Let $K$ be an algebraically closed field of positive characteristic and let $G$ be a reductive group over $K$ with Lie algebra $\mathfrak{g}$. This paper will show that under certain mild assumptions on $G$, the commuting variety $\mathcal{C}(\mathfrak{g})$ is an irreducible algebraic variety.

This paper will extend the result of [12], under certain mild restrictions on $\mathfrak{g}$, to the case where $K$ is an algebraically closed field of characteristic $p$.

The proof, following [12], is by induction on the dimension of $G$. The induction step is straightforward in all cases except that of a nilpotent element $x$ having no non-central semisimple elements in its centralizer $\mathfrak{z}(x)$. 

1. INTRODUCTION

Let $\mathfrak{g}$ be a Lie algebra over an algebraically closed field $K$. The commuting variety $\mathcal{C}(\mathfrak{g})$ is defined as

$$\{ (x, y) \in \mathfrak{g} \times \mathfrak{g} \mid [x, y] = 0 \}.$$ 

$\mathcal{C}(\mathfrak{g})$ is clearly a Zariski closed subset of $\mathfrak{g} \times \mathfrak{g}$.

In [12], it was proved that if $K$ is a field of characteristic zero and $\mathfrak{g}$ is a reductive Lie algebra over $K$, then $\mathcal{C}(\mathfrak{g})$ is an irreducible variety. In fact it was shown that if $(x, y) \in \mathcal{C}(\mathfrak{g})$ and $N$ is a neighborhood of $(x, y)$ in $\mathfrak{g} \times \mathfrak{g}$, then there exists a maximal torus $t$ of $\mathfrak{g}$ such that $N$ meets $t \times t$. (Earlier, in [9] and independently in [4], it was proved that $\mathcal{C}(\mathfrak{gl}_n)$ is irreducible, where $\mathfrak{gl}_n$ is the Lie algebra of all $n \times n$ matrices over an algebraically closed field of arbitrary characteristic.)

This paper will extend the result of [12], under certain mild restrictions on $\mathfrak{g}$, to the case where $K$ is an algebraically closed field of characteristic $p$.

The proof, following [12], is by induction on the dimension of $G$. The induction step is straightforward in all cases except that of a nilpotent element $x$ having no non-central semisimple elements in its centralizer $\mathfrak{z}(x)$. 

Commuting Varieties of Lie Algebras over Fields of Prime Characteristic

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In Section 4, we prove that this condition is equivalent to \( x \) being distinguished. This allows us, in Section 5, to deal with this final case, using the fact that a distinguished nilpotent element is Richardson (see [1, 2, 10]).

We use the following notation.

The center and the connected component of an algebraic group \( G \) are denoted by \( Z(G) \) and \( G^\circ \), respectively. Similarly, the center of a restricted Lie algebra \( \mathfrak{g} \) will be denoted \( Z(\mathfrak{g}) \). If \( x \) is an element of the Lie algebra \( \mathfrak{g} \) of an algebraic group \( G \), then \( Z_G(x) \) (resp. \( \delta_\mathfrak{g}(x) \)) is the Ad- (resp. \( \text{ad-} \)) centralizer of \( x \). We will also use similar notation for other centralizers, for example, writing \( Z_G(y) \) for the centralizer in \( G \) of an element \( y \in G \). We write \( x = x_s + x_n \) for the Jordan–Chevalley decomposition of \( x \in \mathfrak{g} \), where \( x_s \) is the semisimple part of \( x \), and \( x_n \) is the nilpotent part.

A restricted subalgebra \( \mathfrak{s} \) of the restricted Lie algebra \( \mathfrak{g} = \text{Lie}(G) \) will be called a torus if it is an abelian subalgebra consisting of semisimple elements. In Section 2, we will show that all maximal tori in \( \mathfrak{g} \) are algebraic, that is, equal to \( \text{Lie}(T) \) for some torus \( T \subset G \). Let \( T \) be a maximal torus of \( G \), and let \( \Phi = \Phi(G, T) \) be the root system of \( G \) relative to \( T \). For \( \alpha \in \Phi \), we will also denote by \( \alpha \) the derivative \( (d\alpha)_e : \text{Lie}(T) \rightarrow K \). This will cause no confusion.

2. PRELIMINARIES

From now on we assume that \( K \) is an algebraically closed field of characteristic \( p > 0 \). Let \( G \) be a connected algebraic group over \( K \) and let \( \mathfrak{g} = \text{Lie}(G) \). The Lie algebra \( \mathfrak{g} \) carries a natural restriction map \( X \rightarrow X^{[p]} \). We can express the center of \( \mathfrak{g} \) as a Lie algebra direct sum \( \delta(\mathfrak{g}) = \delta(\mathfrak{g})_s \oplus \delta(\mathfrak{g})_n \) due to the Jordan–Chevalley decomposition. Let \( t_\delta = \delta(\mathfrak{g})_s \). Then \( t_\delta \) is a torus in the restricted Lie algebra \( \mathfrak{g} \).

**Lemma 2.1.** Let \( T \) be a maximal torus of \( G \), and \( t = \text{Lie}(T) \). Then \( t_\delta \subseteq t \).

**Proof.** Let \( v \in t_\delta \). Then by [3, Sect. 9.1], \( \text{Lie}(Z_G(v)) = \delta_\mathfrak{g}(v) = \mathfrak{g} \). So \( G \) stabilizes every point of \( t_\delta \). In particular, \( t_\delta \subseteq \mathfrak{g}^T \), where \( \mathfrak{g}^T = \{ x \in \mathfrak{g} \mid (\text{Ad}t)(x) = x \forall t \in T \} \). But [3, Sects. 9.2, 12.1] show that \( \mathfrak{g}^T = t \oplus \mathfrak{n} \), where \( \mathfrak{n} \) is a restricted subalgebra of \( \mathfrak{g} \) consisting of nilpotent elements, and \( [t, \mathfrak{n}] = 0 \). This decomposition also corresponds to the Jordan–Chevalley decomposition in \( \mathfrak{g}^T \). Thus \( t_\delta \subseteq t \).}

**Theorem 1.** Let \( \bar{s} \) be a maximal torus in the restricted Lie algebra \( \mathfrak{g} \), and let \( T \) be a maximal torus in \( G \). Let \( t = \text{Lie}(T) \). Then \( \bar{s} \) is \( G \)-conjugate to \( t \). Thus the maximal tori in \( \mathfrak{g} \) are all algebraic.

**Proof.** The proof is by induction on \( \dim(G) - \dim(T) \). The theorem is clearly true when \( G = T \), so assume \( \dim(G) - \dim(T) = n > 0 \) and that
the theorem is known to be true for any pair \( G', T' \), with \( \dim(G') - \dim(T') < n \).

Suppose first of all that \( \mathfrak{s} \subseteq \mathfrak{z}(\mathfrak{g}) \). Then clearly \( \mathfrak{s} = t \). But then by Lemma 2.1, \( \mathfrak{s} \subseteq t \), and so \( \mathfrak{s} = t \).

So assume that there is some element \( \mathfrak{v} \in \mathfrak{x}(\mathfrak{g}) \). Then by [3, Sect. 11.8], \( \mathfrak{v} \in \mathfrak{Lie}(T') \) for some torus \( T' \) in \( \mathfrak{g} \), and we may choose \( T' \) to be maximal.

Now \( \mathfrak{g} \) properly contains \( \mathfrak{Z}(\mathfrak{v})' \) and \( \mathfrak{Lie}(\mathfrak{Z}(\mathfrak{v})') = \mathfrak{z}(\mathfrak{v}) \supseteq \mathfrak{s} \) by [3, Sect. 9.1]. So our induction hypothesis applies to \( \mathfrak{Z}(\mathfrak{v})' \) (which contains \( T' \)). Thus \( \mathfrak{s} \) is \( \mathfrak{Z}(\mathfrak{v})' \)-conjugate to \( t' = \mathfrak{Lie}(T') \). The conjugacy of all maximal tori in \( \mathfrak{g} \) (see, for example, [3, Sect. 11.3]) completes the proof.

We can improve Lemma 2.1 for the case in which we are interested.

**Lemma 2.2.** Suppose that \( \mathfrak{g} \) is reductive and \( \mathfrak{sl} = \mathfrak{Lie}(\mathfrak{g}) \). Then \( \mathfrak{z}(\mathfrak{g}) \) is a torus in \( \mathfrak{g} \), hence is contained in every maximal torus of \( \mathfrak{g} \).

**Proof.** It is clearly enough to show that \( \mathfrak{z}(\mathfrak{g}) \) is contained in a maximal torus \( \mathfrak{t} \) of \( \mathfrak{g} \). Choose a maximal torus \( \mathfrak{T} \) of \( \mathfrak{g} \) and let \( \mathfrak{t} = \mathfrak{Lie}(\mathfrak{T}) \).

Write \( \mathfrak{g} = \mathfrak{t} \oplus \sum \mathfrak{g}_\alpha \), where the sum is taken over all roots \( \alpha \) of \( \mathfrak{g} \) relative to \( \mathfrak{T} \). As \( \mathfrak{T} \) acts on each root space, we only need to show that no root space is contained in \( \mathfrak{z}(\mathfrak{g}) \).

Suppose \( \mathfrak{a}_\alpha \subseteq \mathfrak{z}(\mathfrak{g}) \). Choose a basis element \( \mathfrak{e}_\alpha \in \mathfrak{a}_\alpha \). Now consider the one-parameter root subgroups \( \mathfrak{U}_\alpha, \mathfrak{U}_{-\alpha} \) of \( \mathfrak{g} \) described in [7, Sect. 26], \( \mathfrak{U}_\alpha \) is the unique connected one-dimensional \( T \)-stable subgroup \( \mathfrak{A} \) of \( \mathfrak{g} \) such that \( \mathfrak{g}_\alpha \subseteq \mathfrak{Lie}(\mathfrak{A}) \). The subgroup \( \mathfrak{H} \) of \( \mathfrak{g} \) generated by \( \mathfrak{U}_\alpha \) and \( \mathfrak{U}_{-\alpha} \) is semisimple of rank 1. Thus \( \mathfrak{H} \) is isomorphic to either \( \mathfrak{SL}(V) \) or \( \mathfrak{PGL}(V) \), where \( V \) is a vector space of dimension 2. In the former case, the center of \( \mathfrak{h} = \mathfrak{Lie}(\mathfrak{H}) \) is a torus, and in the latter, it is trivial. But \( \mathfrak{e}_\alpha \in \mathfrak{z}(\mathfrak{h}) \) is nilpotent. This provides a contradiction, and so the proof is complete.

The example of \( \mathfrak{g} = \mathfrak{SL}(V) \), where \( V \) is a vector space of dimension 2 and \( \mathfrak{K} \) is an algebraically closed field of characteristic 2, shows that we do not always have \( \mathfrak{z}(\mathfrak{g}) = \mathfrak{Lie}(\mathfrak{Z}(\mathfrak{g})) \), even when \( \mathfrak{g} \) is reductive. However, certain (mildly restrictive) conditions do ensure that this is true (see Lemma 4.1). Furthermore, the following lemma shows that we do nevertheless have a completely general analogy with the fact that, for \( \mathfrak{g} \) reductive, \( \mathfrak{Z}(\mathfrak{g}) \) is the intersection of all maximal tori in \( \mathfrak{g} \).

**Lemma 2.3.** Let \( \mathfrak{g} \) be a connected reductive group and \( \mathfrak{g} = \mathfrak{Lie}(\mathfrak{g}) \). Then \( \mathfrak{z}(\mathfrak{g}) \) is the intersection of all maximal tori of \( \mathfrak{g} \).

**Proof.** We already know from Lemma 2.2 that \( \mathfrak{z}(\mathfrak{g}) \) is contained in every maximal torus of \( \mathfrak{g} \). So suppose that \( \mathfrak{x} \in \mathfrak{t} \) for all maximal tori \( \mathfrak{t} \), but \( \mathfrak{x} \notin \mathfrak{z}(\mathfrak{g}) \). Choose any maximal torus \( \mathfrak{T} \) of \( \mathfrak{g} \), and decompose \( \mathfrak{g} \) as the direct sum of the root spaces \( \mathfrak{g}_\gamma \), \( \gamma \in \Phi(\mathfrak{g}, \mathfrak{T}) \), and the maximal torus \( \mathfrak{t} = \mathfrak{Lie}(\mathfrak{T}) \). As \( x \in \mathfrak{t} \), \( \text{ad}(x) \) acts on each \( \mathfrak{g}_\gamma \), and so for some \( \alpha \in \Phi(\mathfrak{g}, \mathfrak{T}) \) and non-zero
$e_a \in \mathfrak{g}_a$, $[x, e_a] = \lambda e_a$, where $\lambda \neq 0$. Consider again the root subgroup $U_a$ of $G$. We can choose an isomorphism $\epsilon_a: G_a \to U_a$ such that $\text{Ad}(\epsilon_a(r)) \cdot x = x + r[e_a, x]$ for all $r \in G_a$.

Set $g = \epsilon_a(1)$. Since $[e_a, x] = -\lambda e_t \neq 0$, $(\text{Ad} g) \cdot x \neq t$. Then $x \notin (\text{Ad} g^{-1}) \cdot t$, contradicting the assumption that $x$ is in every maximal torus of $\mathfrak{g}$. Therefore, the intersection of the maximal tori of $\mathfrak{g}$ is contained in $\mathfrak{h}(\mathfrak{g})$, and so the proof is complete. \hfill\qed

3. THE THEOREM

In this section, we assume that $G$ is a connected reductive algebraic group over $K$ satisfying the following hypotheses:

(a) The derived subgroup $G^{(1)}$ of $G$ is simply-connected.

(b) $p$ is a good prime for $G$.

(c) The Lie algebra $\mathfrak{g}$ of $G$ has a non-degenerate symmetric bilinear $G$-invariant form $B: \mathfrak{g} \times \mathfrak{g} \to K$.

The main theorem of this paper is the following:

**Theorem 2.** Let $\mathcal{C}(\mathfrak{g}) = \{(x, y) \in \mathfrak{g} \times \mathfrak{g} \mid [x, y] = 0\}$, and let $\mathcal{C}'(\mathfrak{g})$ be the union of all $t \times t$, for $t$ the maximal tori of $\mathfrak{g}$. Then $\mathcal{C}(\mathfrak{g})$ is equal to the closure of $\mathcal{C}'(\mathfrak{g})$ in $\mathfrak{g} \times \mathfrak{g}$.

The proof is by induction on the dimension of $G$. The theorem is clearly true if the dimension of $G$ is zero. From now on, denote the closure of $\mathcal{C}'(\mathfrak{g})$ by $\mathcal{C}(\mathfrak{g})$. Clearly, $t \times t$ is contained in $\mathcal{C}(\mathfrak{g})$ for any maximal torus $t$ of $\mathfrak{g}$ (so that $\mathcal{C}'(\mathfrak{g}) \subseteq \mathcal{C}(\mathfrak{g})$), and hence $\mathcal{C}(\mathfrak{g}) \subseteq \mathcal{C}(\mathfrak{g})$, as $\mathcal{C}(\mathfrak{g})$ is closed. So it is required to prove that if $(x, y) \in \mathcal{C}(\mathfrak{g})$, then $(x, y) \in \mathcal{C}(\mathfrak{g})$. This leads us to the following induction hypothesis:

Let $H$ be a connected reductive algebraic group over $K$ which satisfies (a), (b), and (c), and has smaller dimension than $G$. Suppose $x$ and $y$ are commuting elements of the Lie algebra $\mathfrak{h}$ of $H$. Then $(x, y) \in \mathcal{C}(\mathfrak{h})$.

We start with an easy lemma.

**Lemma 3.1.** Let $M$ be an algebraic group acting morphically on a variety $V$, let $HCM$ be any subset closed under taking inverses, and let $X \subset V$ be a subset such that $h(X) = X \forall h \in H$. Then $X$ is $H$-stable.

**Proof.** If $Y$ is a closed subset of $V$ containing $X$, then $h^{-1}(Y)$ is closed and contains $X$ for each $h \in H$. Thus $\overline{X} \subseteq h^{-1}(\overline{X})$ for each $h \in H$. It follows that $\overline{X} = h(\overline{X})$ for each $h \in H$. \hfill\qed
LEMMA 3.2. Suppose $x, y \in \mathfrak{g}$, and $z, w \in \mathfrak{z}(\mathfrak{g})$, the center of $\mathfrak{g}$. Then

(i) $(x, y) \in \mathcal{E}(\mathfrak{g}) \Leftrightarrow (x + z, y + w) \in \mathcal{E}(\mathfrak{g})$.

(ii) $(x, y) \in \mathcal{E}(\mathfrak{g}) \Leftrightarrow (x + z, y + w) \in \mathcal{E}(\mathfrak{g})$.

Proof. (i) is obvious. For (ii), we use the fact, from Lemma 2.2, that $\mathfrak{z}(\mathfrak{g})$ is contained in each maximal torus of $\mathfrak{g}$. Now apply Lemma 3.1 to the action of $\mathfrak{z}(\mathfrak{g}) \times \mathfrak{z}(\mathfrak{g})$ on $\mathfrak{g} \times \mathfrak{g}$ given by $(z, w) \cdot (x, y) = (x + z, y + w)$. □

LEMMA 3.3. Suppose $(x, y) \in \mathcal{E}(\mathfrak{g})$ and one of $x$ or $y$ is not in $\mathfrak{z}(\mathfrak{g})$. Then $(x, y) \in \mathcal{E}(\mathfrak{g})$.

Proof. If $\sigma$ is the element of $\text{GL}(\mathfrak{g} \oplus \mathfrak{g})$ sending $(x, y)$ to $(y, x)$, then $\sigma$ stabilizes $\mathcal{E}(\mathfrak{g})$ and is self-inverse, so we can apply Lemma 3.1 to see that $\sigma$ stabilizes $\mathcal{E}(\mathfrak{g})$. It is also clear that $\sigma$ stabilizes $\mathcal{E}(\mathfrak{g})$.

So we need to prove this lemma in the case $x \notin \mathfrak{z}(\mathfrak{g})$. By [3, Sect. 9.1; 13, I.4.3, II.3.19, II.5.3], $\mathfrak{z}_0(x_s) = \text{Lie}(Z_G(x_s))$, and $Z_G(x_s)$ is a connected reductive algebraic group satisfying (a), (b), and (c). As $x_s$ is not central, we can use the induction hypothesis on $\mathfrak{z}_0(x_s)$. Indeed, $x$ and $y$ are commuting elements of $\mathfrak{z}_0(x_s)$, by standard properties of the Jordan–Chevalley decomposition, hence $(x, y) \in \mathcal{E}(\mathfrak{z}_0(x_s))$.

Now any maximal torus of $\mathfrak{z}_0(x_s)$ is contained in a maximal torus of $\mathfrak{g}$. So $\mathcal{E}(\mathfrak{z}_0(x_s))$ is contained in $\mathcal{E}(\mathfrak{g})$ and hence $\mathcal{E}(\mathfrak{z}_0(x_s))$ is contained in $\mathcal{E}(\mathfrak{g})$. But then $(x, y) \in \mathcal{E}(\mathfrak{g})$. □

LEMMA 3.4. Suppose $(x, y) \in \mathcal{E}(\mathfrak{g})$, and one of $\mathfrak{z}_0(x)$, $\mathfrak{z}_0(y)$ contains a non-central semisimple element $s$ of $\mathfrak{g}$. Then $(x, y) \in \mathcal{E}(\mathfrak{g})$.

Proof. Using Lemma 3.1 and the map $\sigma$, we may reduce the proof to the case where $s \in \mathfrak{z}_0(x)$. Let $U = \{\lambda s + (1 - \lambda)y \mid \lambda \in K\}$ and $D = \{z \in U \mid (\text{ad} z)_{\mathfrak{z}_0(x)} \neq 0\}$. Clearly, $D$ is open in $U$ and $s \in D$, so $D$ is a dense subset of $U$. Furthermore, $(x, z) \in \mathcal{E}(\mathfrak{g})$ for every $z$ in $D$, by Lemma 3.3. Thus $(x, z) \in \mathcal{E}(\mathfrak{g})$ for every $z$ in $U$, as $\mathcal{E}(\mathfrak{g})$ is closed. In particular, $(x, y) \in \mathcal{E}(\mathfrak{g})$. □

For $z \in \mathfrak{g}$, consider the following property:

\[(\mathfrak{z}_0(z))_s = \mathfrak{z}(\mathfrak{g}). \tag{1}\]

Lemmas 3.2 to 3.4 lead us to the final case in the proof, where $x$ and $y$ are nilpotent elements of $\mathfrak{g}$ satisfying (1). In the next section, we will show that a nilpotent element of $\mathfrak{g}$ satisfies (1) if and only if it is distinguished.
4. DISTINGUISHED NILPOTENT ELEMENTS

Let $G$ be as in Section 3, and let $x$ be a nilpotent element of $\mathfrak{g}$. Denote by $\mathfrak{g}'$ the Lie algebra of the derived subgroup $G^{(1)}$ of $G$. Then by [10, Sect. 2.9], $x \in \mathfrak{g}'$.

Recall that $x$ is called distinguished if $Z_{G^{(1)}}(x)^o$ is a unipotent group.

**Theorem 3.** The element $x$ is distinguished if and only if it satisfies the condition (1) given at the end of Section 3.

**Remark.** If $\text{char}(K) = 0$, then $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{z}(\mathfrak{g})$ and $\mathfrak{z}(x) = \text{Lie}(Z_G(x))$. So the statement of the theorem is true by Engel’s theorem and its analogue for unipotent groups.

**Proof.** Recall that $G^{(1)}$ is simply-connected and semisimple. Let $G_1, G_2, \ldots, G_r$ be the simple simply-connected normal subgroups of $G^{(1)}$. Write $\mathfrak{g}_i = \text{Lie}(G_i)$. Then $G^{(1)} = G_1 \times G_2 \times \cdots \times G_r$, and $\mathfrak{g}' = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_r$. Define groups $\tilde{G}_1, \tilde{G}_2, \ldots, \tilde{G}_r$ by setting

$$\tilde{G}_i = \begin{cases} \text{GL}(V_i), & \text{if } G_i = \text{SL}(V_i) \text{ and } p | \dim V_i, \\ G_i, & \text{otherwise.} \end{cases}$$

Let $\tilde{\mathfrak{g}}_i = \text{Lie}(\tilde{G}_i)$. Write $\tilde{G} = \tilde{G}_1 \times \tilde{G}_2 \times \cdots \times \tilde{G}_r$ and $\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}_1 \oplus \tilde{\mathfrak{g}}_2 \oplus \cdots \oplus \tilde{\mathfrak{g}}_r$. According to [5, Sect. 6.2], there exist tori $T_0, T_1$ with respective Lie algebras $\mathfrak{t}_0, \mathfrak{t}_1$ such that $\mathfrak{g} \hookrightarrow \tilde{\mathfrak{g}} \oplus \mathfrak{t}_0 = \tilde{\mathfrak{g}} = \text{Lie}(\tilde{G})$, and $\mathfrak{g}_i \subseteq \tilde{\mathfrak{g}}_i$. (We identify $\tilde{G}^{(1)}$ and $G^{(1)}$ with their respective images in $\tilde{G}$ and $G$.) Furthermore, $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{t}_1$.

As a consequence, $\tilde{G}^{(1)} = G^{(1)} = \tilde{G}^{(1)}$. So $x$ is distinguished in $G$ if and only if it is distinguished in $\tilde{G}$ (or $\tilde{G}$). Furthermore, $\delta_{\tilde{\mathfrak{g}}}(x) = \delta_{\tilde{\mathfrak{g}}}(x) \oplus \mathfrak{t}_0 = \delta_{\mathfrak{g}}(x) \oplus \mathfrak{t}_1$, and $\mathfrak{t}_0$ and $\mathfrak{t}_1$ are both contained in the center of $\tilde{\mathfrak{g}}$. Hence (1) holds for $x$ in $\mathfrak{g}$ if and only if it holds for $x$ in $\tilde{\mathfrak{g}}$ (or $\tilde{\mathfrak{g}}$). This makes it possible to say that $x$ is distinguished (or satisfies (1)) without any real ambiguity.

Write $x = x_1 + x_2 + \cdots + x_r$, $x_i \in \mathfrak{g}_i$. Then

$$Z_{G^{(1)}}(x)^o = Z_{G_1}(x_1)^o \times Z_{G_2}(x_2)^o \times \cdots \times Z_{G_r}(x_r)^o$$

and

$$\delta_{\mathfrak{g}}(x) = \delta_{\mathfrak{g}}(x_1) \oplus \delta_{\mathfrak{g}}(x_2) \oplus \cdots \oplus \delta_{\mathfrak{g}}(x_r).$$

So $x$ is distinguished if and only if each $x_i$ is distinguished (in $\mathfrak{g}_i$ or $\tilde{\mathfrak{g}}_i$), and $x$ satisfies (1) if and only if each $x_i$ satisfies (1) in $\tilde{\mathfrak{g}}_i$. Thus it will be sufficient to prove the theorem in the following two cases:

(i) $G$ is simple, simply-connected, and not of type $A_{kp-1}$ for $k \in \mathbb{Z}$;

(ii) $G = \text{GL}(kp)$, any $k \in \mathbb{Z}$. 

By [6, Sect. 2.7] and [13, I.4.3], if $G$ is of the first type (and $p$ is good), then $\mathfrak{g}$ is simple as a Lie algebra. So in this case $\delta(\mathfrak{g})$ is trivial, and $x$ satisfies (1) if and only if $\delta_\beta(x)$ consists entirely of nilpotent elements. Hence if $x$ satisfies (1), then $x$ is distinguished, for if $Z_G(x)^\circ$ contains a non-trivial torus, then so must $\delta_\beta(x)$. Furthermore, [13, I.5.6] shows that $\delta_\beta(x) = \text{Lie}(Z_G(x))$ in the case where $G$ is not of type $A_n$. In fact, a look at the condition of [13, I.5.1], which is required for this result, shows that this can easily be extended to the case where $G = \text{SL}(V)$, for $V$ a vector space of dimension prime to $p$. So if $x$ is distinguished, then $x$ satisfies (1).

It therefore remains to prove the theorem under the assumption that $G = \text{GL}(V)$, where $p \mid \dim V$.

Suppose $x \in \mathfrak{g} = \text{gl}(V)$ does not satisfy (1). Then $\delta_\beta(x)$ contains a torus $\mathfrak{t}$ of dimension 2 such that $K \cdot \text{Id}_V \subset \mathfrak{t}$. So the intersection of $\mathfrak{t}$ with $G$ is a non-empty open subset of $\mathfrak{t}$, hence generates a torus $T$ (contained in $Z_G(x)$) of dimension greater than 1. Now since $Z_G(y) = Z(G) \cdot Z_{\text{SL}(V)}(y)$, $y \in \mathfrak{g}$ is distinguished if and only if all maximal tori in $Z_G(y)$ are of dimension 1. Thus $x$ is not distinguished.

Now assume $x$ is not distinguished. Then $Z_{\text{SL}(V)}(x)$ contains a non-trivial torus, hence $Z_G(x)$ contains a torus of dimension 2. Thus $\delta_\beta(x)$ contains a torus of dimension 2, which implies that $x$ does not satisfy (1).

This completes the proof of the theorem.  

For use in the next section, we record a result which is easily obtainable from the proof above.

**Lemma 4.1.** Let $G$ be a connected reductive algebraic group over $K$ with Lie algebra $\mathfrak{g}$ satisfying (a), (b), and (c) of Section 3. Then $\delta(\mathfrak{g}) = \text{Lie}(Z(G))$.

**Proof.** The result follows from a consideration of the dimension of $\delta(\mathfrak{g})$. It is clear from Section 2 that $\text{Lie}(Z(G)) \subseteq \delta(\mathfrak{g})$, and so it will suffice to show that the dimensions are equal. We use the description of $\hat{G}$ given in the proof of Theorem 3.

We have

$$Z(\hat{G}) = Z(\hat{G}) \times T_0,$$

$$\delta(\hat{\mathfrak{g}}) = \delta(\hat{\mathfrak{g}}) \oplus t_0 = \delta(\mathfrak{g}) \oplus t_1 \quad \text{and} \quad \dim Z(\hat{G}) + \dim T_1 = \dim Z(G).$$

So equality of dimensions in $\hat{G}$ will show equality of dimensions in $\hat{\mathfrak{g}}$ (and hence $G$). But now the decomposition of $\hat{G}$ reduces the proof to a verification that the result holds in each of the $\hat{G}_i$, which is trivial. 

**Corollary 4.2.** Let $G$, $\mathfrak{g}$ be as above, and choose a maximal torus $T$ of $G$ with Lie algebra $\mathfrak{t}$. Then for any basis $B$ of the root system of $G$ relative to $T$, the set of $\hat{\mathfrak{g}}$ derivatives of elements of $B$ is a linearly independent subset of $\mathfrak{t}$.
Proof. Let $\Phi$ be the root system of $G$ relative to $T$. Decompose $\mathfrak{g}$ as the direct sum of $t$ and the root spaces $\mathfrak{g}_\alpha$, for $\alpha \in \Phi$. Lemma 2.2 shows that $\mathfrak{z}(\mathfrak{g})$ is contained in $t$. Thus $\mathfrak{z}(\mathfrak{g}) = \{ t \in t \mid \alpha(t) = 0 \ \forall \alpha \in B \}$. The result now follows from Lemma 4.1 and the fact that $\dim T - \dim \mathcal{Z}(G) = |B|$. 

The following result is not relevant to the proof of Theorem 2, but nevertheless fits in very well to our discussion of distinguished nilpotent elements. It extends a result of [14]. The proof given here is very similar to Tauvel's proof for the case where $K$ is an algebraically closed field of characteristic zero.

**Theorem 4.** Let $G, \mathfrak{g}, K$ be as in Section 3, and let $x$ be a nilpotent element of $\mathfrak{g}$. Then $x$ is distinguished if and only if $\delta_\mathfrak{g}(x)$ is nilpotent.

Proof. Choose a maximal torus $T$ of $G$, and let $\Phi$ be the root system of $G$ relative to $T$. Let $\Phi^+$ be a positive system in $\Phi$ and let $\Delta = \{ \alpha_1, \alpha_2, \ldots, \alpha_r \}$ be the basis of simple roots. We have integer-valued functions $m_i : \Phi \rightarrow \mathbb{Z}$ such that $\alpha = \sum m_i(\alpha)\alpha_i$, for every root $\alpha$. Let $\mathfrak{n}$ (respectively $\mathfrak{n}^+$) denote the span of the root spaces $\mathfrak{g}_\alpha$ (respectively $\mathfrak{g}_{-\alpha}$) with $\alpha \in \Phi^+$. Then $\mathfrak{n} = \mathfrak{n}^- \oplus \mathfrak{n} \oplus \mathfrak{n}^+$, where $\mathfrak{n} = \text{Lie}(T)$. By [3, Sects. 11.3, 14.17], $x$ is $G$-conjugate to an element in $\mathfrak{n}$. Thus we may assume without loss of generality that $x \in \mathfrak{n}$. For each $z \in \mathfrak{n}$, we write $z = \sum z_\alpha$, where $z_\alpha \in \mathfrak{g}_\alpha$ (and $\alpha \in \Phi^+$). For each simple root $\alpha_i$, we define a function $|.|_i : \mathfrak{n} \rightarrow \mathbb{Z}$, given by $|z|_i = \sum z_\alpha m_i(\alpha)$.

We claim that if $\delta_\mathfrak{g}(x)$ is nilpotent, then $|x|_i \neq 0$ for all $i$.

Indeed, suppose $|x|_i = 0$ for some $i$. Let $\beta = \alpha_i$. It is clear that $\mathfrak{g}_{-\beta} \subset \delta_\mathfrak{g}(x)$. Furthermore, by Corollary 4.2, we may choose $h \in t$ such that $\beta(h) = 1$, but $\alpha_j(h) = 0$ for every $\alpha_j \in \Delta \setminus \{ \beta \}$. Thus $h \in \delta_\mathfrak{g}(x)$. But $[h, \mathfrak{g}_{-\beta}] = \mathfrak{g}_{-\beta}$, and so $\delta_\mathfrak{g}(x)$ is not nilpotent. This contradicts the initial assumption, proving the claim. Now let us return to the proof of Theorem 4. We see first of all that, from Theorem 3, we certainly have that if $x$ is distinguished, then $\delta_\mathfrak{g}(x)$ is nilpotent. So suppose that $\delta_\mathfrak{g}(x)$ is nilpotent, but $x$ is not distinguished. Let $h$ be a non-central semisimple element of $\delta_\mathfrak{g}(x)$. Then $\delta_\mathfrak{g}(h)$ is a Levi subgroup of $\mathfrak{g}$ which contains $x$. We may assume that $\delta_\mathfrak{g}(h)$ is the Levi subalgebra corresponding to a standard parabolic subgroup of $G$ (see [7, Sect. 30.1]). In other words, we may assume that there is a proper subset $I \subset \Delta$ such that $\delta_\mathfrak{g}(h) = t \oplus \sum_{\alpha \in \Phi^-} \mathfrak{g}_\alpha$, where $\Phi^-_I$ is the subsystem of $\Phi$ generated by $I$. Let $\Phi_I^- = \Phi_I \cap \Phi^-$, and $n_I = \sum_{\alpha \in \Phi_I^-} \mathfrak{g}_\alpha$. Then, after conjugating if necessary, we may assume that $x \in n_I$. Let $\alpha_i \in \Delta \setminus I$. Then $|x|_i = 0$. Therefore, $\delta_\mathfrak{g}(x)$ is not nilpotent by the above discussion, and so we have a contradiction. Thus the proof is complete. \[\square\]
5. COMPLETION OF THE PROOF

Let $G, K, \mathfrak{g}$ be as in Section 3, and suppose that $P$ is a parabolic subgroup of $G$, with unipotent radical $U$. Let $\mathfrak{p}$ and $\mathfrak{u}$ be the Lie algebras of $P$ and $U$, respectively. Recall that (nilpotent) $x \in \mathfrak{u}$ is Richardson if the $P$-conjugacy class of $x$ is an open subset of $\mathfrak{u}$.

Now suppose further that $P$ is a proper subgroup of $G$. Let $M$ be a Levi subgroup of $P$, with Lie algebra $\mathfrak{m}$, so that $P$ is the semidirect product of $M$ and $U$ (and $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{u}$). Define $A = Z(M)^\circ$. Then $A$ is a torus and $\text{Rad} P = AU$. Let $\alpha = \text{Lie}(A)$. Then [13, II.5.4] and Lemma 4.1 show that $\alpha = \delta(\mathfrak{m})$. Set $\mathfrak{r} = \alpha \oplus \mathfrak{u}$ and $\alpha' = \{ a \in \alpha \mid \delta_\mathfrak{p}(a) = \mathfrak{m} \}$.

**Lemma 5.1.** $\alpha'$ is a non-empty open subset of $\alpha$.

**Proof.** Let $T$ be a maximal torus in $G$ and let $\Phi = \Phi(G, T)$ be the root system of $G$ relative to $T$. Let $\Phi^+$ be a positive system in $\Phi$ and let $\Delta = \{ \alpha_1, \alpha_2, \ldots, \alpha_r \}$ be the basis of simple roots contained in $\Phi^+$. Then

$$\mathfrak{g} = \sum_{\alpha \in \Phi^+} \mathfrak{g}_{-\alpha} \oplus \mathfrak{t} \oplus \sum_{\alpha \in \Phi^+} \mathfrak{g}_\alpha.$$

For a subset $I$ of $\Delta$, we denote by $\Phi_I$ and $\Phi_I^+$ the respective subsets of $\Phi$ and $\Phi^+$ consisting of those roots expressible as $\mathbb{Z}$-sums of elements of $I$. We may assume $\mathfrak{p}$ is a standard parabolic subalgebra of $\mathfrak{g}$. So for some $I \subset \Delta$,

$$\mathfrak{p} = \sum_{\alpha \in \Phi_I^+} \mathfrak{g}_{-\alpha} \oplus \mathfrak{t} \oplus \sum_{\alpha \in \Phi_I^+} \mathfrak{g}_\alpha,$$

$$\mathfrak{m} = \sum_{\alpha \in \Phi_I^+} \mathfrak{g}_{-\alpha} \oplus \mathfrak{t} \oplus \sum_{\alpha \in \Phi_I^+} \mathfrak{g}_\alpha,$$

$$\mathfrak{u} = \sum_{\alpha \in \Phi^+ \setminus \Phi_I^+} \mathfrak{g}_\alpha.$$

Then

$$\alpha = \{ t \in \mathfrak{t} \mid \alpha_i(t) = 0 \quad \forall \alpha_i \in I \},$$

$$\alpha' = \{ t \in \alpha \mid \alpha(t) \neq 0 \quad \forall \alpha \in \Phi^+ \setminus \Phi_I^+ \}.$$

Corollary 4.2 shows that $\alpha_1, \alpha_2, \ldots, \alpha_r$ are linearly independent as elements of $\mathfrak{t}^*$, and so, as $p$ is a good prime, each $\alpha \in R^+ \setminus R_I^+$ is non-trivial on $\alpha$. Therefore, $\alpha \setminus \alpha'$ is a union of closed subsets of codimension 1, and so $\alpha'$ is non-empty. \hfill \square

Let $m = \dim \mathfrak{m}$ and $d = \dim \mathfrak{r}$.

**Lemma 5.2.** Suppose $r = a + u \in \mathfrak{r}$, with $a \in \alpha'$, $u \in \mathfrak{u}$. Then $r$ is $P$-conjugate to $a$. In particular, $r$ is semisimple and $\dim \delta_\mathfrak{p}(r) = m$. 

Proof. Define a length function $l$ on $R$ by setting $l(\alpha_i) = 1$ for each simple root, and extending additively. Then there is a $\mathbb{Z}$-grading $$\mathfrak{g} = \mathfrak{g}(-n) \oplus \mathfrak{g}(-(n - 1)) \oplus \cdots \oplus \mathfrak{g}(n),$$ where $\mathfrak{g}(0) = \mathfrak{t}$ and $\mathfrak{g}(i) = \sum_{\alpha \in \Phi}(\mathfrak{g}_\alpha)^i$. Clearly, $\mathfrak{u} \subseteq \mathfrak{g}(1) \oplus \cdots \oplus \mathfrak{g}(n)$. Write $\mathfrak{u} = \mathfrak{u}_1 + \mathfrak{u}_2 + \cdots + \mathfrak{u}_n$, with $\mathfrak{u}_i \in \mathfrak{g}(i) \cap \mathfrak{u}$.

All that is required now is an induction step to remove the shortest part of $\mathfrak{u}$. Suppose that $\mathfrak{u}_i$ is the shortest non-zero part of $\mathfrak{u}$. Clearly, $\mathfrak{u}_i = \sum \mathfrak{u}_\alpha$, where the sum is taken over all roots $\alpha$ of length $i$. Then we can remove any $\mathfrak{u}_\alpha$ without adding any more $\mathfrak{u}_j$ (for $j < i$), or any $\mathfrak{u}_\beta$ (for $\beta \neq \alpha$ of length $i$). To do this, we again use the root subgroup $U_\alpha$ of $G$.

We pick an isomorphism $e_\alpha: \mathfrak{g}_\alpha \rightarrow U_\alpha$ such that $te_\alpha(x)t^{-1} = e_\alpha(\alpha(t)x)$ for all $t \in T$, $x \in \mathfrak{g}_\alpha$. Then there is a non-zero element $e_\alpha \in \mathfrak{g}_\alpha$ such that

$$\text{Ad}(e_\alpha(x))(a) = a - x\alpha(a)e_\alpha.$$

Furthermore, $\text{Ad}(e_\alpha(x))$ sends $\mathfrak{g}_\beta$ into $\mathfrak{g}_\beta \oplus \mathfrak{l}$, where $\mathfrak{l}$ is a sum of root spaces for roots of length strictly greater than $\beta$.

It is an easy task now to find an element of $P$ which removes $\mathfrak{u}_\alpha$ in the required way: if $\mathfrak{u}_\alpha = \lambda e_\alpha$, then set $x = \lambda(\alpha(a))^{-1}$. Then $e_\alpha(x)$ is the required element. 

Now let $\tau' = \{r \in \tau \mid \dim \mathfrak{g}_\alpha(r) = m\}$. Then $\tau'$ is a $P$-stable subset of $\tau$, and $\tau' + \mathfrak{u}$ is a dense open subset in $\tau$ by Lemmas 5.1 and 5.2. Suppose $x \in \mathfrak{u}$ is such that the $P$-conjugacy class of $x$ is dense in $\mathfrak{u}$. Then the $P$-stabilizer of $x$ has the same dimension as $\mathfrak{u}$, and so $x \in \tau'$.

Our next lemma plays a crucial role in the proof of Theorem 2. The argument goes back to Richardson (see [12, p. 317]). It is characteristic free.

Lemma 5.3. Let $\mathcal{R}$ be the intersection of $\tau' \times \mathfrak{v}$ with $\mathfrak{c}(\mathfrak{g})$. Let $\pi: \mathcal{R} \rightarrow \tau'$ denote the restriction of the projection from $\tau' \times \mathfrak{v}$ onto $\tau'$. Then $\pi$ is an open map.

Proof. Let $c \in \tau'$ and let $\mathfrak{d} = \delta_\phi(c)$. Choose a vector space $\mathfrak{f}$ such that $\mathfrak{d} \oplus \mathfrak{f} = \mathfrak{v}$. Denote by $\text{Gr}_m(\mathfrak{v})$ the Grassman variety of all $m$-dimensional subspaces of $\mathfrak{v}$, and let $\mathcal{F}$ be its principal open subset consisting of all subspaces having trivial intersection with $\mathfrak{f}$. Let $\tau'' = \{r \in \tau' \mid \delta_\phi(r) \in \mathcal{F}\}$, a nonempty open subset of $\tau'$. There is an isomorphism of algebraic varieties $\phi: \text{Hom}_k(\mathfrak{b}, \mathfrak{f}) \rightarrow \mathcal{F}$, given by sending a map $\alpha$ to the subspace $\{d + \alpha(d) \mid d \in \mathfrak{d}\}$.

The map from $\tau'$ to $\text{Gr}_m(\mathfrak{v})$ which sends $r$ to $\delta_\phi(r)$ is clearly a morphism of varieties. We use it to define a morphism $\tau$ from $\tau'' \times \mathfrak{d}$ to $\mathcal{R}$ by setting

$$\tau(r, d) = (r, d + \phi^{-1}(\delta_\phi(r))(d)) \quad \text{for all } r \in \tau'', \quad d \in \mathfrak{d}.$$
The diagram

\[
\begin{array}{ccc}
\mathfrak{t}'' \times \mathfrak{d} & \xrightarrow{\tau} & \mathcal{R} \\
pr_1 \downarrow & & \downarrow \pi \\
\mathfrak{t}' & & \\
\end{array}
\]

is commutative. Since \( \mathfrak{t}'' \) is open in \( \mathfrak{t}' \), \( pr_1 : \mathfrak{t}'' \times \mathfrak{d} \to \mathfrak{t}' \) is an open map. But then \( \pi \) is open, too.

**Lemma 5.4.** Let \( G, \mathfrak{g}, K \) be as in Section 3, and let \( (x, y) \in \mathcal{C}(\mathfrak{g}) \) be such that either \( x \) or \( y \) is Richardson. Then \( (x, y) \in \mathcal{C}^\prime(\mathfrak{g}) \).

**Proof.** Using Lemma 3.1, we may reduce to the case where \( x \) is Richardson. Let \( P \) be a (proper) parabolic subgroup of \( G \) with unipotent radical \( U \) such that the \( P \)-conjugacy class of \( x \) is dense in \( U \), the Lie algebra of \( U \). Pick an open neighborhood \( N \) of \( (x, y) \) in \( \mathfrak{g} \times \mathfrak{g} \).

Let \( N' \) be the intersection of \( N \) and \( \mathcal{R} \). Note that \( x \in \mathfrak{t}' \). Then \( (x, y) \in \mathcal{R} \), hence \( N' \) is a non-empty open subset of \( \mathcal{R} \). Thus \( \pi(N') \) is a non-empty open subset of \( \mathfrak{t}' \). In particular, \( \pi(N') \) meets \( a' + \mathfrak{u} \). So \( \pi(N') \) contains a semisimple element \( s \in \mathfrak{t}' \). Clearly, \( s \notin \mathfrak{g}(\mathfrak{g}) \). Now choose \( t \in \mathfrak{g}(s) \) such that \( (s, t) \in N' \). Then by Lemma 2.3, \( (s, t) \in \mathcal{C}(\mathfrak{g}) \).

So every open neighborhood of \( (x, y) \) in \( \mathfrak{g} \times \mathfrak{g} \) meets \( \mathcal{C}(\mathfrak{g}) \). Thus \( (x, y) \in \mathcal{C}(\mathfrak{g}) \).

By [1, 2, 10, Sect. 1.4], every distinguished nilpotent element of \( \mathfrak{g} \) is Richardson. So, by the above proposition, we have the following:

**Lemma 5.5.** Suppose \( (x, y) \in \mathcal{C}(\mathfrak{g}) \), where one of \( x \) or \( y \) is a distinguished nilpotent element of \( \mathfrak{g} \). Then \( (x, y) \in \mathcal{C}(\mathfrak{g}) \).

This completes the proof of Theorem 2.

**Theorem 5.** Let \( G, \mathfrak{g}, K \) be as before. Then the commuting variety of \( \mathfrak{g} \) is an irreducible variety.

**Proof.** The adjoint action of \( G \) on \( \mathfrak{g} \) induces the diagonal action \( \text{Ad} \times \text{Ad} \) of \( G \) on \( \mathcal{C}(\mathfrak{g}) \). Choose a maximal torus \( t \) of \( \mathfrak{g} \). From Theorem 1, the maximal tori in \( \mathfrak{g} \) are conjugate, and so \( (\text{Ad} \times \text{Ad})(G) \cdot (t \times t) = \mathcal{C}(\mathfrak{g}) \), so that \( \mathcal{C}(\mathfrak{g}) \) is irreducible. Hence \( \mathcal{C}(\mathfrak{g}) \) is also irreducible.

**Remark.** The equality \( \mathcal{C}(\mathfrak{g}) = \mathcal{C}(\mathfrak{g}) \) may fail if the assumption (C) of Section 3 is relaxed. As an example, consider \( \mathfrak{g} = \mathfrak{sl}(2, K) \), with \( p = 2 \). Then \( \dim(\mathcal{C}(\mathfrak{g})) = 5 \), while \( \dim(\mathcal{C}(\mathfrak{g})) = 2 \). It can be shown that if \( \text{char}(K) = p \) is an odd prime, then \( \mathcal{C}(\mathfrak{sl}_p(K)) = \mathcal{C}(\mathfrak{sl}_p(K)) \).
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