JOURNAL OF COMBINATORIAL THEORY 9, 129–135 (1970)

Characterizations of Derived Graphs

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Communicated by F. Harary

Received December 1968

ABSTRACT

The derived graph of a graph G has the edges of G as its vertices, with adjacency determined by the adjacency of the edges in G. A new characterization of derived graphs is given in terms of nine excluded subgraphs. A proof of the equivalence of all known characterizations is also given.

The *derived graph* ∂G of a graph G is defined as that graph having the edges of G as its vertices, with two vertices being adjacent if and only if the corresponding edges are adjacent in G. This concept has been rediscovered in various contexts and thus has many other names: interchange graph [8], line graph [7], adjoint [5], and edge-to-vertex dual [10] are a few. The purpose of this note is to present criteria for a graph to be a derived graph. One of the three characterizations given is new, and a unified complete proof of all is presented.

Each of the graphs in Figure 1 has the first as its derived graph, and by a theorem of Whitney [12] these are the only two connected graphs having the same derived graph. They are examples of the following classes of graphs: the *complete graph* K_p has p vertices with every pair of vertices adjacent, the *bicomplete* (or *complete bipartite*) *graph* $K_{m,n}$ has $m + n$ vertices with each of m of the vertices adjacent to precisely the other n. Thus, the first graph in Figure 1 is denoted K_3, the second is $K_{1,3}$; the latter has an especially important role in the study of derived graphs.

FIGURE 1

We mention in passing that the derived graphs of these two classes of graphs have been characterized by Conner [1], Hoffman [2, 3], Moon [6], and Shrikhande [11].

Other examples of derived graphs are given in Figure 2, where F_1 is derived from the unique graph obtained by adding a new edge to $K_{1,3}$, F_2 is ∂F_1, and F_3 is ∂K_4. All of these also have a part in the proof of the theorem characterizing derived graphs.

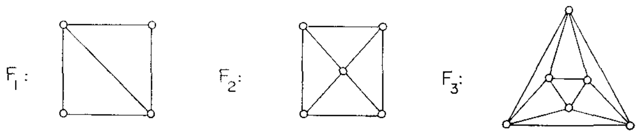

FIGURE 2

Some definitions needed for the theorem are the following. A *clique* of a graph is a maximal complete subgraph. A triangle in a graph G is called *odd* if some vertex of G is adjacent to an odd number of the vertices of the triangle, and *even* otherwise. A subgraph H of graph G is called *induced* (by its vertices) if it is the maximal subgraph on its vertices, that is, if two vertices of H are adjacent in G they are also adjacent in H. We often write $u \sim v$ if vertices u and v are adjacent, and $u \not\sim v$ if they are not.

In the following theorem characterizing derived graphs, statement (2) is due to Krausz [4] and (3) to van Rooij and Wilf [9]. The last criterion, although new, has been independently discovered by N. Robertson (unpublished).

THEOREM. *The following statements are equivalent for a graph G.*

(1) *G is the derived graph of some graph.*

(2) *The edges of G can be partitioned into complete subgraphs in such a way that no vertex belongs to more than two of the subgraphs.*

(3) *The graph $K_{1,3}$ is not an induced subgraph of G; and if abc and bcd are distinct odd triangles, then a and d are adjacent.*

(4) *None of the nine graphs in Figure* 3 *is an induced subgraph of G.*

PROOF: It is assumed throughout that G is connected.

(1) *implies* (2). Assume that G is the derived graph of H. The edges at each vertex of H determine a complete subgraph of G, and every edge of G lies in exactly one of these. Since each edge of H has two vertices, the corresponding vertex of G is in at most two of these complete subgraphs.

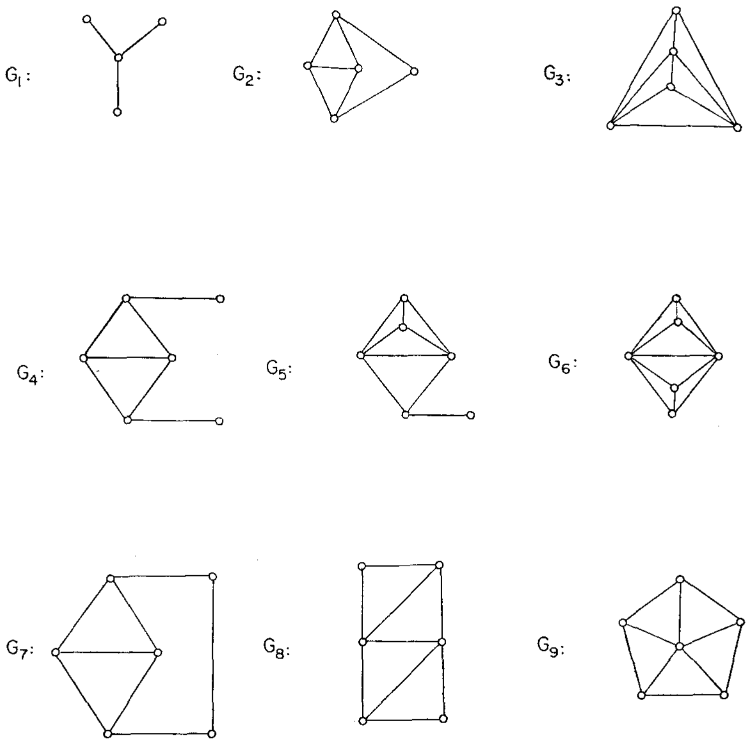

FIGURE 3

(2) *implies* (4). It is easily seen that, when any of the nine graphs of Figure 3 has its edges partitioned into complete subgraphs, some vertex is in at least three of the subgraphs. Therefore none of these can be a derived graph. Because every induced subgraph of a derived graph must itself be derived, the result follows.

(4) *implies* (3). Suppose that G does not satisfy (3) and yet does not have $G_1 = K_{1,3}$ as an induced subgraph. We will show that G must have one of the other eight graphs of Figure 3 as an induced subgraph. It follows from (3) that G has two odd triangles abc and bcd with $a \not\sim d$. There are two cases to consider, depending on whether or not some vertex is adjacent to an odd number of vertices of both triangles.

If there is such a vertex v, there are two possibilities: v is adjacent either to exactly one vertex of each triangle or to more than one vertex of one of the triangles. In the latter case, it must be adjacent to all four of the vertices, giving G_3 as an induced subgraph. In the former, v is either adjacent only to b or to c, so that G_1 would be induced, or it is adjacent to both a and d, which means that G_2 is induced.

Now assume there is no vertex adjacent to an odd number of vertices of both triangles. Let u be adjacent to an odd number in abc and v to an odd number in bcd. Two facts are now noted:

(α) If u or v is adjacent to b or to c, then it is also adjacent to a or to d since otherwise G_1 is an induced subgraph.

(β) Neither u nor v can be adjacent to both a and d, since it would then be adjacent to an odd number of vertices of both triangles.

There are now three possibilities to consider:

(i) Each of u and v is adjacent to only one vertex of the corresponding triangle.

(ii) Each is adjacent to all three vertices of the corresponding triangle.

(iii) One is adjacent to all three vertices of a triangle, the other to only one of the other triangle.

The first of these is the most complicated, and all possible subcases are considered:

$u \sim a$ and $v \sim d$: This gives G_4 or G_7 as an induced subgraph (because of (β)) depending on whether or not $u \sim v$.

$u \sim c$ and $v \sim d$: From (α) and (β) it follows that $u \sim d$ and $v \not\sim a$. If $u \not\sim v$, then the induced subgraph $\langle b, d, u, v\rangle$ is G_1, while, if $u \sim v$, graph G_8 is obtained.

$u \sim c$ and $v \sim b$: Necessarily $u \sim d$ and $v \sim a$, so that, if $u \not\sim v$, a graph isomorphic to G_8 is obtained, while, if $u \sim v$, graph G_3 is induced.

$u \sim c$ and $v \sim c$: Again $u \sim d$ and $v \sim a$, so that, if $u \sim v$, then G_9 is obtained, and, if $u \not\sim v$, then G_1 is an induced subgraph.

Except for interchanging roles of vertices, this exhausts the possibilities of (i).

In (ii), $u \not\sim d$ and $v \not\sim a$. If $u \sim v$, then a subgraph isomorphic to G_3 is induced, while, if $u \not\sim v$, then G_6 occurs.

For (iii), assume that u is adjacent to a, b, and c, and thus not to d. There are two possibilities, depending on which vertex of triangle bcd is adjacent to v. If $v \sim d$, then G_2 or G_5 is obtained according as u is or is not adjacent to v. If $v \sim c$ or b, then either G_3 or G_1 is induced, depending on whether or not v is adjacent to both a and u.

(3) *implies* (2). Assume that G satisfies (3). We first show that if, in addition, G has two even triangles with a common edge, then it must be isomorphic to one of the three graphs of Figure 2. Let abc and bcd be

even triangles, in which case $a \not\sim d$. If G is not graph F_1, it must have a fifth vertex u adjacent to one of the others. Since both triangles are even, we can assume that u is adjacent either to just b and c or to the three vertices a, b, and d. The former case cannot occur since then $K_{1,3}$ would be induced as $\langle b, a, d, u\rangle$; the latter gives graph F_2. If G has a sixth vertex v, then the same argument implies that v is adjacent to a, d, and either b or c. If $v \sim b$, then $u \not\sim v$ implies that the induced subgraph $\langle b, c, u, v\rangle$ is $K_{1,3}$, while $u \sim v$ implies that abu and bud are odd triangles with $a \not\sim d$; both violate the hypotheses. Therefore $v \sim c$. Also, $v \sim u$ since otherwise abu and bud are again odd triangles. Hence graph F_3 is obtained. There cannot be a seventh vertex in G since it would have to be adjacent to precisely the same vertices as v and b, and then $K_{1,3}$ would again be induced.

It is readily verified that each of these three graphs can have its edges partitioned to satisfy (2).

Now assume that G has no pair of even triangles with a common edge. Let S be the family of cliques which are not even triangles and let T be the family of edges which lie on a unique and even triangle.

We now show that these subgraphs in $S \cup T$ determine a partition of the edges of G. Clearly every edge is in at least one member. If an edge abc is in two, then both must be cliques which are not even triangles. There are vertices a and d each in exactly one of the two cliques, and hence not adjacent. But abc and bcd are odd triangles, each being in a clique that has at least four vertices or being itself a clique that is an odd triangle. Hence, the edges are partitioned into complete subgraphs.

By considering three cases, we next show that any vertex lies in at most two of the members of $S \cup T$.

First, let v be a vertex which lies on exactly one member of T, say edge vw of the even triangle uvw. Then edge uv must be on an odd triangle uva. Any point adjacent to v must also be adjacent to u since uvw is even. Furthermore, any two such points b and c must be adjacent since both triangles uvb and uvc have an edge in common with uvw and are thus odd. Hence v lies in precisely one member of S.

Next, assume v lies on two members of T. If these are edges uv and $u'v$ of different even triangles uvw and $u'vw'$, then u must be adjacent to u' or w'. But this means uv is on two even triangles, which cannot occur. Hence both members of T containing v must lie on the same even triangle uvw. In this case v cannot be on any other line since that would mean that uvw is odd or that uv or vw lies on two triangles. Hence v lies on only the two members of T, none of S.

Now suppose that v lies on three members of S, say cliques A, B, and C. Let a, b, and c be other vertices in these respective cliques. Because no edge is in more than one of these cliques, none of these vertices lies in any of the other cliques. Also two, say a and b, are adjacent, since otherwise $K_{1,3}$ would be induced. This implies that triangle abv is even, since otherwise it would be in a member of S containing both a and b. Therefore c must be adjacent to a or to b, say the latter. But the same argument as above implies that cbv is an even triangle, which contradicts the assumption that no two even triangles have a common edge. Therefore, no vertex lies in more than two members of S.

(2) *implies* (1). Let U be the family of complete subgraphs given in (2) together with the graphs consisting of the single vertices which appear in only one of the complete subgraphs. Then each vertex of G is in exactly two members of U. Define the graph H to have U as its set of vertices with two vertices being adjacent whenever the corresponding subgraphs have a common vertex. We now show that G is the derived graph of H. There is certainly a one-to-one correspondence f from the edges of H to the vertices of G: for each edge x in H, let $f(x)$ be the vertex of G which is in the two subgraphs of U which x joins in H.

What remains to be shown is that adjacency is preserved between ∂H and G. Let x and y be distinct edges in H. Assume x joins A and B and y joins B and C, that is, $x \sim y$ in ∂H. Clearly, $f(x) \sim f(y)$ in G since B is a complete subgraph. On the other hand, assume x joins A and B and y joins C and D, that is, $x \not\sim y$ in ∂H. Then $f(x)$ is in only A and B and $f(y)$ in only C and D, so that $f(x) \not\sim f(y)$ in G. This completes the proof.

The theorem thus gives several answers to the characterization question posed by Seshu and Reed [10] and Ore [8]. In closing we mention some results giving answers to other questions they raise. These solutions have been found by Menon [5, 6], van Rooij and Wilf [9], and others. Iterated derived graphs are defined inductively as expected:

$$\partial^1 G = \partial G \qquad \text{and} \qquad \partial^{n+1} G = \partial(\partial^n G).$$

The only connected graphs which are isomorphic to their derived graphs are the cycles. Thus, if G is a cycle, then $G = \partial^n G$ for all n, while, if $G = K_{1,3}$, then $\partial G = \partial^n G$ for all n, but $G \neq \partial G$. If G is the graph of the path on n vertices, then $\partial^{n-1} G$ is a single vertex and $\partial^n G$ does not exist. For any other connected graph G, the number of vertices in $\partial^n G$ becomes arbitrarily large as n becomes large. Therefore, these results classify graphs by their iterated derived graphs.

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