Step 1. Let be given a circle $K'(r, x_m, y_m)$ with midpoint (x_m, y_m) and the radius r, respectively. If K' intersects the segment $v(P)$ (see Fig. 11), then it holds:

$$p_2 - 0.5 < \pm\sqrt{r^2 - (p_1 - x_m)^2} + y_m \leq p_2 + 0.5. \tag{1}$$

Assume r, x_m, y_m are variables, then holds:

$$p_2 - 0.5 - z_3 < \pm\sqrt{z_1^2 - (p_1 - z_2)^2} \leq p_2 + 0.5 - z_3. \tag{2}$$

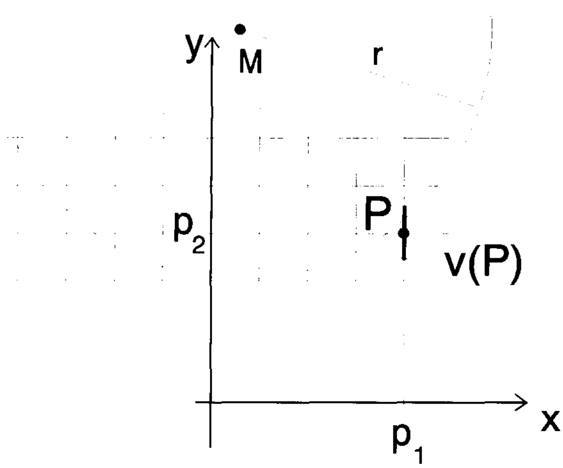

Fig. 11.

Some simple consequences of (2) are the following:

(E1) For fixed p_1 and y, respectively, the equation

$$z_1^2 = (p_1 - z_2)^2 + (y - z_3)^2$$

determining a double-cone. We can select one cone C_{y,p_1} for the sign + (and one for the sign −) by inequation (2).

(E2) If we choose the sign + (or −) in (2), then the union

$$U_{y,p_1} := \bigcup_{p_2-0.5<y\leq p_2+0.5} C_{y,p_1}$$

of cones forms a convex body.

(E3) The point $(z_1, z_2, z_3) = (r, x_m, y_m)$ is an element of U_{y,p_1}, if and only if $K'(r, x_m, y_m) \cap v(P) \neq \emptyset$, where $P = (p_1, p_2)$.

(E4) Analogously the properties (E1) up to (E3) hold with respect to $h(P)$ for the union U_{x,p_2}, where $p_1 - 0.5 < x \leq p_1 + 0.5$ and $P = (p_1, p_2) \in \xi$.

(E5) The sets U_{y,p_1}, U_{x,p_2} are bounded by the diameter diam(ξ) of ξ, i.e., it holds:

$$r < 2 \cdot \mathrm{diam}(\xi) \text{ if and only if } |z_1| < 2 \cdot \mathrm{diam}(\xi).$$

Step 2. Observe a set SEGMENTS ⊆ ALLSEGMENTS of horizontal or vertical segments $h(P)$ or $v(P)$, with $P \in \xi$.
Let be M the number of points of ξ.
Similarly Step 1, we transform the set SEGMENTS into M convex and bounded polytops U_{x,p_2}, U_{y,p_1} where these polytops are in $\mathbb{R}^3$.

The theorem of Helly and property (E3) prove the first direction of Theorem 1.

The conversion is trivial and the proof of Theorem 1 is finished. □

Corollary. *If the number of points of ξ is N, then we can solve the digital circle problem in* $O(N^4)$.

2.2 The main result

We prepare the next theorem about the recognition of a digital circle by giving some properties of boundary points of Euclidean circles C.

Let SUPPORT ⊆ C be a non-empty subset of points of a Euclidean circle C. Let denote τ an arbitrary translation and ι the identity.

Definition 1. We call SUPPORT a *supporting set* of C, iff for any translation $\tau \neq 1$ in the plane $\mathbb{R}^2$ there is a point $P \in$ SUPPORT with $P \notin \tau(C)$.

Definition 2. A supporting set SUPPORT of C is called *minimal* (MSS), iff for every point $P \in$ SUPPORT the set SUPPORT\\{P} is no supporting set of K. Let be SUPPORT an arbitrary MMS of the circle C with midpoint M.

It is easy to show the following properties:

(S1) SUPPORT contains no point of the interior int(C) of C.

(S2) The number of points of SUPPORT is equal 2 or equal 3.

(S3) M is element of the convex hull conv(SUPPORT) of SUPPORT.

(S4) Let HALF $\subseteq$ bd(C) be an arbitrary half-circle without at least one endpoint. Then follows SUPPORT $\nsubseteq$ HALF.

Note, the computing test for supporting sets SUPPORT of a given circle C is possible in O(N), if the number of points of the set SUPPORT is N.

In the next claim we gather some properties of digital circles.

Lemma 3. *Let be given a digital circle ξ in the lattice $\mathbb{Z}^2$. Suppose,* BOX *is the minimal covering lattice box of ξ with midpoint $M(x_m, y_m)$ and side length a, b. Let*

$$\text{MID} := \{(x, y) \in \mathbb{R}^2 \mid x_m - 0.5 < x \leq x_m + 0.5 \wedge y_m - 0.5 < y \leq y_m + 0.5\}$$

and let $\overline{\text{MID}} := \text{cl(MID)}$ *the closure of* MID. *Assume, C is an arbitrary Euclidean circle with midpoint M and the boundary K.*

If $\text{dig}(K) = \xi$ *then hold the following properties*:

(C1) $\text{O} \in \text{MID}$

(C2) *The radius r of K is bounded by*

$$\max(a, b) - 1 < r < \min(a, b) + 1.$$

(C3) *We consider regions* $\text{REG}_1, \ldots, \text{REG}_4$ *with slope $\frac{1}{2}$ or $-\frac{1}{2}$ and width $\sqrt{2}$ (see Fig. 12). Then holds:*

In every region $\text{REG}_1, \ldots, \text{REG}_4$ *there are at most three points of ξ.*

(C4) *If there are three points P, Q, R in a region* $\text{REG}_1, \ldots, \text{REG}_4$, *then the curvature of minimal arc of the projective circle containing P, Q, R 'corresponds' to the curvature of K (see the Figs* 13a *and* 13b, *respectively).*

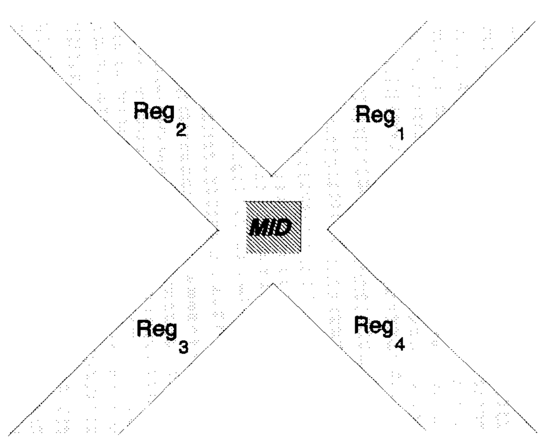

Fig. 12.

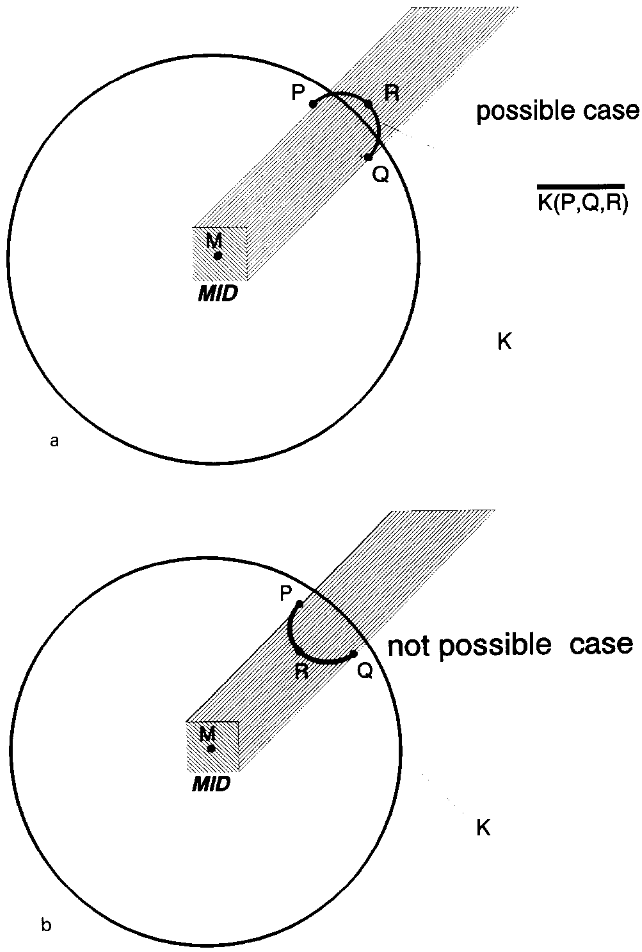

Fig. 13.

(C5) *Let be given three points P, Q, R contained in a region, say* REG_1. *Let denote* $P_1, P_2, Q_1, Q_2, R_1, R_2$ *the closest point to* 0 *of the crosses* cl(cross(P)), cl(cross(Q)) *and* cl(cross(R)). *Clearly, if* P_1 *or* P_2 *is a point of* $\overline{\mathrm{MID}}$, *then holds* $P_1 = P_2$. *Analogously, if* Q_1 *or* Q_2 *contained in* $\overline{\mathrm{MID}}$, *then let* $Q_1 = Q_2$. *Then holds*:

(a) *There is only one configuration of the three points P, Q, R.*

(b) *Let be given the configuration of P, Q, R as in Fig.* 14 *shown. Then all points* P_1, P_2, Q_1, Q_2 *are included in the considered circle C.*

(C6) *Let be given exactly two points P, Q contained in a region, say* REG_1. *Let be* P_1, P_2 *and* Q_1, Q_2 *the two closest endpoints of* cl(cross(P)) *and* cl(cross(Q)). *It can occur, that all points* P_1, P_2, Q_1, Q_2 *lying on a line segment. In this case or other possible cases (see the points P, R in Fig.* 14) *two points of* P_1, P_2, Q_1, Q_2 *are included in C, if the other both points contained in the circle C.*

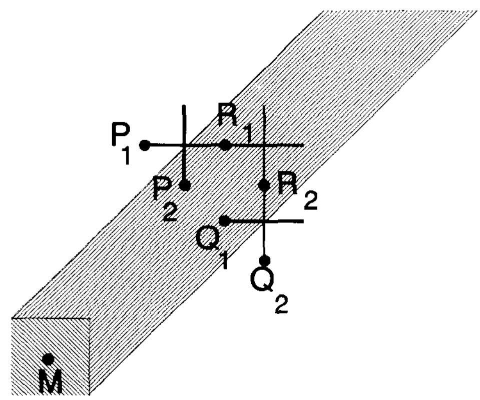

Fig. 14.

The proofs of (C1), . . . , (C6) are omitted here. For the demonstration of (C5) we remark, any Euclidean circle C contains R_1 or R_2 must contain P_1, P_2, Q_1, and Q_2, respectively (see Fig. 15). Other configurations of the points P, Q, R are not possible.

Now we formulate the main theorem.

(CIR4) *If the number of points of a given set* $\xi \subseteq \mathbb{Z}^2$ *is* N, *thn we can decide the digital circle problem in linear time* $O(N)$ *and this time is optimal.*

Proof. (a) *Necessary conditions.* For a given digital circle ξ there is necessarily a minimal Euclidean circle K with $\text{dig}(k) = \xi$. In this part we give an algorithm for finding K. Assume, ξ is a digital circle consisting of N lattice points.

(a1) In a first part we determine the minimal covering circle of the so-called 'inner' points of all crosses cross(P), $P \in \xi$.

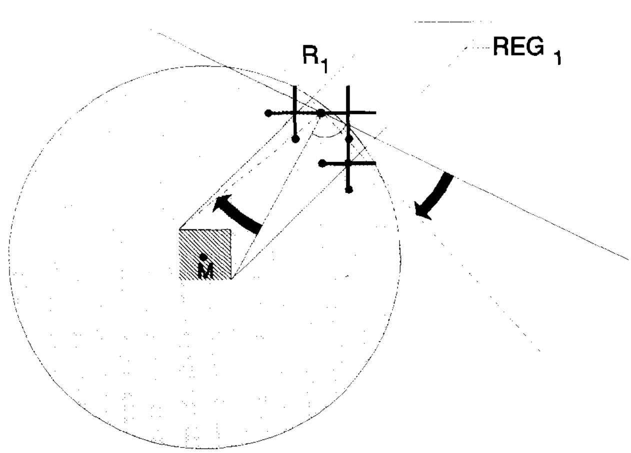

Fig. 15.

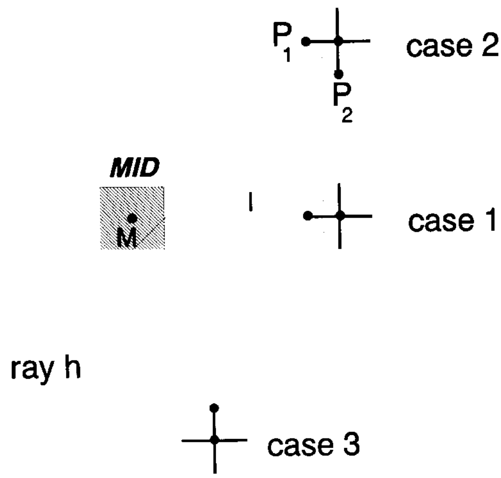

Fig. 16.

Step 1. We compute the minimal covering lattice box BOX of ξ with midpoint $M(x_m, y_m)$. Let be

$$\text{MID} := \{(x, y) \in \mathbb{R}^2 \mid x_m - 0.5 < x \leq x_m + 0.5 \wedge y_m - 0.5 < y \leq y_m + 0.5\}$$

and let $\overline{\text{MID}} := \text{cl(MID)}$ the closure of MID.

Step 2. For every $P \in \xi$ we choose one or two endpoints of the closure cl(cross(P)) with the help of the criterions (see Fig. 16):

Case 1: If there is a gridline 1 with $Q \in 1$ meets $\overline{\text{MID}}$, then we insert the closest point Q_1 of cl(cross(Q)) to M into the set INNER.

Case 2: If case 1 does not hold then let be P_1, P_2 two endpoints with distinct coordinates of cl(cross(P)). If the straight line (ray) $h \perp g(P_1, P_2)$ contained the midpoint of the segment $\overline{P_1P_2}$ meets $\overline{\text{MID}}$, then we insert P_1, P_2 into INNER, if P_1, P_2 are closest points of cl(cross(P)) to M having this property.

If we insert two so-called double-points P_1, P_2 of a cross cl(cross(P)) in the set INNER, then we insert the triple $(P; P_1, P_2)$ in a set DOUBLE.

Case 3: If Cases 1 or 2 do not hold, then we insert the closest point R_1 of cl(cross(R)) to M into INNER.

We remark, in case of digital circles follows from (C3) of Lemma 3, the number of triples in the set DOUBLE are at most 12.

Step 3. We compute the Euclidean minimal spanning circle C with boundary K for the set INNER with help of Megiddo's linear time algorithm (see [7]).

Let be SUPPORT $:= K \cap$ INNER the set of common points of the set INNER and the circle K.

Step 4. If no triple $(P; P_1, P_2) \in$ DOUBLE exists with $P_1, P_2 \in$ SUPPORT then the circle K is absolutely minimal and the algorithm is at the end.

In other cases we modify the set INNER in view of the double-points.

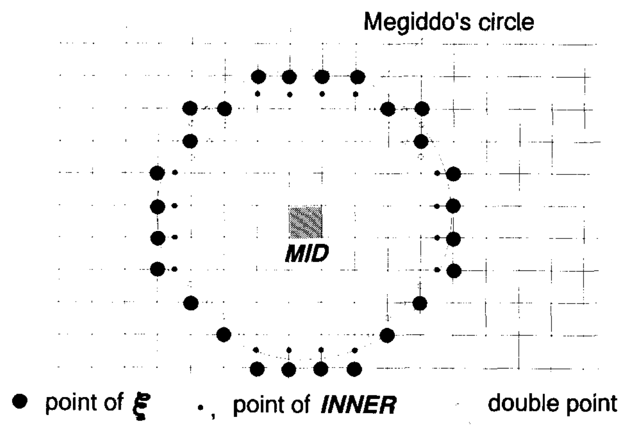

Fig. 17.

We compute circles K_i with the help of Megiddo's algorithm on the basis of sets INNER_i, where at least one of every double-point P_1, P_2 with $(P; P1, P2) \in$ DOUBLE is included in INNER_i. Note, from property (C3) of Lemma 3 it follows, there are at most 3^{12} such sets INNER_i, i.e. it holds $1 \leqslant i \leqslant 3^{12}$. One of these circle K_i, say $K_{\min}$, must be minimal.

Now we have the following situation:

The minimal covering circle $C_{\min}$ contains at least one point of every cross cl(cross(P)), where $P \in \xi$. If we assume, the minimal circle $K_{\min}$ we obtain by Step 1 up to Step 4, and the minimal circle K' for a given digital circle ξ with $\text{dig}(K') = \xi$ are not the same, we consider the minimal supporting sets SUPPPORT and SUPPPORT′ of $K_{\min}$ and K', respectively. At least one half-circle H or H' of $K_{\min}$ or K', say H, lying outside of the other one. Then at least one supporting point of $K_{\min}$ lying outside of K'. This is a contradiction for minimal covering circles $K_{\min}$ and K'.

Fig. 17 shows the points of the sets INNER and DOUBLE, respectively. The minimal circle $K_{\min}$ and a set SUPPORT of $K_{\min}$ is shown in Fig. 18.

(a2) We can give a lower bound for the number N of points of a digital circle ξ. It holds

$$N \geqslant 4\lfloor \min(a, b) \rfloor - 4,$$

where a, b are the length of the smallest lattice box containing ξ.

(b) *Sufficient conditions.* Let be given a circle $K \subseteq \mathbb{R}^2$ and a set of points $\xi \subseteq \mathbb{Z}^2$. Suppose, the number of points of ξ is N.

(b1) If for any cross(P), $P \in \xi$, there is one common point of cross(P) and K, then it holds $\xi \subseteq \text{dig}(K)$.

(b2) If for any point $P' \in \text{dig}(K \cap \Gamma(\mathbb{Z}^2))$ there is one point $P \in \xi$ with $P' = P$, then it holds $\xi \supseteq \text{dig}(K)$, where $\Gamma(\mathbb{Z}^2)$ denote the grid lines in the plane.

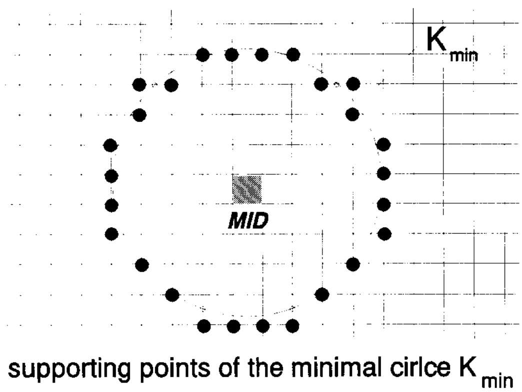

Fig. 18.

Now we can devide the digital circle problem for a given set ξ of points in two parts:

Firstly we determine the minimal covering circle K on basis of the sets INNER_i with the time complexity $O(N)$.

Secondly we test the sufficient conditions for ξ with respect to K in the same time complexity $O(N)$. The proof of Theorem 2 is finished. □

Conclusion. The considerations of digital straight lines and digital circles have shown, there is no way to transform one onto the other in general case. Therefore it is not possible to use well-known properties of digital straight lines for the recognition of digital circles.

The main result is the decision in linear time, wether or not a given planar set of gridpoints is a digital circle. The recognition of digital circles is possible in the same time as the recognition of digital straight line segments, if the number of points are the same.

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