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Integral Representations and Liouville Theorems for Solutions of Periodic Elliptic Equations

Peter Kuchment

Mathematics and Statistics Department, Wichita State University, Wichita, Kansas 67260-0033
E-mail: kuchment@twsuvm.uc.twsu.edu

and

Yehuda Pinchover

Department of Mathematics, Technion—Israel Institute of Technology, Haifa 32000, Israel

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The paper contains integral representations for certain classes of exponentially growing solutions of second order periodic elliptic equations. These representations are the analogs of those previously obtained by S. Agmon, S. Helgason, and other authors for solutions of the Helmholtz equation. When one restricts the class of solutions further, requiring their growth to be polynomial, one arrives to Liouville type theorems, which describe the structure and dimension of the spaces of such solutions. The Liouville type theorems previously proved by M. Avellaneda and F.-H. Lin and J. Moser and M. Struwe for periodic second order elliptic equations in divergence form are significantly extended. Relations of these theorems with the analytic structure of the Fermi and Bloch surfaces are explained. © 2001 Academic Press
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1. INTRODUCTION

The topic of this paper stems from two sources. The first of them are representation theorems for certain classes of eigenfunctions of the Laplace operator in $\mathbb{R}^n$ or, equivalently, of solutions of the Helmholtz equation

$$-\Delta u - k^2 u = 0 \qquad \text{in } \mathbb{R}^n, \tag{1.1}$$

where $k \in \mathbb{C}^* := \mathbb{C} \backslash \{0\}$. Such theorems for arbitrary solutions of (1.1) were obtained in $\mathbb{R}^2$ and in the hyperbolic plane by S. Helgason [22, 23] and in

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$\mathbb{R}^n$ by M. Hashizume *et al.* [21], M. Morimoto [37], L. A. Caffarelli and W. Littman [44], and recently by S. Agmon [4]. We remark that it should also be possible to deduce similar results from L. Ehrenpreis' *fundamental principle*. The zero set of the symbol of the operator in the left hand side of (1.1) is

$$\Sigma = \{\xi \in \mathbb{C}^n \mid \xi^2 = k^2\},$$

where $\xi^2 = \sum_{j=1}^n \xi_j^2$. L. Ehrenpreis' fundamental principle in the particular case of equation (1.1) claims that any solution of (1.1) can be represented as a combination (i.e., an integral with respect to the parameter ξ) of the exponential solutions

$$e_\xi(x) := \exp(i\xi \cdot x), \qquad \xi \in \Sigma,$$

where $\xi \cdot x = \sum_{j=1}^n \xi_j x_j$ (see the details and more precise formulation in [16] or [39]). The set Σ is an irreducible analytic subset of $\mathbb{C}^n$, which is uniquely determined when $k \neq 0$ by its spherical subset

$$S = \{\xi \in \mathbb{C}^n \mid \xi = k\omega,\ \omega \in S^{n-1} \subset \mathbb{R}^n\}.$$

Here S^{n-1} denotes the unit sphere in $\mathbb{R}^n$. It is clear then that due to the uniqueness of analytic continuation, the exponential representation of solutions $u(x)$ of (1.1) should be reducible to one that involves the solutions e_ξ with $\xi \in S$ only. Namely, consider the restriction mapping from functions analytic on the whole characteristic variety Σ to the sphere S. Due to the irreducibility and the uniqueness of analytic continuation, this mapping is one-to-one. Hence, there is a function space on the sphere S which is the isomorphic image of the space of all analytic functions on Σ. It follows that any hyperfunction (analytic functional) on Σ can be rewritten as a functional on S. Since the fundamental principle essentially expresses all solutions of (1.1) as applications of such analytic functionals to the analytic family of exponential solutions, we get our conclusion.

Now, depending on how fast the solution $u(x)$ grows at infinity, the corresponding representing functional on S is actually a measure, a distribution, a hyperfunction, or a functional of a more general kind. For instance (see [4, 21, 37]), an arbitrary solution in $\mathbb{R}^n$ can be represented as

$$u(x) = \langle \phi(\xi), e_\xi(x) \rangle, \tag{1.2}$$

where $\phi(\xi)$ is a functional on S which belongs to the dual space to the space $\mathscr{E} := \lim_{R\to\infty} \mathscr{E}_R(S^{n-1})$. Here for every $R>0$ the Hilbert space $\mathscr{E}_R(S^{n-1})$ is defined as follows,

$$\mathscr{E}_R(S^{n-1}) := \left\{ \psi \mid \psi(\omega) = \sum_{l,m} a_{l,m} \frac{(R/2)^l}{\Gamma(l+(n+1)/2)} Y_l^m(\omega), \text{ s.t.} \right.$$
$$\left. \|\psi\|_{\mathscr{E}_R} := \left(\sum_{l,m} |a_{l,m}|^2 \right)^{1/2} < \infty \right\},$$

where $Y_l^m(\omega)$ denote spherical harmonics, and $\mathscr{E}(S^{n-1})$ is equipped with the inductive limit topology of $\lim_{R\to\infty} \mathscr{E}_R(S^{n-1})$.

The representation (1.2) can be formally rewritten as

$$u(x) = \int_S e_\xi(x)\, d\phi(\xi),$$

where $\phi(\xi)$ is a suitable functional.

The functional ϕ is a hyperfunction (analytic functional) on S if and only if for arbitrary $\varepsilon>0$ the solution $u(x)$ grows not faster than

$$O(\exp((|Im\, k| + \varepsilon)\, |x|))$$

(see [4]). One can also describe other classes of solutions, for instance, solutions which are represented by a distribution or a measure (see [2, 3, 4, 36] and the references therein).

As we have already mentioned, these results could be probably extracted from the fundamental principle [16, 39]. The crucial factors are that S is sufficiently massive and Σ is irreducible, so S determines Σ uniquely. Besides, S is a rather simple analytic manifold. These features allow more or less explicit descriptions of the needed spaces of test functions and functionals. It is easy to understand that if Σ were reducible, it would not be possible to obtain the representation of *all* solutions using only $\xi \in S$. The reason is that the solution e_ξ with ξ that belongs to a component not touching S would not be representable this way. On the other hand, if one wants to deal only with solutions growing not faster than $O(\exp(|Im\, k| + \varepsilon)\, |x|)$ for all $\varepsilon>0$, then the irreducibility of Σ is not needed. In this case, it is only required that Σ is irreducible in a vicinity of S, so other components of Σ do not meet S.

The fundamental principle was extended in [30] to solutions of certain growth (for instance, of exponential growth) of elliptic and hypoelliptic *periodic* equations (see also the extensions of the results of [30] provided in [40]). The role of the exponential solutions is played here by the so called *Floquet–Bloch* solutions (see Definition 1), and an analog of the characteristic manifold Σ is the variety F sometimes called the *Fermi surface* (see Definition 2). This raises the hope of finding representations similar to the ones discussed above for the more general case of a second order elliptic operator with periodic coefficients. This is not straightforward, however, due to several reasons. First of all, it is not that clear what should be a natural analog of S. An appropriate variety, as we explain later, is provided for second order equations by the analysis of the cone of positive solutions done by S. Agmon and by V. Lin and Y. Pinchover (see [2, 36, 30], and the references therein). The disadvantage is that the whole consideration must be done *below* the spectrum of the operator (more precisely, below *the generalized principal eigenvalue* Λ_0, see (2.8)). Secondly, proving the irreducibility of F happens to be a very hard nut to crack (this problem arises also in direct and inverse spectral problems, see for instance [9, 18, 28, 31, 32]). Fortunately enough, by appropriately restricting the growth of the solutions, one can sometimes work near a single irreducible component, and hence avoid proving the irreducibility of F. Consequently, we prove a representation theorem (Theorem 18) that characterizes all the solutions which have integral expansion into *positive* Bloch solutions with a hyperfunction as a "measure."

The fundamental principle also suggests a point of view that is crucial for establishing representation theorems for solutions of equations with constant or periodic coefficients. Namely, it is to one's advantage to treat solutions of the original equation in the dual sense, i.e., as functionals on appropriate spaces that are orthogonal to the range of the dual operator. We adopt this approach throughout the paper.

If one attempts now to further restrict the growth of solutions and considers the problem of the structure of all polynomially growing (or bounded) solutions, one arrives at the second topic of our study, Liouville type theorems. The classical Liouville theorem characterizes the space of all harmonic functions in $\mathbb{R}^n$ of polynomial growth of order N. The validity of an analog of this classical theorem has been studied in many situations (see for instance [14, 33, 34] for recent results, surveys, and further references). An interesting case was considered by M. Avellaneda and F.-H. Lin [8], and also by J. Moser and M. Struwe [38]. In these papers the authors dealt with polynomially growing solutions of a second order elliptic equation $Lu = 0$ in divergence form with periodic coefficients and obtained a comprehensive answer (for related results see also [14, 33] and the references therein). More precisely, using the formalism of homogenization theory [10, 25], it was proved that any solution v of the equation $Lu = 0$ in $\mathbb{R}^n$ of polynomial growth is representable as a finite sum of the form

$$v(x) = \sum_{j = (j_1, \ldots, j_n) \in \mathbb{Z}^n_+} x^j p_j(x), \tag{1.3}$$

where the functions $p_j(x)$ are periodic with respect to the group of periods of the equation. Moreover, the space of all solutions of the equation $Lu=0$ of polynomial growth of order at most N is of dimension $h_{n,N}$, where

$$h_{n,N} := \binom{n+N}{N} - \binom{n+N-2}{N-2} \tag{1.4}$$

is the dimension of the space of all harmonic polynomials of degree $\leqslant N$ in n variables. We will also use the notation

$$q_{n,N} := \binom{n+N}{N} \tag{1.5}$$

for the dimension of the space of all polynomials of degree at most N in n variables. Notice that $q_{n-1,N}$ also coincides with the dimension of the space of all homogeneous polynomials of degree N in n variables, so in particular, $h_{n,N} = q_{n-1,N-1} + q_{n-1,N}$.

We remark that the method of [8, 38] can be slightly modified to provide an extension of this Liouville theorem for general second order elliptic equations with periodic coefficients under the assumption that the generalized principal eigenvalue is zero (see Appendix A and also the recent paper [35], where a partial result of this type was independently obtained).

One can make an observation that these Liouville theorems are actually of the same nature as the representation theorems discussed above. In this case the analog of the set S is the single point $\xi = 0$ and the representing functional ϕ is a distribution supported at this point.

Let us recall the following standard notion of Floquet theory (see [15, 30, 42]):

DEFINITION 1. A solution $u(x)$ representable as a finite sum of the form

$$u(x) = e^{ik \cdot x} \left(\sum_{j=(j_1, \dots, j_n) \in \mathbb{Z}^n_+} x^j p_j(x) \right) \tag{1.6}$$

with functions $p_j(x)$ periodic with respect to the group of periods of the equation is called a *Floquet solution with a quasimomentum* $k \in \mathbb{C}^n$. Here $k \cdot x = \sum k_l x_l$. The maximum value of $|j| = \sum_{l=1}^n j_l$ in the representation (1.6) is said to be the *order* of the Floquet solution. Floquet solutions of zero order are called *Bloch solutions*.

One sees that the representation (1.3) corresponds to a Floquet solution with a zero quasimomentum. A Liouville theorem of the type mentioned above implies in particular that the only *real* quasimomentum that can occur for the equation under consideration is $k=0$ (modulo the action of the lattice reciprocal to the group of periods). We show in the present paper that the finiteness of the set of *real* quasimomenta for a periodic elliptic equation is equivalent to the finite dimensionality of the spaces of solutions having a given polynomial growth and to their representation similar to, albeit more general than (1.3). This statement is very general and holds for any periodic elliptic equation (it is also true for hypoelliptic equations and systems, although we do not address them here). If some additional information is available on the analytic behaviour of the dispersion relations, one can find the *exact* dimensions of these spaces (see Theorem 23). We present in Theorem 28 some classes of second order equations (including Schrödinger, magnetic Schrödinger, and general second order elliptic equations with real periodic coefficients) for which one can achieve all these sharp results. We show that the problem of calculating the dimensions of the spaces of Floquet solutions of a given polynomial growth reduces to a purely function theoretic question and can be resolved in a very general setting (Theorem 10).

The proofs of the results of this paper are largely dependent upon the techniques of the Floquet theory developed in [30].

The outline of the paper is as follows. The next section introduces necessary notations and preliminary results from the Floquet theory and the theory of positive solutions of periodic elliptic equations. In particular, we obtain a new general result (Theorem 10) on the dimensions of the spaces of Floquet solutions, which plays crucial role in our approach to Liouville theorems. Section 3 contains the proof of the integral representation (Theorem 18) analogous to Theorem 5.1 in [4]. In Section 4, we discuss Liouville type theorems. In particular, Theorems 23 and 28 are established. In order to make the reading of the paper easier, we postpone the proofs of all the technical lemmas to Section 5. Some conclusions and remarks are provided in Section 6. The Appendix contains an alternative proof of a part of Theorem 28 using the homogenization technique similar to the one used in [8, 38].

Results of this paper related to Liouville theorems were presented in March 2000 at the University of Toronto and at the Weizmann Institute.

When the paper was being prepared for submission, P. Li informed the authors that the statement of the third part of Theorem 28 for the special case of an operator of the form $L = -\sum a_{ij}(x)\,\partial_i\,\partial_j$ was simultaneously and independently obtained in [35] using homogenization formalism.

2. NOTATIONS AND PRELIMINARY RESULTS

Consider a linear (scalar) elliptic partial differential operator $P(x, D)$ of order m in $\mathbb{R}^n$, $n \geqslant 2$ (in some parts of the paper we will restrict the class of operators further). Here we employ the standard notation $D = \frac{1}{i}\frac{\partial}{\partial x}$. The ellipticity is understood in the sense of the nonvanishing of the principal symbol $P_m(x, \xi)$ of the operator P for all $\xi \in \mathbb{R}^n \setminus \{0\}$. The dual operator (the formal adjoint) P^* has similar properties. Here we use the duality provided by the bilinear (rather than the sesquilinear) form

$$\langle g, f\rangle = \int_{\mathbb{R}^n} f(x)\, g(x)\, dx.$$

We assume that the coefficients of P are smooth and periodic with respect to a lattice Γ in $\mathbb{R}^n$. The smoothness condition can be significantly reduced (see Section 6). In fact, so far we only need that both operators P and P^* define Fredholm mappings between the Sobolev space H^m and L^2 on the torus $\mathbb{T}^n = \mathbb{R}^n/\Gamma$.

An additional condition is required that would guarantee the discreteness of the spectrum of the "shifted" operators $P(x, D+k)$ on the torus $\mathbb{T}^n$ for all $k \in \mathbb{C}^n$. We need to exclude the possible pathological situation when the spectrum of P on the torus coincides with the whole complex plane (like in the case of the operator $\exp(i\phi)\, d/d\phi$ on the circle). For instance, self-adjointness of P could be such a condition. Another example is second order uniformly elliptic operators of the form (2.5). For more sufficient conditions see for example [1].

In what follows, the particular choice of the lattice is irrelevant and can always be reduced to the case $\Gamma = \mathbb{Z}^n$, which we will assume from now on. We will always use the word "periodic" in the meaning of "Γ-periodic."

We denote by $K = [0, 1]^n$ the standard fundamental domain (the *Wigner–Seitz cell*) of the lattice $\Gamma = \mathbb{Z}^n$, and by $B = [-\pi, \pi]^n$ the *first Brillouin zone*, which is a fundamental domain of the reciprocal (dual) lattice $\Gamma^* = (2\pi\mathbb{Z})^n$. We identify Γ-periodic functions, in the natural way, with functions on $\mathbb{T}^n$.

We introduce now the set that plays the role of the characteristic variety Σ discussed in the introduction.

DEFINITION 2. The (complex) *Fermi surface* F_P of the operator P (at the zero energy level) consists of all vectors $k \in \mathbb{C}^n$ (called *quasimomenta*) such that the equation $Pu = 0$ has a nonzero Bloch solution $u(x) = e^{ik\cdot x}p(x)$, where $p(x)$ is a Γ-periodic function.

It would be useful later on to realize that in this definition the positivity of the solution is not required, and in fact the solution is usually complex.

In many cases, it is convenient to introduce a spectral parameter λ. This leads to the notion of the *Bloch variety*:

Definition 3. The (complex) *Bloch variety* B_P of the operator P consists of all pairs $(k, \lambda) \in \mathbb{C}^{n+1}$ such that the equation $Pu = \lambda u$ has a nonzero Bloch solution $u(x) = e^{ik \cdot x} p(x)$ with the quasimomentum k.

It is clear that the Fermi surface is just the projection onto the k-space of the intersection of the Bloch variety with the hyperplane $\lambda = 0$.

One can consider the Bloch variety B_P as the graph of a (multivalued) function $\lambda(k)$, which is usually called the *dispersion relation*. Then the Fermi surfaces become the level surfaces of the dispersion relation. Since the spectra of all operators $P(x, D+k)$ on the torus $\mathbb{T}^n$ are discrete, we can single out continuous branches λ_j of this multivalued dispersion relation. These branches are usually called the *band functions* (see [42, 30]).

The following analyticity property of the Fermi and Bloch varieties is important:

Lemma 4 [30, Theorems 3.1.7 and 4.4.2]. *The Fermi and Bloch varieties are the sets of all zeros of entire functions of a finite order in $\mathbb{C}^n$ and $\mathbb{C}^{n+1}$, respectively.*

Another property of the Bloch and Floquet varieties that we will need later is the relation between the corresponding varieties of the operators P and P^*.

Lemma 5 [30, Theorem 3.1.5]. *A quasimomentum k belongs to F_{P^*} if and only if $-k \in F_P$. Analogously, $(k, \lambda) \in B_{P^*}$ if and only if $(-k, \lambda) \in B_P$. In other words, the dispersion relations $\lambda(k)$ and $\lambda^*(k)$ for the operators P and P^* are related as follows:*

$$\lambda^*(k) = \lambda(-k). \tag{2.1}$$

We note that the Fermi surface F_P is periodic with respect to the reciprocal lattice $\Gamma^* = (2\pi\mathbb{Z})^n$. Therefore, it is sometimes useful to factor out the periodicity by considering the (analytic) exponential mapping $\rho\colon \mathbb{C}^n \to (\mathbb{C}^*)^n$, where

$$z = \rho(k) = \rho(k_1, ..., k_n) = (\exp ik_1, ..., \exp ik_n).$$

This mapping can be identified in a natural sense with the quotient map $\mathbb{C}^n \to \mathbb{C}^n/\Gamma^*$. We also introduce the complex torus

$$T = \rho(\mathbb{R}^n) = \{z \in \mathbb{C}^n \mid |z_j| = 1,\ j = 1, 2, ..., n\}. \tag{2.2}$$

DEFINITION 6. The image $\Phi_P = \rho(F_P)$ of the Fermi surface F_P under the mapping ρ is called the *Floquet surface* of the operator P.

The reader familiar with the Floquet theory immediately recognizes the Floquet surface as the set of all Floquet multipliers of the equation $Pu = 0$.

The main tool in the Floquet theory is an analog of the Fourier transform (see [30, Section 2.2], [42]), which we will call the *Floquet transform* $\mathcal{U}$ (it is sometimes also called the *Gelfand transform*),

$$f(x) \to \mathcal{U}f(z, x) = \sum_{\gamma \in \Gamma} f(x - \gamma)\, z^{\gamma}, \qquad z \in (\mathbb{C}^*)^n, \tag{2.3}$$

where we denote $z^{\gamma} = z_1^{\gamma_1} z_2^{\gamma_2} \cdots z_n^{\gamma_n}$.

It is often convenient to use for the Floquet transform $\mathcal{U}$ the quasimomentum coordinate k instead of the multiplier z, where

$$z = \rho(k) = (\exp ik_1, ..., \exp ik_n).$$

We need to recall now some definitions from [30]. For a point $z \in (\mathbb{C}^*)^n$, we denote by $E_{m,z}$ the closed subspace of the Sobolev space $H^m(K)$ formed by the restrictions of functions $v \in H^m_{\text{loc}}(\mathbb{R}^n)$ that satisfy the Floquet condition $v(x + \gamma) = z^{\gamma} v(x)$ for any $\gamma \in \Gamma$. One can show (see Theorem 2.2.1 in [30]) that

$$\mathcal{E}_m := \bigcup_{z \in (\mathbb{C}^*)^n} E_{m,z} \tag{2.4}$$

forms a holomorphic sub-bundle of the trivial bundle $(\mathbb{C}^*)^n \times H^m(K)$. As any infinite dimensional analytic Hilbert bundle over a Stein domain, it is trivializable (see Theorems 1.3.2, 1.3.3, and 1.5.23 in [30]). One can also notice that for $m = 0$ the bundle $\mathcal{E}_0$ coincides with the whole $(\mathbb{C}^*)^n \times L^2(K)$.

The following standard auxiliary result for the transform $\mathcal{U}$ collects several statements from Theorem XIII.97 in [42] and Theorem 2.2.2 in [30]:

LEMMA 7. 1. *For any nonnegative integer m the operator*

$$\mathcal{U}\colon H^m(\mathbb{R}^n) \to L^2(T, \mathcal{E}_m)$$

is an isometric isomorphism, where $L^2(T, \mathcal{E}_m)$ denotes the space of square integrable sections over the complex torus T of the bundle $\mathcal{E}_m$, equipped with the natural topology of a Hilbert space.

2. *Let the space*

$$\Theta^m = \{ f \in H^m_{loc}(R^n) \mid \sup_{\gamma \in \Gamma} \|f\|_{H^m(K+\gamma)} \exp(b\,|\gamma|) < \infty, \forall b > 0\}$$

be equipped with the natural Fréchet topology. Then

$$\mathscr{U}: \Theta^m \to \Gamma((\mathbb{C}^*)^n, \mathscr{E}_m)$$

is an isomorphism, where $\Gamma((\mathbb{C}^*)^n, \mathscr{E}_m)$ *is the space of all analytic sections over* $(\mathbb{C}^*)^n$ *of the bundle* $\mathscr{E}_m$*, equipped with the topology of uniform convergence on compacta.*

3. *Let the elliptic operator* P *be of order* m*. Then under the transform* $\mathscr{U}$ *the operator*

$$P: H^m(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$$

becomes the operator of multiplication by a holomorphic Fredholm morphism $P(z)$ *between the fiber bundles* $\mathscr{E}_m$ *and* $\mathscr{E}_0$*. Here* $P(z)$ *acts on the fiber of* $\mathscr{E}_m$ *over the point* $z \in T$ *as the restriction to this fiber of the operator* P *acting between* $H^m(K)$ *and* $L^2(K)$*.*

Here is another standard way of looking at the morphism $P(z)$. Let $z = \exp ik$, then commuting with the exponent $\exp ik \cdot x$ one can (locally) trivialize the bundle $\mathscr{E}_m$ reducing it to the trivial bundle with the fiber $H^m(\mathbb{T}^n)$, where as before $\mathbb{T}^n = \mathbb{R}^n/\Gamma$. At the same time the operator $P(z)$ takes the form $P(x, D+k)$ between Sobolev spaces on the torus $\mathbb{T}^n$.

Let us discuss the structure of the Floquet solutions (see Definition 1) and of functions of Floquet type (1.6) in general. For illustration, consider the constant coefficient case, where the role of the Floquet solutions is played by the exponential polynomials

$$e^{ik \cdot x} \sum_{|j| \leqslant N} p_j x^j.$$

It is well known that, considered as distributions, all such functions are Fourier transformed into distributions supported at the point $(-k)$. Moreover, the converse statement is also true. A simple but extremely important and relatively unnoticed observation is that under the Floquet transform, each Floquet type function of the form (1.6) corresponds, in a similar way, to a (vector valued) distribution supported at the quasimomentum $(-k)$. We collect below this fact and some other previously known properties of Floquet solutions, as well as a new result on the dimensions of the spaces of such solutions, which will play the crucial role in establishing the Liouville type theorems.

First of all, every Floquet type function u (see (1.6)), being of exponential growth, determines a (continuous linear) functional on the space Θ^0. If, additionally, it satisfies the equation $Pu = 0$ for a periodic elliptic operator of order m, then as such a functional it is clearly orthogonal to the range of the dual operator $P^*\colon \Theta^m \to \Theta^0$. According to Lemma 7, after the Floquet transform any such functional becomes a functional on $\Gamma((\mathbb{C}^*)^n, \mathcal{E}_0)$, which is orthogonal to the range of the Fredholm morphism $P^*(z)\colon \mathcal{E}_m \to \mathcal{E}_0$ generated by the dual operator $P^*\colon \Theta^m \to \Theta^0$. We are now ready to formulate the following auxiliary result.

LEMMA 8. 1. *A continuous linear functional u on Θ^0 is generated by a function of the Floquet form* (1.6) *with a quasimomentum k if and only if after the Floquet transform it corresponds to a functional on $\Gamma((\mathbb{C}^*)^n, \mathcal{E}_0)$ which is a distribution ϕ that is supported at the point $v = \exp(-ik)$, i.e., has the form*

$$\langle \phi, f \rangle = \sum_{|j| \leqslant N} \left\langle q_j, \left. \frac{\partial^{|j|} f}{\partial z^j} \right|_v \right\rangle, \qquad f \in \Gamma((\mathbb{C}^*)^n, \mathcal{E}_0),$$

where $q_j \in L^2(K)$. The orders N of the Floquet function (1.6) *and of the corresponding distribution ϕ are the same.*

2. *Let a_k be the dimension of the kernel of the operator*

$$P(x, D+k)\colon H^m(\mathbb{T}^n) \to L^2(\mathbb{T}^n).$$

Then the dimension of the space of Floquet solutions of the equation $Pu = 0$ of order at most N with a quasimomentum k is finite and does not exceed $a_k q_{n,N}$.

The estimate on the dimension given in the second part of Lemma 8 is very crude and in many cases can be significantly improved. In fact, as the following theorem shows, we obtain an explicit formula for the dimension of the space of Floquet solutions with a given quasimomentum in the case of a simple eigenvalue. This theorem seems to be new and constitutes the crucial part of the Liouville theorem proved in Section 4 (Theorem 23).

In order to formulate this result, we need to prepare some notions and notations.

DEFINITION 9. Let Q be a homogeneous polynomial in n complex variables. A polynomial $p(x)$ in $\mathbb{R}^n$ is called *Q-harmonic*, if it satisfies the differential equation $Q(D)\, p = 0$.

Let $\mathcal{P}$ denote the vector space of all polynomials in n variables, and let P_l be the subspace of all homogeneous polynomials of degree l. Denote by $\mathcal{P}_N = \bigoplus_{l=0}^{N} P_l$ the subspace of all polynomials of degree at most N. So, $\mathcal{P} = \bigoplus_{l=0}^{\infty} P_l$. If $Q(k)$ is a nonzero homogeneous polynomial of degree s, then the differential operator $Q(D)\colon P_{l+s} \to P_l$ is surjective for any l (this simple statement will also follow from the proof of the theorem below). Hence, the mapping $Q(D)\colon \mathcal{P} \to \mathcal{P}$ has a (nonuniquely defined) linear right inverse R that preserves the homogeneity of polynomials.

THEOREM 10. *Assume that zero is an eigenvalue of algebraic multiplicity 1 of the operator* $P(x, D+k_0)\colon H^m(\mathbb{T}^n)\to L^2(\mathbb{T}^n)$ *on the torus* $\mathbb{T}^n$. *Let* $\lambda(k)$ *be an analytic function in a neighborhood of* k_0 *such that* $\lambda(k)$ *is a simple eigenvalue of the operator* $P(x, D+k)\colon H^m(\mathbb{T}^n)\to L^2(\mathbb{T}^n)$ *and* $\lambda(k_0)=0$. *Let*

$$\lambda(k)=\sum_{l=l_0}^{\infty}\lambda_l(k-k_0)$$

be the Taylor expansion of $\lambda(k)$ *around the point* k_0 *into homogeneous polynomials such that* λ_{l_0} *is the first nonzero term of this expansion. Then for any* $N\in\mathbb{N}$ *the dimension of the space of Floquet solutions of the equation* $Pu=0$ *in* $\mathbb{R}^n$ *of order at most* N *and with the quasimomentum* k_0 *is equal to the dimension of the space of all* λ_{l_0}*-harmonic polynomials of degree at most* N. *Moreover, given a linear right inverse* R *of the mapping* $\lambda_{l_0}(D)\colon \mathcal{P}\to\mathcal{P}$ *that preserves homogeneity, one can construct an explicit isomorphism between these spaces.*

Proof. It is sufficient to consider the case $k_0=0$, since the general case reduces to this by a change of variables. Consider the operator family

$$A(k)=P^*(x, D-k)-\lambda(-k)\colon H^m(\mathbb{T}^n)\to L^2(\mathbb{T}^n)$$

which is analytic in a neighborhood of 0. At each point k of this neighborhood $A(k)$ has by the construction a one-dimensional kernel. Then, according to Theorem 1.6.13 in [30], there exists an analytic non-vanishing vector $\psi(k)\in \mathit{Ker}\,A(k)$. In other words, $P^*(x, D-k)\,\psi(x, k)=\lambda(-k)\,\psi(x, k)$. Let us choose a closed complementary subspace M to $\mathit{Ker}\,A(0)$ in $H^m(\mathbb{T}^n)$. Then it is complementary to $\mathit{Ker}\,A(k)$ in a neighborhood of 0. Since $P^*(x, D)$ has zero kernel on M and is Fredholm, we conclude that $P^*(x, D-k)$ has zero kernel on M for all k in a neighborhood of 0. We denote by $\Pi(k)$ the closed subspace in $L^2(\mathbb{T}^n)$ defined as $\Pi(k)=P^*(x, D-k)(M)$. Applying Theorem 1.6.13 of [30] again, we conclude that $\Pi(k)$ depends holomorphically on k in a neighborhood of 0 (i.e., forms a Banach bundle) and hence it is complementary to $\mathit{Ker}\,A(k)$. Representing now the operator $P^*(x, D-k)$ in the block form according to the decompositions

$$H^m(\mathbb{T}^n)=M\oplus \mathit{Ker}\,A(k)$$

and

$$L^2(\mathbb{T}^n)=\Pi(k)\oplus \mathit{Ker}\,A(k),$$

we get

$$P^*(x, D-k) = \begin{pmatrix} B(k) & 0 \\ 0 & \lambda(-k) \end{pmatrix},$$

where $B(k)$ is an analytic invertible operator-function between M and $\Pi(k)$. If we now have a functional ϕ on $\Gamma((\mathbb{C}^*)^n, \mathscr{E}_0)$ supported at $v = \exp(0)$, such that it is orthogonal to the range of the operator of multiplication by $P^*(k)$, then it must be equal to zero on all sections of the bundle $\Pi(k)$. This means that the restriction of such functionals to the sections of the one-dimensional bundle *Ker* $A(k)$ is an one-to-one mapping. This reduces the problem to the following scalar one: find the dimension of the space of all distributions of order N supported at the origin such that they are orthogonal to the ideal generated by $\lambda(-k)$ in the ring of germs of analytic functions. One can change variables to eliminate the minus sign in front of k. Due to the finiteness of the order of the distribution, the problem further reduces to the following: find the dimension of the cokernel of the mapping

$$\Lambda_N : \mathscr{P}_N \to \mathscr{P}_N,$$

where $\Lambda_N(p)$ for $p \in \mathscr{P}_N$ is the Taylor polynomial of order N at 0 of the product $\lambda(k)\, p(k)$. Let us write the block matrix Λ_{ij} of the operator Λ_N that corresponds to the decomposition $\mathscr{P}_N = \bigoplus_{l=0}^N P_l$. It is obvious that $\Lambda_{ij} = 0$ for $i - j < l_0$. For $i - j \geqslant l_0$ the entry Λ_{ij} is the operator of multiplication by λ_{i-j} acting from P_j into P_i. Since $\lambda_{l_0} \neq 0$, for $i - j = l_0$ the operator Λ_{ij} of multiplication by λ_{l_0} has zero kernel. Being interested in the cokernel of Λ_N, we need to find the kernel of the adjoint matrix Λ_N^*. The adjoint matrix acts in the space $\bigoplus_{l=0}^N P_l^*$, where P_l^* can be naturally identified with the space of linear combinations of the derivatives of order l of the Dirac's delta-function at the origin. Here we have $\Lambda_{ij}^* = 0$ for $j - i < l_0$, and for $j - i \geqslant l_0$ the entry Λ_{ij}^* is the dual to the operator of multiplication by λ_{j-i} acting from P_i into P_j. In particular, since for $j - i = l_0$ the latter operator is injective, we conclude that the operators Λ_{ij}^* are surjective. This enables one to find the dimension of the kernel of the matrix Λ_N^* and even to describe its structure. Namely, let

$$\psi = (\psi_0, ..., \psi_N) \in \bigoplus_{l=0}^{N} P_l^*$$

be such that $\Lambda^* \psi = 0$. Due to the triangular structure of Λ_N^*, it is easy to solve this system. Indeed, it can be written as follows:

$$\sum_{j \geqslant i + l_0} \Lambda_{ij}^* \psi_j = 0, \qquad i = 0, ..., N - l_0.$$

Taking the Fourier transform, we can rewrite this system in the form

$$\sum_{j \geqslant i+l_0} \lambda_{j-i}(D)\,\widehat{\psi_j}=0, \qquad i=0, ..., N-l_0,$$

where $\hat{\psi}$ denotes the Fourier transform of ψ. Therefore, $\widehat{\psi_j}$ is a homogeneous polynomial of degree j in $\mathbb{R}^n$. For $i=N-l_0$ we have

$$\lambda_{l_0}(D)\,\widehat{\psi_N}=0.$$

This equality means that $\widehat{\psi_N}$ can be chosen as an arbitrary λ_{l_0}-harmonic homogeneous polynomial of order N. Moving to the previous equation, we analogously obtain

$$\lambda_{l_0}(D)\,\widehat{\psi_{N-1}}+\lambda_{l_0+1}(D)\,\widehat{\psi_N}=0,$$

or

$$\lambda_{l_0}(D)\,\widehat{\psi_{N-1}}=-\lambda_{l_0+1}(D)\,\widehat{\psi_N}.$$

The right hand side is already determined, and the nonhomogeneous equation, as we concluded before, always has a solution, for instance

$$-R(\lambda_{l_0+1}(D)\,\widehat{\psi_N}).$$

This means that

$$\widehat{\psi_{N-1}}+R(\lambda_{l_0+1}(D)\,\widehat{\psi_N})$$

is a λ_{l_0}-harmonic homogeneous polynomial of order $N-1$. We see that the solution $\widehat{\psi_{N-1}}$ exists and is determined up to an addition of any homogeneous λ_{l_0}-harmonic polynomial of degree $N-1$. Continuing this process until we reach $\widehat{\psi_0}$, we conclude that the mapping

$$\psi=(\psi_0, ..., \psi_N) \rightarrow \phi=(\phi_0, ..., \phi_N),$$

where

$$\phi_j=\widehat{\psi_j}+R\sum_{i>j} \lambda_{i-j+l_0}(D)\,\hat{\psi}_i$$

establishes an isomorphism between the cokernel of the mapping Λ_N and the space of λ_{l_0}-harmonic polynomials of degree at most N. This proves the theorem. ∎

In the cases of the simplest structures of the Taylor series, the theorem implies the following:

COROLLARY 11. *Under the hypotheses of Theorem* 10 *the following hold*:

1. *If k_0 is a noncritical point of the band function $\lambda(k)$, then the dimension of the space of Floquet solutions of the equation $Pu=0$ in $\mathbb{R}^n$ of order at most N with a quasimomentum k_0 is equal to the dimension $q_{n-1,N}$ of the space of all polynomials of degree at most N in $\mathbb{R}^{n-1}$.*

2. *If the Taylor expansion of the band function $\lambda(k)$ at a point k_0 starts with a nondegenerate quadratic form, then the dimension of the space of Floquet solutions of the equation $Pu=0$ in $\mathbb{R}^n$ of order at most N with a quasimomentum k_0 is equal to the dimension $h_{n,N}$ of the space of harmonic* (*in the standard sense*) *polynomials of degree at most N in $\mathbb{R}^n$. In particular, this condition is satisfied at nondegenerate extrema.*

In both cases an isomorphism can be provided explicitly as in the previous theorem.

Proof. 1. By our assumptions, the Taylor expansion of $\lambda(k)$ starts with a nonzero linear term $\lambda_1(k)=\sum_{j=1}^{n} a_j k_j$, $a_j \in \mathbb{C}$. The corresponding differential operator is

$$\lambda_1(D) = -i \sum_{j=1}^{n} a_j \frac{\partial}{\partial x_j} = \sum_{j=1}^{n} \alpha_j \frac{\partial}{\partial x_j} + i \sum_{j=1}^{n} \beta_j \frac{\partial}{\partial x_j},$$

where α_j and β_j are real. Consider first the case when the vectors $\alpha=(\alpha_j)$ and $\beta=(\beta_j)$ are collinear. Then $\lambda_1(D)$ becomes $\gamma_0 \sum \gamma_j \frac{\partial}{\partial x_j}$, where $\gamma_0 \neq 0$ is a complex number and $\gamma=(\gamma_j)$ is a nonzero real vector. A linear change of coordinate system brings $\lambda_1(D)$ to the operator $\frac{\partial}{\partial x_1}$ (up to an irrelevant constant factor). Thus, the λ_1-harmonic polynomials are exactly those independent on x_1. Invoking Theorem 10, we get our conclusion in this case. Consider now the situation when α and β are linearly independent. Then a linear change of variables brings λ_1 to the form $\partial/\partial \bar{z}$, where $z=x_1+ix_2$. Since any polynomial in variables $(x_1, ..., x_n)$ is a polynomial of the same degree in $(z, \bar{z}, x_3, ..., x_n)$, the λ_1-harmonic polynomials are the ones depending on $(z, x_3, ..., x_n)$ only (i.e., the ones analytic in z). This again reduces the number of variables to $n-1$.

2. By our assumptions, the first nonzero homogeneous term is a nondegenerate quadratic form $\lambda_2(k-k_0)$, which is reducible to the sum of squares of coordinates by a linear change of variables. Therefore, in the new coordinates $\lambda_2(D)=-\Delta$. Using Theorem 10, we obtain the desired result. ∎

In the remaining part of this section we restrict further the form of the operator. Namely, we consider now second order operators with *real* periodic coefficients of the form

$$L = -\sum_{i,\, j=1}^{n} a_{ij}(x)\, \partial_i\, \partial_j + \sum_{i=1}^{n} b_i(x)\, \partial_i + c(x). \tag{2.5}$$

It is assumed that the uniform ellipticity condition

$$\sum_{i,\, j=1}^{n} a_{ij}(x)\, \zeta_i \zeta_j \geqslant a \sum_{i=1}^{n} \zeta_i^2$$

is satisfied for all $x, \zeta \in \mathbb{R}^n$, where a is a positive constant.

For such operators, we introduce the function that will play the crucial role in our considerations. Its properties were studied in detail in [2, 36], and [41]. Consider the function $\Lambda(\xi)\colon \mathbb{R}^n \to \mathbb{R}$ defined by the condition that the equation

$$Lu = \Lambda(\xi)\, u$$

has a *positive* Bloch solution of the form

$$u_\xi(x) = e^{\xi \cdot x} p_\xi(x), \tag{2.6}$$

where $p_\xi(x)$ is Γ-periodic.

LEMMA 12. 1. *The value $\Lambda(\xi)$ is uniquely determined for any $\xi \in \mathbb{R}^n$.*

2. *The function $\Lambda(\xi)$ is bounded from above, strictly concave, analytic, and has a nonzero gradient at all points except at its maximum point.*

3. *Consider the operator*

$$L(\xi) = e^{-\xi \cdot x} L \circ e^{\xi \cdot x} = L(x, D - i\xi)$$

on the torus $\mathbb{T}^n$. Then $\Lambda(\xi)$ is the principal eigenvalue of $L(\xi)$ with a positive eigenfunction p_ξ. Moreover, $\Lambda(\xi)$ is algebraically simple.

4. *The Hessian of $\Lambda(\xi)$ is nondegenerate at all points.*

One should note that since the function $\Lambda(\xi)$ is analytic, it is actually defined in a neighborhood of $\mathbb{R}^n$ in $\mathbb{C}^n$. This remark will be used in what follows.

Let us denote

$$\Lambda_0 = \max_{\xi \in \mathbb{R}^n} \Lambda(\xi). \tag{2.7}$$

It follows from [2, 36] that an alternative definition of Λ_0 is

$$\Lambda_0 = \sup\{\lambda \in \mathbb{R} \mid \exists u > 0 \text{ such that } (L-\lambda)\, u = 0 \text{ in } \mathbb{R}^n\}, \tag{2.8}$$

and that in the self-adjoint case Λ_0 coincides with the bottom of the spectrum of the operator L. The common name for Λ_0 is *the generalized principal eigenvalue* of the operator L in $\mathbb{R}^n$.

We will often need to assume that Λ_0 is either nonnegative or strictly positive. In the self-adjoint case such an assumption has a clear spectral interpretation. In the next lemma, we provide some known sufficient conditions for the nonnegativity or positivity of Λ_0 for operators of the form (2.5).

LEMMA 13. *Consider an operator L of the form* (2.5)

1. $\Lambda_0 \geqslant 0$ *if and only if the operator L admits a positive* (*super*)*solution. This condition is satisfied in particular when $c(x) \geqslant 0$.*

2. $\Lambda_0 \geqslant 0$ *if and only if the operator L admits a positive solution of the form* (2.6).

3. $\Lambda_0 = 0$ *if and only if the equation $Lu = 0$ admits exactly one normalized positive solution in $\mathbb{R}^n$.*

4. *If $c(x) = 0$, then $\Lambda_0 = 0$ if and only if $\int_{\mathbb{T}^n} b(x)\, \psi(x)\, dx = 0$, where ψ is the principal eigenfunction of L^* on $\mathbb{T}^n$. In particular, divergence form operators satisfy this condition. Here $b(x) = (b_i(x), ..., b_n(x))$.*

5. *Let $\xi \in \mathbb{R}^n$, and assume that $u_\xi(x) = e^{\xi \cdot x} p_\xi(x)$ and $u^*_{-\xi}$ are positive Bloch solutions of the equations $Lu = 0$ and $L^*u = 0$, respectively. Denote by ψ the periodic function $u_\xi u^*_{-\xi}$. Consider the function*

$$\tilde{b}_i(x) = b_i(x) - 2 \sum_{j=1}^{n} a_{ij}(x)\{\xi_j + (p_\xi(x))^{-1}\, \partial_j p_\xi(x)\},$$

and denote

$$\gamma = (\gamma_1, ..., \gamma_n) := \left(\int_{\mathbb{T}^n} \tilde{b}_1(x)\, \psi(x)\, dx, ..., \int_{\mathbb{T}^n} \tilde{b}_n(x)\, \psi(x)\, dx \right).$$

Then $\Lambda_0 = 0$ if and only if $\gamma = 0$.

Let us discuss also some additional properties that will play an important role in the sequel. Assume that $\Lambda_0 \geqslant 0$. Then Lemma 12 implies that the zero level set

$$\Xi = \{\xi \in \mathbb{R}^n \mid \Lambda(\xi) = 0\} \tag{2.9}$$

is either a strictly convex compact analytic surface in $\mathbb{R}^n$ of dimension $n-1$ (this is the case if and only if $\Lambda_0 > 0$), or a singleton (this is the case if and only if $\Lambda_0 = 0$). The manifold Ξ consists of all $\xi \in \mathbb{R}^n$ such that the equation $Lu = 0$ admits a positive Bloch solution $u_\xi(x) = e^{\xi \cdot x} p_\xi(x)$, $p_\xi(0) = 1$. Moreover, the set of all such positive Bloch solutions is the set of all *minimal positive solutions* of the equation $Lu = 0$ in $\mathbb{R}^n$ [2, 36]. It is also established that a function u is a positive solution of the equation $Lu = 0$ in $\mathbb{R}^n$ if and only if there exists a positive finite measure μ on Ξ such that

$$u(x) = \int_\Xi u_\xi(x)\, d\mu(\xi).$$

We denote by G the convex hull of Ξ, and by $\mathring{G}$ its interior. Note that if $\Lambda_0 \geqslant 0$ then $\Lambda_0 = 0$ if and only if $\Xi = G$ and hence $\mathring{G} = \emptyset$.

Lemma 14. *Suppose that $\Lambda_0 > 0$. There exists a neighborhood W of G in $\mathbb{C}^n$ and an analytic function*

$$W \ni \xi \mapsto p_\xi(\cdot) \in H^2(\mathbb{T}^n)$$

such that for any $\xi \in W$ the function of x

$$u_\xi(x) = \exp(\xi \cdot x)\; p_\xi(x)$$

is a nonzero Bloch solution of the equation $Lu = \Lambda(\xi)\, u$ with a quasimomentum $-i\xi$. Moreover, one can choose the function p in such a way that it is positive for all $\xi \in \Xi$.

Comparing the definitions of Ξ and of the Fermi surface F_L, it follows that

$$-i\Xi \subset F_L.$$

The next lemma specifies further the relation between these two varieties:

Lemma 15. *Let $\Lambda_0 \geqslant 0$. Then*

1. *The intersection of the complex Fermi surface F_L with the tube*

$$\mathscr{T} = \{k \in \mathbb{C}^n \mid Im\, k = (Im\, k_1, ..., Im\, k_n) \in -G\} \tag{2.10}$$

coincides with the union of the surface $-i\Xi$ with its translations by the vectors of the reciprocal lattice Γ^*, i.e., consists of vectors $k = -i\xi + \gamma$ where $\xi \in \Xi$ and $\gamma \in \Gamma^*$. Moreover, up to a multiplicative constant, any nonzero Bloch solution with a quasimomentum in the above intersection is a positive Bloch solution.

2. *If $\Lambda_0 > 0$, then the intersection of F_L with a sufficiently small neighborhood of $-i\Xi$ is a (smooth) analytic manifold that coincides with the set of zeros of the function $\Lambda(ik)$.*

Analogously to the definition of the Floquet surface $\Phi = \Phi_L$, we define the surface

$$\Psi = \rho(-i\Xi) = \{z \mid z = (\exp \xi_1, ..., \exp \xi_n), \xi \in \Xi\}, \tag{2.11}$$

and the tubular domain

$$V = \rho(\mathscr{T}), \tag{2.12}$$

where $\mathscr{T}$ was defined in (2.10). The results of Lemmas 14 and 15 can be rephrased in terms of these objects:

LEMMA 16. *Let $\Lambda_0 \geqslant 0$. Then*

1. $\Phi \cap V = \Psi$.

If $\Lambda_0 > 0$, then

2. *The intersection of Φ with a sufficiently small neighborhood of Ψ is a (smooth) connected analytic manifold.*

3. *The intersections of Φ with neighborhoods of the tube V form a basis of neighborhoods of Ψ in Φ.*

4. *For a sufficiently small neighborhood Φ_ε of Ψ in Φ there exists an analytic function p: $\Phi_\varepsilon \to H^2(\mathbb{T}^n)$ such that for any $z \in \Phi_\varepsilon$ the function of x*

$$u_z(x) = z^x p(z, x)$$

is a nonzero Bloch solution of the equation $Lu = 0$.

We will also employ the following lemma:

LEMMA 17. *Consider an operator L of the form* (2.5)

1. *Assume that $c(x) \geqslant 0$. Then the only solutions of the equation $Lu = 0$ of the type* $\exp(ik \cdot x)\, p(x)$, *where $k \in \mathbb{R}^n$ and p is a Γ-periodic function are the constants. If such a nontrivial solution exists, then $c(x) = 0$, and $\Lambda(0) = 0$ (i.e., $0 \in \Xi$).*

2. *Suppose that the operator L admits a positive periodic supersolution $\psi \in C^{2,\alpha}(\mathbb{R}^n)$. Assume that $v(x) = \exp(ik \cdot x)\, p(x)$ is a nontrivial solution of the equation $Lu = 0$, where $k \in \mathbb{R}^n$ and p is a periodic function. Then there exists $C \in \mathbb{C}$ such that $v = C\psi$, the function ψ is a positive periodic solution, and $\Lambda(0) = 0$.*

3. REPRESENTATION OF SOLUTIONS BY HYPERFUNCTIONS

The main result of this section (Theorem 18) is analogous to Theorem 5.1 in [4], which characterizes the class of solutions of the Helmholtz equation that can be represented by means of hyperfunctions on S (see also the introduction to our paper).

In order to state it, we need to introduce a new object. Let us denote by $h(\omega)$, $\omega \in S^{n-1}$ the indicator function of the convex set G. Namely,

$$h(\omega) = \sup_{\xi \in G} (\omega \cdot \xi), \tag{3.1}$$

where $\omega \cdot \xi = \sum_{j=1}^{n} \omega_j \xi_j$ is the inner product in $\mathbb{R}^n$. The next Theorem will be stated in terms of this function.

THEOREM 18. *Suppose that $\Lambda_0 > 0$. Let u be a solution of the equation $Lu = 0$ in $\mathbb{R}^n$ satisfying for any $\varepsilon > 0$ the estimate*

$$|u(x)| \leqslant C_\varepsilon \exp((h(x/|x|) + \varepsilon)\, |x|), \tag{3.2}$$

where C_ε is a constant depending only on ε and u. Then u can be represented as

$$u(x) = \langle \mu(\xi), u_\xi(x) \rangle, \tag{3.3}$$

where u_ξ is the analytic positive Bloch solution corresponding to $\xi \in \Xi$ (see Lemma 14), and $\mu(\xi)$ is a hyperfunction (analytic functional) on Ξ. The converse statement is also true: for any hyperfunction μ on Ξ, the function $u(x)$ in (3.3) is a solution of the equation $Lu = 0$ in $\mathbb{R}^n$ which satisfies the growth condition (3.2).

Remark 19. Using a standard elliptic argument it follows that the pointwise growth condition (3.2) is equivalent to the growth condition

$$u(x) \exp(-(h(x/|x|) + \varepsilon)|x|) \in L^2(\mathbb{R}^n). \tag{3.4}$$

Proof of Theorem 18. Assume first that a solution u has the representation (3.3). We need to prove that u satisfies the growth condition (3.2). Due to the real analyticity of u_ξ with respect to ξ and according to Lemmas 14 and 15, u_ξ can be extended to an analytic vector function $u_\xi(x) = \exp(\xi \cdot x)\, p_\xi(x)$ on an ε-neighborhood U_ε of Ξ in iF_L. Since μ is a hyperfunction (analytic functional) on Ξ, we have an estimate

$$|u(x)| \leqslant C_\varepsilon \max_{\xi \in U_\varepsilon} |u_\xi(x)|.$$

Hence we have

$$|u(x)| \leqslant C_\varepsilon \max_{\xi \in U_\varepsilon} |e^{\xi \cdot x}| = C_\varepsilon e^{|x| \, (h(x/|x|) + \delta(\varepsilon))},$$

where $\lim_{\varepsilon \to 0} \delta(\varepsilon) = 0$, which gives (3.2).

Suppose now that u satisfies (3.2). We need to prove that u can be represented as in (3.3). Let G_ε be the ε-neighborhood of G and $h_\varepsilon = h + \varepsilon$ be the indicator function of $\overline{G_\varepsilon}$. Consider the following Fréchet spaces of test functions,

$$W_{m,\varepsilon} = \{\phi \in H^m_{loc}(\mathbb{R}^n) \mid \langle \phi \rangle_{m,\delta} < \infty, \forall\, 0 < \delta < \varepsilon\},$$

where

$$\langle \phi \rangle_{m,\delta} := \sup_{\gamma \in \Gamma} \{ \|\phi\|_{H^m(K+\gamma)} e^{(h_\delta(\gamma/|\gamma|)\, |\gamma|)} \}.$$

It is obvious that the operator L^* maps continuously $W_{2,\varepsilon}$ into $W_{0,\varepsilon}$. Consequently, the linear functional

$$\langle u, \phi \rangle := \int_{\mathbb{R}^n} u(x)\, \phi(x)\, dx$$

is continuous on the space $W_{0,\varepsilon}$ for any $\varepsilon > 0$. Since $Lu = 0$, Schauder elliptic estimates together with the periodicity of the operator show that estimates similar to (3.2) hold also for the derivatives of u. One observes that u is a continuous functional on $W_{0,\varepsilon}$ which annihilates the range of the operator L^*: $W_{2,\varepsilon} \to W_{0,\varepsilon}$. Now Floquet theory arguments analogous to the ones used in [30, Section 3.2] can be applied to yield (3.3). Let us make this part more precise.

Our first goal is to obtain a Paley–Wiener type theorem for the Floquet transform in the spaces $W_{m,\varepsilon}$. Let us denote by V_ε the domain in $(\mathbb{C}^*)^n$

$$V_\varepsilon = \{z = (z_1, \ldots, z_n) \in (\mathbb{C}^*)^n \mid z_j = \exp ik_j \text{ such that}$$
$$\operatorname{Im} k = (\operatorname{Im} k_1, \ldots, \operatorname{Im} k_n) \in (-G_\varepsilon)\}$$

and let

$$V_\varepsilon^* = \{z = (z_1, \ldots, z_n) \in (\mathbb{C}^*)^n \mid z^{-1} = (z_1^{-1}, \ldots, z_n^{-1}) \in V_\varepsilon\},$$

The domains V_ε form a basis of neighborhoods of the tube V, where V is defined by (2.12). The following statement is a Paley–Wiener type theorem for the transform $\mathcal{U}$ which is suitable for our purpose.

LEMMA 20. 1. *The operator*

$$\mathscr{U}\colon W_{m,\varepsilon} \to \Gamma(V_\varepsilon^*, \mathscr{E}_m)$$

is an isomorphism, where $\Gamma(V_\varepsilon^*, \mathscr{E}_m)$ is the space of holomorphic sections over V_ε^* of the bundle $\mathscr{E}_m$, equipped with the topology of uniform convergence on compacta.

2. *Under the transform $\mathscr{U}$, the operator*

$$L^*\colon W_{2,\varepsilon} \to W_{0,\varepsilon}$$

becomes the operator $\mathscr{L}(z)$ of multiplication by a holomorphic Fredholm morphism between the fiber bundles $\mathscr{E}_2$ and $\mathscr{E}_0$:

$$\Gamma(V_\varepsilon^*, \mathscr{E}_2) \xrightarrow{\mathscr{L}(z)} \Gamma(V_\varepsilon^*, \mathscr{E}_0).$$

Here $\mathscr{L}(z)$ acts on each fiber of $\mathscr{E}_2$ as the restriction to this fiber of the operator L^* acting between $H^2(K)$ and $L^2(K)$.

Let us choose a value $\varepsilon_0 > 0$ such that the intersection of Φ with V_ε is smooth and connected. This is possible according to Lemma 16. From now on, we will only consider the values $0 < \varepsilon < \varepsilon_0$.

Since the image $\mathscr{U}u$ of the solution u under the Floquet transform $\mathscr{U}$ is a continuous linear functional on $\Gamma(V_\varepsilon^*, \mathscr{E}_0)$ which is in the cokernel of the operator

$$\Gamma(V_\varepsilon^*, \mathscr{E}_2) \xrightarrow{\mathscr{L}(z)} \Gamma(V_\varepsilon^*, \mathscr{E}_0),$$

our task is to describe all such functionals. Several theorems of this kind were proven in [30]. In our current situation such a representation can be obtained rather easily, due to the simplicity of the structure of the Floquet variety inside V_ε. Namely, let $u_z(\cdot) = z^x p(z, \cdot)$ be the Bloch solution of the equation $Lu = 0$ introduced in Lemma 16. Let also $\mathscr{H}(\Phi_\varepsilon)$ be the space of holomorphic functions on $\Phi_\varepsilon = \Phi \cap V_\varepsilon$ equipped with the topology of uniform convergence on compacta. We introduce the mapping

$$t\colon \Gamma(V_\varepsilon^*, \mathscr{E}_0) \to \mathscr{H}(\Phi_\varepsilon)$$

which for any section $f(z, x)$ of the bundle $\mathscr{E}_0$ produces

$$t_f(z) = \langle f(z^{-1}, \cdot), u_z \rangle = \int_{\mathbb{T}^n} f(z^{-1}, x)\, u_z(x)\, dx.$$

Here $z^{-1} = (z_1^{-1}, \ldots, z_n^{-1})$.

LEMMA 21. *Let $0<\varepsilon<\varepsilon_0$, where ε_0 is the value defined above. Then the mapping t is a topological homomorphism and the following sequence is exact*:

$$\Gamma(V_\varepsilon^*, \mathscr{E}_2) \xrightarrow{\mathscr{L}(z)} \Gamma(V_\varepsilon^*, \mathscr{E}_0) \xrightarrow{t} \mathscr{H}(\Phi_\varepsilon) \to 0.$$

This lemma practically finishes the proof of the theorem. Namely, the solution u after the Floquet transform leads to a continuous linear functional on $\Gamma(V_\varepsilon^*, \mathscr{E}_0)$ that annihilates the range of the operator of multiplication by $\mathscr{L}(z)$. Lemma 21 implies that such a functional can be pushed down to the space $\mathscr{H}(\Phi_\varepsilon)$. Since this functional, due to the estimate (3.2), is continuously extendable to $\mathscr{H}(\Phi_\varepsilon)$ for arbitrarily small values of ε, it is in fact a hyperfunction (analytic functional) μ on $\Phi=\bigcap_{\varepsilon>0}\Phi_\varepsilon$. Hence, the action $\langle u, \phi\rangle$ of the functional u on a function $\phi\in W_{0,\varepsilon}$ can be obtained as

$$\langle u, \phi\rangle = \langle \mu(z), t(z)(\mathscr{U}\phi)\rangle.$$

Applying now the explicit formulas for the transforms $\mathscr{U}$ and t, one arrives to the representation (3.3). Indeed,

$$\begin{aligned} t_{(\mathscr{U}\phi)}(z) &= \int_K \mathscr{U}\phi(z^{-1}, x)\, u_z(x)\, dx \\ &= \sum_{\gamma\in\Gamma} \int_{K-\gamma} \phi(x)\, z^{-\gamma} u_z(x+\gamma)\, dx \\ &= \int_{\mathbb{R}^n} \phi(x)\, u_z(x)\, dx. \end{aligned} \tag{3.5}$$

In this calculation we used the property of the Bloch solutions

$$u_z(x+\gamma) = z^\gamma u_z(x).$$

Therefore,

$$\langle u, \phi\rangle = \langle\langle \mu(z), u_z\rangle, \phi\rangle,$$

which concludes the proof of the theorem. ∎

4. LIOUVILLE-TYPE THEOREM

In this section we discuss Liouville theorems for periodic equations. We will consider at the moment an arbitrary linear elliptic operator $P(x, D)$

with smooth Γ-periodic coefficients which satisfies the assumptions made in Section 2 (as above, without loss of generality we can reduce the consideration to the case $\Gamma = \mathbb{Z}^n$).

DEFINITION 22. We say that *the Liouville theorem holds true for the operator P*, if for any $N \in \mathbb{N}$ the space $V_N(P)$ of solutions of the equation $Pu = 0$ in $\mathbb{R}^n$ that can be estimated as

$$\|u\|_{L^2(K+\gamma)} \leqslant C(1 + |\gamma|)^N \qquad \text{for all} \quad \gamma \in \Gamma$$

is finite dimensional.

In the case when the Liouville theorem holds, we will be also interested in the dimensions d_N of the spaces $V_N(P)$ and in representations of their elements analogous to (1.3).

The result below explains under what conditions on the operator P a Liouville-type theorem holds. These conditions will then be verified for some specific classes of operators.

As was mentioned in the introduction, solutions representable as (1.3) are just Floquet solutions with zero quasimomentum. So, the Liouville theorem of [8, 38] cited in the introduction states that any polynomially growing solution is a Floquet solution with a zero quasimomentum. Let us also mention that any Bloch solution $e^{ik \cdot x} p(x)$ with a real quasimomentum k is automatically bounded. This means that the validity of the Liouville theorem for an operator P implies that the number of the real quasimomenta of solutions of the equation $Pu = 0$ must be finite (modulo the action of the reciprocal lattice). In other words, the Fermi surface for P intersects the real space at a finite number of points (modulo the reciprocal lattice). In terms of the Floquet variety it means that the set $\mathscr{Z} := \Phi_P \cap T$ is finite. We denote the cardinality of a set A by $\# A$. As the second statement of the next theorem shows, the finiteness of $\mathscr{Z}$ is in fact the only claim of the Liouville theorem.

THEOREM 23. 1. *The equation $Pu = 0$ has a nonzero polynomially growing solution if and only if it has a nonzero bounded Bloch solution, i.e., if and only if the intersection $F_L \cap \mathbb{R}^n$ of the Fermi surface for P with the real space is not empty* (*or equivalently, $\mathscr{Z} = \Phi_P \cap T \neq \varnothing$*).

2. *The Liouville theorem holds for the operator P if and only if the intersection $F_P \cap \mathbb{R}^n$ is a finite set modulo the reciprocal lattice* (*or equivalently, $\# \mathscr{Z} < \infty$*). *Moreover, if $\# \mathscr{Z} = \infty$ then the Liouville theorem does not hold even for bounded solutions, i.e., $d_0 = \dim(V_0) = \infty$.*

3. *If the Liouville theorem holds, then each solution* $u \in V_N(P)$ *can be represented as a finite sum of Floquet solutions*:

$$u(x) = \sum_{q \in F_P \cap \mathbb{R}^n} e^{iq \cdot x} \sum_{|j| \leqslant N} x^j p_{j,q}(x). \tag{4.1}$$

4. *If the Liouville theorem holds, then for all* $N \geqslant 0$ *we have*

$$d_N \leqslant d_0 q_{n,N} < \infty,$$

where $q_{n,N}$ *is the dimension of the space of all polynomials of degree at most* N *in* n *variables.*

5. *Assume that the Liouville theorem holds and that for each real quasimomentum* q (*i.e., for each* $q \in F_P \cap \mathbb{R}^n$) *the conditions of Theorem* 10 *are satisfied. Then for each* $N \geqslant 0$ *the dimension* d_N *of the space* $V_N(P)$ *is equal to the sum over* $q \in (F_P \cap \mathbb{R}^n)/\Gamma^*$ *of the dimensions of the spaces of* λ_q*-harmonic polynomials of order at most* N (*see Definition* 9), *where* λ_q *is the first nonzero homogeneous term in the Taylor expansion at the point* q *of the dispersion relation* (*band function*) $\lambda(k)$.

Proof. Statements 4 and 5 follow from 3 together with Lemma 8 and Theorem 10. So, we first prove statements 2 and 3 and conclude with the proof of the first statement.

In order to prove 2 let us notice that if $\#\mathscr{Z} = \infty$ then each point $z = \exp ik \in \mathscr{Z}$ provides a bounded Bloch solution with the quasimomentum k, and these solutions are linearly independent. This means that the Liouville theorem cannot hold in this case.

Assume now that $\#\mathscr{Z} < \infty$. We need to prove that the Liouville theorem and representation (4.1) hold true. Obviously, if u has a representation of the form (4.1), then u is of a polynomial growth. The proof that any polynomially growing solution is of the form (4.1) follows the same simple strategy as in the proofs of Theorem 18 and of the main Floquet representation [30, Theorem 3.2.1] (which, in turn, comes from the approach of [16] and [39]). As in the case with the fundamental principle (see [16] and [39]), it is more convenient to deal with a dual formulation, as it is done in [30]. Namely, any polynomially growing solution $u(x)$ can be interpreted in the dual way, as a functional on an appropriate functional space, which belongs to the cokernel of the dual operator P^*. Consequently, a representation theorem for all such functionals must be obtained. In order to make this idea precise, we need to introduce appropriate test functions spaces.

Consider the Fréchet spaces

$$C_m = \{\phi \in H^m_{\mathrm{loc}}(\mathbb{R}^n) \mid \sup_{\gamma \in \Gamma} \|\phi\|_{H^m(K+\gamma)} (1+|\gamma|)^N < \infty, \forall N\}.$$

Let the order of the operator P be m, then it is clear that P^* maps continuously C_m into C_0. Due to the polynomial growth of $u(x)$, the linear functional

$$\langle u, \phi \rangle = \int_{\mathbb{R}^n} u(x)\, \phi(x)\, dx$$

is continuous on C_0. Since $Pu = 0$, one easily observes that u annihilates the range of the operator P^*: $C_m \to C_0$. We need now a Paley–Wiener type theorem for the spaces C_m with respect to the Floquet transform.

Lemma 24. 1. *The operator*

$$\mathscr{U}: C_m \to C^\infty(T, \mathscr{E}_m)$$

is an isomorphism, where $C^\infty(T, \mathscr{E}_m)$ *is the space of* C^∞ *sections of the bundle* $\mathscr{E}_m$ *over the complex torus* T, *equipped with the standard topology.*

2. *Under the transform* $\mathscr{U}$, *the operator*

$$P^*: C_m \to C_0$$

becomes the operator $\mathscr{P}(z)$ *of multiplication by a holomorphic Fredholm morphism between the fiber bundles* $\mathscr{E}_m$ *and* $\mathscr{E}_0$:

$$C^\infty(T, \mathscr{E}_m) \xrightarrow{\mathscr{P}(z)} C^\infty(T, \mathscr{E}_0).$$

Here $\mathscr{P}(z)$ *acts on each fiber of* $\mathscr{E}_m$ *as the restriction to this fiber of the operator* P^* *acting between* $H^m(K)$ *and* $L^2(K)$.

3. *The operator* $\mathscr{P}(z)$ *is invertible for a point* $z \in T$ *if and only if* $z^{-1} \notin \Phi$.

The next lemma is an analog of the classical theorem on the structure of distributions supported at a single point. Together with the previous lemma it essentially leads to the statement of the theorem.

Lemma 25. *Let* T *be a* C^∞*-manifold and* $\mathscr{P}: T \to L(B_1, B_2)$ *be a* C^∞*-function with values in the space* $L(B_1, B_2)$ *of bounded linear operators between Banach spaces* B_1 *and* B_2. *Assume that for each* $z \in T$ *the operator* $\mathscr{P}(z)$ *is a Fredholm operator. Then*

1. *If $\mathscr{P}(z)$ is surjective for all points z in T, then the multiplication operator*

$$C^{\infty}(T, B_1) \xrightarrow{\mathscr{P}(z)} C^{\infty}(T, B_2)$$

is surjective.

2. *If $\mathscr{P}(z)$ is surjective for all points z except a finite subset $\mathscr{Z} \subset T$, then any continuous linear functional g on the space of smooth vector functions $C^{\infty}(T, B_2)$ that annihilates the range of the multiplication operator*

$$C^{\infty}(T, B_1) \xrightarrow{\mathscr{P}(z)} C^{\infty}(T, B_2)$$

has the form

$$\langle g, \phi \rangle = \sum_{z \in \mathscr{Z}} \left[\sum_{j \leqslant N} D_{j,z}(\langle g_{j,z}, \phi \rangle) \right]_z . \tag{4.2}$$

Here $g_{j,z}$ are continuous linear functionals on B_2, $\langle g_{j,z}, \phi \rangle$ denotes the duality between B_2^* and B_2, $D_{j,z}$ are linear differential operators on T, and $N \in \mathbb{N}$.

We are ready now to finish the proof of the nontrivial part of the third statement of Theorem 23.

If u is a solution of polynomial growth, it belongs, as it has been mentioned already, to the cokernel of the operator $P^*: C_m \to C_0$. After the Floquet transform we are dealing with the cokernel of the operator

$$C^{\infty}(T, \mathscr{E}_2) \xrightarrow{\mathscr{P}(z)} C^{\infty}(T, \mathscr{E}_0).$$

By Lemma 24, the only points $z \in T$ where $\mathscr{P}(z)$ is not invertible are those points where z^{-1} belongs to the Floquet variety. Since by our assumption the set $\mathscr{Z} = T \cap \Phi$ is finite, it follows that the operator function $\mathscr{P}(z)$ satisfies all the assumptions of Lemma 25. The fact that we are dealing with Banach bundles instead of fixed Banach spaces is irrelevant, since these bundles are trivial. This means that we have the representation (4.2) with $g_j \in L^2(\mathbb{T}^n)$. According to Lemma 8, functionals of the form (4.2) correspond under the inverse Floquet transform exactly to functions of the form (1.3).

It remains to prove the first statement of the theorem. Let u be a polynomially growing solution. Assume that $\mathscr{Z} = \varnothing$, i.e., the intersection of the Floquet variety Φ with the complex torus T is empty. Therefore, the last statement of Lemma 24 implies the invertibility of $P(z)$ for all $z \in T$. Now, the first statement of Lemma 25 guarantees the surjectivity of the mapping

$$C^{\infty}(T, \mathscr{E}_m) \xrightarrow{\mathscr{P}(z)} C^{\infty}(T, \mathscr{E}_0)$$

and hence the absence of any nontrivial functionals on $C^\infty(T, \mathscr{E}_0)$ that annihilate the image of this mapping. Since under the Floquet transform $\mathscr{U}$, a polynomially growing solution $u(x)$ is mapped to such a functional, we conclude that $u = 0$. ∎

Remark 26. The first statement of Theorem 23 is a part of the analog of the Bloch theorem provided in Theorem 4.3.1 of [30]. Namely, the existence of a sub-exponentially (in particular, polynomially) growing solution implies the existence of a Bloch solution with a real quasimomentum, and hence the nonemptiness of the real Fermi variety. For completeness, we gave above an independent proof of this statement.

One realizes now that the cases when a Liouville-type theorem holds in a nonvacuous way are extremely rare. Namely, Theorem 23 shows that this happens only when the Fermi variety touches the real subspace at a finite set of points (modulo the reciprocal lattice). This means in particular, that in the *selfadjoint case*, one should expect this to happen only at the edges of the spectral gaps. Although it is possible to imagine interior points of the spectrum where such a thing could occur, it is hard to believe that these cases could be anything more than accidents.

One can expect the following conjecture to be true:

Conjecture 27. Let P be a "generic" self-adjoint second order elliptic operator with periodic coefficients and (λ_-, λ_+) be a nontrivial gap in its spectrum. Then each of the gap's endpoints is a unique (modulo the dual lattice) and nondegenerate extremum of a single band function $\lambda_j(k)$.

The validity of this conjecture together with Theorem 23 would imply that generically at the gap ends the dimension of the space V_N is equal to the dimension $h_{n,N}$ of the space of all harmonic polynomials of order at most N in n variables. Unfortunately, the only known theorem of this kind is the recent result of [27], which states that generically a gap edge is an extremum of a single band function.

At the bottom of the spectrum, however, much more is known. The theorem below combines some results of [17, 26, 41] with the statement of Theorem 23 to obtain the structure and dimension of the space of polynomially growing solutions in this case. Below the spectrum, the Liouville theorem holds vacuously, according to the first statement of Theorem 23 and Theorem 4.5.1 in [30].

THEOREM 28. 1. *Let $H = -\Delta + V(x)$ be a Schrödinger operator with a periodic real valued potential $V \in L^{r/2}(\mathbb{T}^n)$, $r > n$. Then the lowest band function $\lambda_1(k)$ has a unique nondegenerate minimum Λ_0 at $k = 0$. All other band* *functions are strictly greater than* Λ_0. *Every solution* $u \in V_N(H-\Lambda_0)$ *is representable in the form* (1.3). *The dimension of the space* $V_N(H-\Lambda_0)$ *is equal to* $h_{n,N}$.

2. *Let* V *be like in the previous statement, then there exists* $\varepsilon > 0$ *such that for any periodic real valued magnetic potential* A *such that*

$$\|A\|_{L^r(\mathbb{T}^n)} < \varepsilon$$

and

$$\int_{\mathbb{T}^n} A(x)\, dx = 0 \tag{4.3}$$

the following statements hold true: *The lowest band function* $\lambda_1(k)$ *of the magnetic Schrödinger operator* $H = (i\nabla + A)^2 + V$ *attains a unique nondegenerate minimum* Λ_0 *at a point* k_0. *All other band functions are strictly greater than* Λ_0. *Every solution* $u \in V_N(H-\Lambda_0)$ *is representable in the Floquet form*

$$v(x) = e^{ik_0 \cdot x} \sum_{|j| \leqslant N} x^j p_j(x)$$

with periodic functions $p_j(x)$. *The dimension of the space* $V_N(H-\Lambda_0)$ *is equal to* $h_{n,N}$.

3. *Suppose that* L *is a second order elliptic operator of the form* (2.5) *such that* $\Lambda_0 \geqslant 0$.

If $\Lambda(0) = 0$ (*i.e.*, $0 \in \Xi$), *then the Liouville theorem holds and every solution* $u \in V_N(L)$ *is representable in the form* (1.3). *The dimension of the space* $V_N(L)$ *is equal to* $h_{n,N}$ *in the case when* $\Lambda_0 = 0$, *and to* $q_{n-1,N}$ *when* $\Lambda_0 > 0$.

If $\Lambda(0) > 0$, *then the equation* $Lu = 0$ *does not admit a nontrivial polynomially growing solution. So, the Liouville theorem holds vacuously.*

Proof. 1. The result of [26] says that the lowest band function $\lambda_1(k)$ has a unique nondegenerate minimum Λ_0 at $k = 0$ and that all other band functions are strictly greater than Λ_0. Now Theorem 23 implies the rest of the claims of this statement.

2. When both the electric and magnetic potentials are sufficiently small, then the result of [17] states that the lowest band function $\lambda_1(k)$ of the magnetic Schrödinger operator $H = (i\nabla + A)^2 + V$ attains a unique nondegenerate minimum Λ_0 at a point k_0, while all other band functions are strictly greater than Λ_0. This statement, however, can be easily extended to the case of arbitrary electric and small magnetic potential. Indeed, when the magnetic potential is equal to zero, one can refer, as in

the previous case, to [26]. At this moment one has to use analyticity of the Bloch variety. Namely, the statement of Lemma 4 (see also [30, Theorem 4.4.2]) can be easily extended to include analyticity with respect to the potentials (see, for instance, [17]). More precisely, there exists an entire function $f(k, \lambda, A, V)$ of all its arguments such that $f(k, \lambda, A, V) = 0$ is equivalent to

$$(k, \lambda) \in B_{(i\nabla + A)^2 + V},$$

where B_H is the Bloch variety of the operator H. Now, the result of [26] for $A = 0$ together with the stated analyticity property imply the required features of the lowest band function for sufficiently small magnetic potentials. The last step is to use again Theorem 23. Note that the normalization (4.3) can always be achieved by a gauge transformation which does not affect the spectrum and the Liouville property.

3. The assumption $\Lambda(0) \geqslant 0$ implies that the operator L admits a positive periodic supersolution. It follows from Lemma 17 that the Fermi surface F_L can touch the real space only at the origin (modulo the reciprocal lattice Γ^*) and in this case $\Lambda(0) = 0$. Therefore, by the first part of Theorem 23, the Liouville Theorem holds vacuously if $\Lambda(0) > 0$.

Suppose now that $\Lambda(0) = 0$. Lemma 12 implies that if $\Lambda_0 > 0$ then the point $k = 0$ is a noncritical point of the dispersion relation, and if $\Lambda_0 = 0$ then $k = 0$ is a nondegenerate extremum. Now Theorem 23, as before, completes the proof. ▌

5. PROOFS OF THE LEMMAS

Proof of Lemma 8. The first claim of the lemma corresponds to Theorem 3.1.3 in [30]. In order to prove the second part of the lemma, let us fix a $k_0 \in \mathbb{C}^n$, and choose a closed subspace $M \subset H^m(\mathbb{T}^n)$ complementary to the kernel of the operator $P^*(x, D - k_0)$. Consider the (analytically depending on k in a neighborhood of k_0) subspace

$$\Pi(k) := P^*(x, D - k)(M) \subset L^2(\mathbb{T}^n)$$

and

$$\mathcal{N} := [\Pi(k_0)]^{\perp}.$$

Then $dim(\mathcal{N}) = a_{k_0}$, and for values of k close to k_0 the space $\mathcal{N}$ remains a complementary subspace to $\Pi(k)$. Representing the operators $P^*(x, D-k)$ in the matrix form according to the decompositions

$$H^m(\mathbb{T}^n) = M \oplus Ker\ P^*(x, D-k_0)$$

and

$$L^2(\mathbb{T}^n) = \Pi(k) \oplus \mathcal{N},$$

we get the matrix

$$\begin{pmatrix} B(k) & * \\ 0 & C(k) \end{pmatrix},$$

where $B(k)$ is an invertible analytic operator function, and $C(k)$ is an analytic matrix function of the size $a_{k_0} \times a^*_{k_0}$. Here $a^*_{k_0}$ is the dimension of the kernel of the operator $P^*(x, D-k_0)$. (Notice that $a_{k_0} = a^*_{k_0}$ if $ind\ P = 0$, which is true for instance, when dealing with scalar elliptic operators, due to the Atiyah–Singer theorem.) Now, the space of all distributions orthogonal to the range of P^* and supported at $\exp(-ik_0)$ reduces to the space of all distributions supported at k_0, acting on $\mathbb{C}^{a_k}$-valued vector functions, and orthogonal to the range of the operator of multiplication by $C(k)$. If we drop the orthogonality condition, the dimension of the space of all such distributions of order at most N is obviously equal to $a_k q_{n,N}$, which proves the estimate. We point out that a direct proof of this estimate for scalar operators can be also easily derived using the Leibnitz's rule. ∎

Proof of Lemma 12. Statements 1 through 3 of the lemma are contained in [36], except the statement that the geometric rather than the algebraic multiplicity of the eigenvalue $\Lambda(\xi)$ is equal to one. The latter follows easily from Lemma 5.2 of [36]. Alternatively, it can be deduced from general theorems on positive operators defined on an ordered Banach space (see for instance, [29, Theorem 2.10]). Statement 4 is proven in [41, Theorem 5]. ∎

Proof of Lemma 13. Statements 1–3 follow from the results of [2, 36], while statements 4–5 follow from [41, Theorem 5]. ∎

Proof of Lemma 14. Consider the following family of operators on the torus: $L(x, D - i\xi) - \Lambda(\xi)$. It follows from Lemma 12 that this family is analytic in a complex neighborhood W of the set G and its values are Fredholm operators between the appropriate Sobolev spaces. The same lemma implies that the dimension of the kernel of all these operators is equal to 1. Hence, these kernels form an analytic fiber bundle over W (see Theorem 1.6.13 and the corresponding references in [30]). One can always assume that the domain W is convex (in the geometric sense). Then the kernel bundle (as all vector bundles on W) is topologically trivial. Since W, being convex, is a domain of holomorphy (see for instance Corollary 2.5.6 in [24]), therefore, the result of [19] (an instance of the so called Oka's principle) implies that the bundle is also analytically trivial. This means the existence of a nowhere zero analytic section u_ξ. Positivity of u_ξ for $\xi \in \Xi$ can be achieved as follows. Let us choose any nonzero analytic solution u_ξ as above. Then for some small neighborhood $W_1 \subset W$ of G, we have $u_\xi(0) \neq 0$. So, we may normalize u_ξ by dividing it by $u_\xi(0)$. The resulting solution is clearly positive for $\xi \in \Xi$. ∎

Proof of Lemma 15. 1. Let $u(x) = e^{ik \cdot x} p(x)$ be a nonzero Bloch solution, where $p(x)$ is a Γ-periodic function, and $k \in F \cap \mathscr{T}$. Assume first that $Im\, k \in -\mathring{G}$, so, $\Lambda_0 > 0$. We need to prove that $Re\, u = Im\, u = 0$. We show for instance, that $u_1 := Re\, u = 0$. Suppose that $u_1 \neq 0$. We may assume that $u_1(x_1) > 0$, for some $x_1 \in \mathbb{R}^n$. Consider the positive solution

$$v(x) = \int_{\Xi} u_\xi(x)\, d\sigma(\xi),$$

where $d\sigma$ is the $(n-1)$-dimensional surface area element on Ξ. For every $M > 0$ there exists $R > 0$ such that $v(x) - Mu_1(x) > 0$ for all $|x| > R$. By the generalized maximum principle, $v(x) > Mu_1(x)$ in $\mathbb{R}^n$. Since M is arbitrarily large and $u_1(x_1) > 0$, we arrived at a contradiction. Note that this argument applies also to *any Floquet* solution with a quasimomentum k such that $Im\, k \in -\mathring{G}$.

Suppose now that $Im\, k \in -\Xi$ and $\Lambda \geqslant 0$. Clearly, it is enough to show that there exists a real constant C and $\xi \in \Xi$ such that $u_1 := Re\, u = Cu_\xi$. Let $\xi = -Im\, k$. Then for a sufficiently small $\varepsilon > 0$ the function $v_\varepsilon := \frac{u_\xi}{2} - \varepsilon u_1$ is a positive solution of the equation $Lu = 0$, which is smaller than u_ξ. Recall that u_ξ is a minimal positive solution of the equation $Lu = 0$. Therefore, there exists $c > 0$ such that $v_\varepsilon = cu_\xi$, which implies that $u_1 = Cu_\xi$ for some $C \in \mathbb{R}$.

2. Consider the zero set F_1 of the analytic function $\Lambda(ik)$ in a small complex neighborhood of $-i\Xi$. Since $\Lambda_0 > 0$, it follows that the gradient of $\Lambda(ik)$ is not zero on $-i\Xi$. Therefore, F_1 is a smooth analytic variety. We will show that the Fermi surface F coincides with F_1 in a neighborhood of $-i\Xi$, which will conclude the proof of the lemma. Indeed, obviously $F_1 \subset F$. Consider a point $k_0 = -i\xi_0 \in -i\Xi$. By Lemma 12, zero is a simple eigenvalue of the operator $L(x, D + k_0) = L(x, D - i\xi_0)$. This means that

the spectral projector that corresponds to a neighborhood of zero is one-dimensional for all complex k close to k_0. We conclude that for all k in a complex neighborhood of $-i\Xi$ there is exactly one eigenvalue close to zero of the operator $L(x, D+k)$. By Lemma 14, we know this eigenvalue, namely $\Lambda(ik)$. Let now k belongs to a small neighborhood of $-i\Xi$ and assume that $k \notin F_1$. Then $\Lambda(ik) \neq 0$, and hence zero cannot be the eigenvalue of $L(x, D+k)$. This means that k does not belong to the Fermi surface F. ▮

Proof of Lemma 17. 1. If $c \gneqq 0$, the assertion of the lemma follows from [36, Theorem 4.5]. On the other hand, if $c=0$, then $0 \in -i\Xi$, and in particular, $0 \in (-G)$. It follows from Lemma 15 that any Bloch solution with a real quasimomentum is the constant solution.

2. This assertion follows directly from part 1 using the operator $\psi^{-1} L \circ \psi$. ▮

Proof of Lemma 20. The second statement of the lemma coincides with Theorem 2.2.3 in [30]. So, we need to prove only the first statement.

Let $\varphi \in W_{m,\varepsilon}$. We will show that the series (2.3) converges uniformly on compacta in V_ε^* as a series of functions on V_ε^* with values in $H^m(K)$. This would imply that $\mathscr{U}\varphi \in \Gamma(V_\varepsilon^*, H^m(K))$, and that the corresponding (one-to-one) mapping $\mathscr{U}\colon W_{m,\varepsilon} \to \Gamma(V_\varepsilon^*, H^m(K))$ is continuous. Let $0<\delta<\delta_1<\varepsilon$. Let $z = \exp ik \in V_\delta^*$, which means that $Im\, k \in G_\delta$. We have

$$\|\mathscr{U}\varphi(z, \cdot)\|_{H^m(K)} \leqslant \sum_{\gamma \in \Gamma} \|\varphi\|_{H^m(K-\gamma)}\, e^{-Im\, k \cdot \gamma} = \sum_{\gamma \in \Gamma} \|\varphi\|_{H^m(K+\gamma)}\, e^{Im\, k \cdot \gamma}$$

$$\leqslant \sum_{\gamma \in \Gamma} \|\varphi\|_{H^m(K+\gamma)}\, e^{(h(\gamma/|\gamma|)+\delta)\,|\gamma|} \leqslant C_\delta \langle \varphi \rangle_{m,\delta_1} < \infty.$$

We need to check now that the mapping $\mathscr{U}$ acts from $W_{m,\varepsilon}$ into $\Gamma(V_\varepsilon^*, \mathscr{E}_m)$. This amounts to showing that $\mathscr{U}\varphi$ satisfies the appropriate Floquet boundary conditions and hence is in fact a section of the sub-bundle $\mathscr{E}_m \subset V_\varepsilon^* \times H^m(K)$. This is a straightforward calculation (see also Theorem 2.2.2 in [30]).

On the other hand, let us assume that $s(z) \in \Gamma(V_\varepsilon^*, \mathscr{E}_m)$. If $z = \exp ik$, then s as a function of k is periodic with respect to the reciprocal lattice Γ^*. Expanding it into the Fourier series, we get

$$s(z) = \sum_{\gamma \in \Gamma} s_\gamma z^\gamma,$$

where $s_\gamma \in H^m(K)$. We can now define a function φ on $\mathbb{R}^n$ such that $\varphi(x-\gamma) = s_\gamma(x)$ for $x \in K$ and $\gamma \in \Gamma$.

The function φ belongs to H^m in the interior of each of the cubes $K+\gamma$. One only needs to check that it belongs to H^m_{loc} at the boundary points of these cubes. The requirement that $s(z)$ is a section of the bundle $\mathscr{E}_m$ rather than just of the bundle $V^*_\varepsilon \times H^m(K)$ does exactly this (see the discussion at the top of p. 96 in [30]).

It remains to show that $\varphi \in W_{m,\varepsilon}$. We use the standard formulas for the Fourier coefficients to get

$$\varphi(\,\cdot\,-\gamma)=s_\gamma=\frac{1}{(2\pi)^n}\int_B s(e^{i(\beta+i\alpha)})\, e^{-i(\beta+i\alpha)\cdot\gamma}\, d\beta, \qquad \forall\alpha\in G_\varepsilon,$$

where B is the first Brillouin zone, and we write $z=\exp ik=\exp(i(\beta+i\alpha))$. Note that

$$\|\phi\|_{H^m(K+\gamma)}\leqslant \max_{z\in V^*_{\delta_1}} \|s(z)\|_{H^m(K)}\, e^{\alpha\cdot(-\gamma)} \qquad \forall\alpha\in G_\varepsilon, \tag{5.1}$$

and therefore

$$\|\phi\|_{H^m(K+\gamma)}\leqslant \max_{z\in V^*_{\delta_1}} \|s(z)\|_{H^m(K)}\, e^{-(h(\gamma/|\gamma|)+\delta_1)}. \tag{5.2}$$

This implies immediately that

$$\begin{aligned}\langle\varphi\rangle_{m,\delta} &= \sup_{\gamma\in\Gamma}\,\{\|\varphi\|_{H^m(K+\gamma)}\, e^{(h(\gamma/|\gamma|)+\delta)|\gamma|}\}\\ &\leqslant C\max_{z\in V^*_{\delta_1}}\|s(z)\|_{H^m(K)}\,\sup_{\gamma\in\Gamma} e^{-(\delta_1-\delta)\,|\gamma|}<\infty,\end{aligned} \tag{5.3}$$

if $\delta_1>\delta$. ∎

Proof of Lemma 21. The statement of this lemma is established in a much more general situation at the beginning of the proof of Theorem 1.7.1 in [30]. However, for the sake of completeness we provide here the proof for our simpler particular situation. First of all, the sequence of the lemma is a complex (i.e., the composition of any two consecutive operators in it is equal to zero). One needs to prove this only in the second term of the sequence, where it follows immediately from the equality (3.5). Indeed, since u_z solves the equation $Lu=0$, (3.5) followed by integration by parts proves the statement.

Let us turn to the exactness. We need to prove it in the second and third terms of the sequence. Consider the second term. Let $f(z,x)\in\Gamma(V^*_\varepsilon,\mathscr{E}_0)$ be such that $t_f(z)=0$. This means that for any $z\in\Phi_\varepsilon$ the function $f(z^{-1},\cdot)$ is orthogonal to the Bloch solution u_z of the equation $Lu=0$. We need to show that $g(z)=\mathscr{L}(z^{-1})^{-1} f(z)$ is analytic, which will mean that f belongs to the range of $\mathscr{L}$. The function $g(z^{-1})$ is automatically analytic outside of Φ_ε, so we only need to make sure that it does not develop any singularities at this subset. We will show that all the necessary and sufficient conditions for the analyticity of g have the form of orthogonality of values of f at certain points to certain functionals. This would resolve the issue, since all such possible orthogonality conditions are the orthogonality of $f(z^{-1})$ to the kernel of L on Bloch functions with a quasimomentum z, and hence to the vanishing of $t_f(z)$. As it was shown in the proof of Theorem 3.3.1 in [30, pp. 113–114], the inverse operator to $\mathscr{L}(z^{-1})$ is the ratio of two analytic functions,

$$\mathscr{L}(z^{-1})^{-1} = B(z)/\Delta(z),$$

where $B(z)$ is an analytic function with values in bounded operators from $L^2(\mathbb{T}^n)$ to $H^2(\mathbb{T}^n)$, and $\Delta(z)$ is a scalar analytic function, which is a regularized determinant of $\mathscr{L}(z^{-1})\,\mathscr{L}(z_0^{-1})^{-1}$ for some point z_0 where the operator is invertible. Such regularized determinants are determined in the standard way by the eigenvalues of the corresponding operators (see for instance Section 2 of Chapter IV in [20] for general definitions and properties of regularized determinants, and for our particular situation the proof of Theorem 3.1.7 and related discussion in Section 1.2 in [30]). The simplicity of the eigenvalue $\Lambda(\xi)$ (Lemma 12) implies that if we introduce instead of z the coordinate ξ such that $z = \exp \xi$, then $\Delta(\exp \xi) = \Lambda(\xi)\, \Delta_1(\xi)$, where $\Delta_1(\xi)$ is an analytic function with no zeros in the domain under our consideration. We recall now that Λ has simple zeros. Hence, the necessary and sufficient condition for f to belong to the range of the operator $\mathscr{L}$ on the space of analytic sections is that the vector-function $B(z)\, f(z)$ vanishes on the set of the zeros of Λ. These conditions obviously have the form of the orthogonality of values of f to some functionals. As it was explained above, this implies exactness at the second term of the sequence.

Let us turn now to proving the exactness at the third term. We need to show that arbitrary analytic function on Φ_ε can be obtained as $t_f(z)$ for some $f \in \Gamma(V_\varepsilon^*, \mathscr{E}_t)$.

Let us denote by Φ_ε^* the manifold

$$\Phi_\varepsilon^* = \{z \mid z^{-1} \in \Phi_\varepsilon\}.$$

Consider the restriction mapping

$$\Gamma(V_\varepsilon^*, \mathscr{E}_0) \to \Gamma(\Phi_\varepsilon^*, \mathscr{E}_0). \tag{5.4}$$

Notice that Φ_ε^* is an analytic subset in V_ε^* and that V_ε and V_ε^* are domains of holomorphy. The latter can be easily proven using power test functions z^a with integer (but not necessarily nonnegative) powers a (a similar derivation can be found in the proof of the implication (iii) $\to$ (i) of Corollary 2.5.8 in [24]). Then Corollary 1 of the Bishop's theorem [43, Theorem 3.3] (see the original theorem in [13]) claims that the restriction mapping (5.4) is surjective (recall that the bundle $\mathscr{E}_0$ is trivial). Hence, it is sufficient to prove that the mapping

$$\tilde{t}\colon \Gamma(\Phi_\varepsilon, \mathscr{E}_0) \to \mathscr{H}(\Phi_\varepsilon).$$

defined as

$$\tilde{t}_f(z) = \langle f(z, \cdot), u_z \rangle = \int_{\mathbb{T}^n} f(z, x)\, u_z(x)\, dx$$

is surjective. Consider the continuous operator $T(z)\colon L^2(K) \to \mathbb{C}$ defined as $T(z)\, y = \langle y, u_z \rangle = \int_{\mathbb{T}^n} y(x)\, u_z(x)\, dx$. Since u_z is not zero, this operator is surjective. It is clear that it depends analytically on z. According to Allan's theorem (see [5] or Theorem 4.4 in [43]), since Φ_ε is a Stein manifold, there exists an analytic right inverse operator $R(z)$. Now, given $\phi(z) \in \mathscr{H}(\Phi_\varepsilon)$, the function $g(z) = R(z)\, \phi(z)$ satisfies $\tilde{t}_g = \phi$. This proves the surjectivity that we need.

The last statement of the lemma about the mapping t being a topological homomorphism is just the open mapping theorem. ∎

Proof of Lemma 24. 1. We first show that the operator $\mathscr{U}$ maps continuously the space C_m into $C^\infty(T, H^m(K))$. Indeed, if $\varphi \in C_m$, then $\|\varphi\|_{H^m(K+\gamma)}$ decays faster than any power of $|\gamma|$. This together with (2.3) leads to the immediate conclusion that $\mathscr{U}\varphi$ belongs to $C^\infty(T, H^m(K))$ and to the continuity of the corresponding mapping. Since $\mathscr{U}\varphi$ is a section of the sub-bundle $\mathscr{E}^m$ (see the Section 2.2 in [30]), this gives us the needed conclusion. Conversely, let

$$s(z) \in C^\infty(T, \mathscr{E}^m) \subset C^\infty(T, H^m(K)).$$

One can expand the $H^m(K)$-valued function $s(z)$ into the Fourier series:

$$s(z) = \sum_{\gamma \in \Gamma} s_\gamma z^\gamma, \qquad z \in T.$$

Here $s_\gamma \in H^m(K)$. Standard estimates of the Fourier coefficients of smooth functions apply, which show that $\|s_\gamma\|$ decays faster than any power of $|\gamma|$. Let us define now a function ϕ on $\mathbb{R}^n$ such that $\phi(x-\gamma) = s_\gamma(x)$ for $x \in K$ and $\gamma \in \Gamma$. The additional information that s is a section of the sub-bundle $\mathscr{E}^m$ leads (as in [30, p. 96]) to the conclusion that $\phi \in H^m_{loc}(\mathbb{R}^n)$. This implies that $\phi \in C_m$ and finishes the proof of the first statement of the lemma.

Statements 2 and 3 are correspondingly parts of Theorem 2.2.3 and 3.1.5 of [30]. ▮

Proof of Lemma 25. The first statement is rather obvious. Indeed, the statement is local, and locally one can construct a smooth one-sided inverse. The second statement can be proven like the similar statement in [30, Corollary 1.7.2]. For completeness, we provide the scheme of the proof here. Under the conditions of the second statement of the lemma, it is easy to see that any functional annihilating the range of the operator of multiplication by $\mathscr{P}(z)$ must be supported at the finite set $\mathscr{Z}$ where $\mathscr{P}(z)$ is not surjective. This also reduces the considerations to a neighborhood U of a point $z_0 \in \mathscr{Z}$. Using the Fredholm property, one can find a closed subspace M of finite codimension in B_1 such that the operators $\mathscr{P}(z)$ have zero kernel on M for all $z \in U$ (see the corresponding lemma in [7], or Lemma 1.2.11 and Remark 2 below it in [30]). Now the problem reduces to a similar one on a finite-dimensional space, where a standard representation of distributions supported at a point implies (4.2). ▮

6. FURTHER REMARKS

Remarks 6.1. 1. Throughout the paper, we have assumed for simplicity that all the coefficients of the operators P and P^* are C^∞-smooth. In fact, we do not need such a restrictive assumption (see the discussion in [30, Section 3.4.D]). For example, a sufficient (but not necessary) condition for all the statements of Section 3 to hold true is that the coefficients of L and L^* are Hölder continuous. Actually, even less is needed. For instance, conditions imposed on the Schrödinger operators in Theorem 28 are sufficient. It is clear that the conditions on the coefficients could be significantly relaxed, if the operators were considered in the weak sense, or by means of their quadratic forms. This should not change the general techniques of the proofs. We did not intend, however, to find the optimal requirements on the coefficients for all our results to hold.

2. It should be possible to describe the class of solutions of the equation $Lu = 0$ that are representable by a distribution rather than by a hyperfunction. We plan to address this problem elsewhere.

3. The Liouville theorem can probably be extended to systems of equations (for instance, to the Maxwell system). In this case one would face the problems of a possibly nonzero index of the corresponding operator and of multiple eigenvalues (the latter can also occur for scalar operators). We believe that the technique of this paper might be adjusted to handle some of these situations. The extensions of the result of [26] to the Pauli and Maxwell

operators obtained in [11] and [12] would provide examples where the needed information on the behavior of the dispersion relations at the bottom of the spectrum is available.

A. APPENDIX

In this appendix we present an alternative proof of the third statement of Theorem 28 in the case when either $\Lambda_0=0$ and $N\geqslant 0$, or $\Lambda_0>0$ and $0\leqslant N\leqslant 1$. The proof relies on some basic notions of homogenization theory [25] and imitates the proof of Theorem 2 in [38], where L is assumed to be an operator in divergence form. Therefore, we skip some details which are essentially the same as in [38].

We need to recall some basic definitions from homogenization theory (see, for example, [10, 25]). Suppose that L is a second order elliptic operator of the form

$$L=-\sum_{i,\,j=1}^{n} a_{ij}(x)\,\partial_i\,\partial_j+\sum_{i=1}^{n} b_i(x)\,\partial_i, \tag{A.1}$$

with periodic coefficients and denote the positive matrix $\{a_{ij}(x)\}$ by $\mathscr{A}(x)$ and the periodic vector $(b_1, ..., b_n)^T$ by b. Let ψ be the positive normalized periodic solution of the equation $L^*u=0$. Let $\Psi(x)=(\Psi_1(x), ..., \Psi_n(x))^T$ be a solution of the equation

$$L\Psi=-b(x)+\int_{\mathbb{T}^n} b(x)\,\psi(x)\,dx \qquad \text{in } \mathbb{T}^n. \tag{A.2}$$

Consider the matrix

$$\mathscr{Q}=\{q_{ij}\}:=\int_{\mathbb{T}^n}(I+\nabla\Psi)^T\,\mathscr{A}(x)(I+\nabla\Psi)\,\psi(x)\,dx, \tag{A.3}$$

were I is the identity matrix. The operator $Q:=-\sum_{i,\,j=1}^{n} q_{ij}\,\partial_i\,\partial_j$ is called the *homogenized operator* of the operator L, and the positive matrix $\mathscr{Q}=\{q_{ij}\}$ is called the *homogenized matrix* (see, [25, Section 2.5]).

The following lemma, which is actually a new formulation of [41, Theorem 5]), establishes a connection between the function Λ and homogenization theory.

LEMMA A.1. *Let L be an operator of the form* (2.5) *and suppose that $\xi\in\Xi$. Let u_ξ and $u^*_{-\xi}$ be the positive Bloch solutions of the equations $Lu=0$*

and $L^*u=0$, *respectively. Denote by* ψ *the periodic function* $u_\xi u^*_{-\xi}$. *Consider the operator*

$$\tilde{L}=(u_\xi(x))^{-1}\, Lu_\xi(x)=-\sum_{i,j=1}^{n} a_{ij}(x)\,\partial_i\,\partial_j+\sum_{i=1}^{n}\tilde{b}_i(x)\,\partial_i, \tag{A.4}$$

let

$$Q=-\sum_{i,j=1}^{n} q_{ij}\,\partial_i\,\partial_j \tag{A.5}$$

be the homogenized operator of the operator $\tilde{L}$, *and* $\mathcal{Q}=\{q_{ij}\}$ *be the homogenized matrix. Then* ψ *is the principal eigenfunction of the operator* $\tilde{L}^*$ *on the torus* $\mathbb{T}^n$ *with an eigenvalue* 0. *Moreover,* $\mathrm{Hess}(\Lambda(\xi))=-\mathcal{Q}$.

Proof. The first statement of the lemma can be checked easily while the second statement follows directly from the formula in [41, Theorem 5], and the definition of the homogenized operator. ∎

Proof of a part of the third statement of Theorem 28. We clearly may assume that $L\mathbf{1}=0$, so,

$$L=-\sum_{i,j=1}^{n} a_{ij}(x)\,\partial_i\,\partial_j+\sum_{i=1}^{n} b_i(x)\,\partial_i.$$

We denote by ψ the normalized positive solution of the equation $L^*u=0$ in $\mathbb{T}^n$. Let Ψ be a solution of the system (A.2), and Q be the homogenized operator of the operator L.

Assume first that $\Lambda_0\geqslant 0$. The case $N=0$ is trivial, and follows from Theorem 23 and Lemma 17. Let $N=1$. Recall that according to Theorem 23, $d_1\leqslant n+1$. Moreover, by Theorem 23 and the Leibnitz's rule, a (real) solution of linear growth is of the form

$$u(x)=\sum_{j=1}^{n} a_j x_j+\phi(x),$$

where $a_j\in\mathbb{R}$ and ϕ is periodic.

By Lemma 13, $\Lambda_0=0$ if and only if for every $1\leqslant k\leqslant n$

$$\alpha_j:=\int_{\mathbb{T}^n} b_j(x)\,\psi(x)\,dx=0. \tag{A.6}$$

For $1\leqslant j\leqslant n$, we write an "Ansatz" for a solution of linear growth of the form

$$F_j(x)=x_j+\phi_j(x), \tag{A.7}$$

where ϕ_j is a periodic function. Clearly, F_j is a solution of $Lu=0$ in $\mathbb{R}^n$ if and only if $\phi_j(x)$ solves the nonhomogeneous equation $Lu=-b_j$ in $\mathbb{T}^n$. By the Fredholm alternative, this equation is solvable in $\mathbb{T}^n$ if and only if $\alpha_j=0$ which holds true for *all* $1\leqslant j\leqslant n$, if and only if $\Lambda_0=0$ (and in this case, $\phi_j=\Psi_j$, see (A.2)). Therefore, $d_1=n+1$ if $\Lambda_0=0$, and $d_1<n+1$ if $\Lambda_0>0$.

In order to finish the proof for $N=1$, we need to prove that if $\Lambda_0>0$, then $d_1\geqslant n$. Without loss of generality, we may assume that $\alpha_n\neq 0$. We construct $(n-1)$ linearly independent solutions of linear growth of the form

$$F_j(x)=x_j-\alpha_j(\alpha_n)^{-1}\,x_n+\phi_j(x),$$

where $1\leqslant j\leqslant n-1$, and ϕ_j solves the equation $Lu=-b_j+\alpha_j(\alpha_n)^{-1}\,b_n$. Note that these $(n-1)$ equations are solvable and therefore, $d_1\geqslant n$.

For $N\geqslant 2$, we assume that $\Lambda_0=0$. Recall that if $u\in V_N$ then by Theorem 23 and the Leibnitz's rule

$$u(x)=u^{(N)}(x)+\sum_{|\nu|<N} x^\nu p_\nu(x),$$

where

$$u^{(N)}(x)=\sum_{|\nu|=N} x^\nu p_\nu,$$

and p_ν are periodic functions if $|\nu|<N$, and $p_\nu\in\mathbb{R}$, if $|\nu|=N$.

Claim. *Assume that $\Lambda_0=0$. Then for all $N\geqslant 0$*

$$Qu^{(N)}=0. \tag{A.8}$$

In particular, $d_N\leqslant h_{n,N}$.

Proof of the claim. Assume first that $N=2$. Then $u\in V_2$ is of the form

$$u(x)=\tfrac{1}{2}\,(Cx\cdot x)+\sum_{j=1}^{n} x_jp_j(x)+p_0(x),$$

where C is a constant symmetric matrix, and $p_0, p_1, ..., p_n$ are periodic functions.

A direct calculation shows that the vector $p=(p_1, ..., p_n)^T$ must satisfy the equation $Lp=-Cb$, which is solvable since $\Lambda_0=0$. Therefore, $p=C\Psi$ (up to a constant vector). Also, p_0 must satisfy

$$Lp_0=f:=tr(A(I+2\nabla\Psi^T)\,C^T)-b\cdot C\Psi.$$

The compatibility condition for this equation is $\int_{\mathbb{T}^n} f(x)\,\psi(x)\,dx=0$, which after some calculations implies that

$$tr(\mathcal{Q}C^T)=0,$$

where $\mathcal{Q}$ is the homogenized matrix of the operator L (see (A.3)). Since $u^{(2)}:=\frac{1}{2}(Cx\cdot x)$ is a homogeneous polynomial of degree 2, it follows that $Qu^{(2)}=tr(\mathcal{Q}C^T)$. Therefore, $u^{(2)}$ solves the equation $Qu=0$. Thus, the case $N=2$ is settled.

For $N>2$, we proceed by induction as in [38]. Namely, assume that the claim (A.8) has been proven for $N-1$, and let $u\in V_N$. Let Δ_i be the difference operator $\Delta_i f(x):=f(x+e_i)-f(x)$, where e_i is the i-th vector of the standard basis of $\mathbb{R}^n$, and $1\leqslant i\leqslant n$. Then $v_i:=\Delta_i u\in V_{N-1}$ and the leading part of v_i is given by $(\Delta_i u)^{(N-1)}=\partial_i u^{(N)}$. By the induction hypothesis, $Q((\Delta_i u)^{(N-1)})=0$. Therefore,

$$\partial_i(Qu^{(N)})=Q(\partial_i u^{(N)})=Q((\Delta_i u)^{(N-1)})=0 \qquad 1\leqslant i\leqslant n.$$

Hence, $Qu^{(N)}=const$, and since $Qu^{(N)}$ is homogeneous of degree $N-2>0$, we obtain that $Qu^{(N)}=0$, and the claim is proved.

It remains to prove that $d_N\geqslant h_{n,N}$. So, for any homogeneous polynomial h of degree N that is Q-harmonic, we need to find a solution $u\in V_N$ such that $u^{(N)}=h$. Let $u\in V_N$ and $\varepsilon>0$. Consider the function

$$\varepsilon^N u\left(\frac{x}{\varepsilon}\right)=\sum_{|\nu|\leqslant N}\varepsilon^{N-|\nu|}x^\nu p_\nu\left(\frac{x}{\varepsilon}\right),$$

which tends to $u^{(N)}$ as $\varepsilon\to 0$. We consider x and $y=\frac{x}{\varepsilon}$ as independent variables and write

$$U(x,y,\varepsilon):=\sum_{|\nu|\leqslant N}\varepsilon^{N-|\nu|}x^\nu p_\nu(y)=U_0(x)+\varepsilon U_1(x,y)+\cdots+\varepsilon^N U_N(x,y).$$

Then the equation $L(x,\partial_x)\,u=0$ implies that

$$(L_0+\varepsilon L_1+\varepsilon^2 L_2)\,U=0,$$

where

$$L_0=L(y,\partial_y);$$

$$L_1=-2\sum_{i,j=1}^{n}a_{ij}(y)\,\partial^2_{x_i,y_j}+\sum_{i=1}^{n}b_i(y)\,\partial_{x_i};\qquad L_2=-\sum_{i,j=1}^{n}a_{ij}(y)\,\partial^2_{x_i,x_j}.$$

We look for a formal differential operator

$$\Phi = \sum_{j=0}^{\infty} \varepsilon^k \Phi_j = \sum_{\nu} \varepsilon^{|\nu|} \phi_\nu(y)\, \partial_x^\nu,$$

where $\phi_\nu(y)$ are periodic functions and $\phi_0 = 1$. This operator should satisfy

$$(L_0 + \varepsilon L_1 + \varepsilon^2 L_2)\, \Phi = M + L_0(y, \partial_y) - 2\varepsilon \sum_{i,\, j=1}^{n} a_{ij}(y)\, \partial_{x_i} \partial_{y_j}, \tag{A.9}$$

where the formal operator

$$M = \sum_{j=2}^{\infty} \varepsilon^j M_j = \sum_{|\nu| \geqslant 2} \varepsilon^{|\nu|} m_\nu\, \partial_x^\nu$$

has constant coefficients.

Comparing the coefficients of ε^s in (A.9) yields the following equations (the equation for $s = 0$ is automatically satisfied).

$$L_0 \Phi_1 + L_1 = -2 \sum_{i,\, j=1}^{n} a_{ij}(y)\, \partial_{x_i} \partial_{y_j}, \qquad s = 1, \tag{A.10}$$

$$L_0 \Phi_s + L_1 \Phi_{s-1} + L_2 \Phi_{s-2} = M_s, \qquad s \geqslant 2. \tag{A.11}$$

It is easily checked that for $s = 1$ the functions $\phi_j(y)$ of Eq. (A.7) are the corresponding solutions for Φ_1. Also, Eq. (A.11) for $s = 2$ is solvable if $M_2 = Q$, where Q is the homogenized operator of L. Similarly, the constant coefficients of the operator M_s, $s > 2$, are determined by the compatibility condition for Eq. (A.11) with $s > 2$.

Let $R\colon \mathscr{P} \to \mathscr{P}$ be a linear right inverse of the homogenized operator Q that preserves the homogeneity of polynomials. Consider the formal operator A which is defined by the equation

$$A - I = R \sum_{j=1}^{\infty} \varepsilon^j M_{j+2},$$

and let A^{-1} be its unique formal inverse. Note that $\varepsilon^2 M_2 A = M$.

Let $U_0(x)$ be a given homogeneous polynomial of degree N which solves the equation $Qu = 0$, and let $V(x) := A^{-1} U_0(x)$. We have

$$MV = \varepsilon^2 M_2 A V = \varepsilon^2 M_2 U_0 = 0.$$

Define $U(x, y, \varepsilon) := \Phi A^{-1} U_0 = \Phi V$, and denote $u(x) := U(x, x, 1)$. By inspection, u has a polynomial growth of order N, and $u^{(N)}(x) = U_0(x)$. Moreover,

$$(L_0 + \varepsilon L_1 + \varepsilon^2 L_2)\, U = (L_0 + \varepsilon L_1 + \varepsilon^2 L_2)\, \Phi V$$
$$= MV + L_0(y, \partial_y)\, V(x) - 2\varepsilon \sum_{i,\, j=1}^{n} a_{ij}(y)\, \partial_{x_i} \partial_{y_j} V(x) = 0,$$

and ΦA^{-1} is the desired mapping. ∎

Remark A.2. 1. Let F_j be the solutions of linear growth defined by Eq. (A.7). A. Ancona [6] proved that the map $F(x) = (F_1(x), ..., F_n(x))$ is a diffeomorphism on $\mathbb{R}^n$ if $n \leqslant 2$, while for $n > 2$ this map is not necessarily a diffeomorphism.

2. Assume that $L\mathbf{1} = 0$ and $\Lambda_0 = 0$. Let $\Lambda(\xi) = \sum_{|\nu| \geqslant 2} a_\nu \xi^\nu$ be the Taylor expansion of the function Λ. We conjecture that $a_\nu = m_\nu$, where m_ν are the coefficients of the operator M.

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