Linear stability of relative equilibria in the charged three-body problem

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Abstract

A relative equilibrium is a periodic orbit of the $n$-body problem that rotates uniformly maintaining the same central configuration for all time. In this paper we generalize some results of R. Moeckel and we apply it to study the linear stability of relative equilibria in the charged three-body problem. We find necessary conditions to have relative equilibria linearly stable for the collinear charged three-body problem, for planar relative equilibria we obtain necessary and sufficient conditions for linear stability in terms of the parameters, masses and electrostatic charges. In the last case we obtain a stability inequality which generalizes the Routh condition of celestial mechanics. We also proof the existence of spatial relative equilibria and the existence of planar relative equilibria of any triangular shape.

Keywords: Central configuration; Relative equilibria; Spectral stability

1. Introduction

It is well known that some observations cannot be explained within the framework of classical Newtonian mechanics. In the large scale are the perihelion advance of Mercury [6], the motions of some active galactic nuclei [10], and certain galaxy interactions [23]; in the small scale are motions of electrons in an atom [11]. Therefore many researchers have tried to explain the dynamics by looking at potentials different from the Newtonian one. Of particular importance in applica-
tions are the so-called quasihomogeneous potentials, which are sums of functions with different degree of homogeneity, such as the Manev potential [6], the Manev-type potentials [16], and the Schwarzschild potential [23].

Another important class of potentials correspond to charged potentials, that is, the potentials used to study the charged \(n\)-body problem. There are many nice contributions in this direction: [1,3–5], to cite only a few. These papers study some features for the global behavior in some particular symmetric problems and in some restricted problems.

The charged \(n\)-body problem concerns the motion of \(n\) point particles endowed with a positive mass \(m_j \in \mathbb{R}^+\) and an electrostatic charge of any sign \(q_j \in \mathbb{R}\), moving under the influence of the respective Newtonian and Coulombian forces. If we denote by \(r_j \in \mathbb{R}^3\) the position of the \(i\)th particle, then the equations of motion are given by

\[
m_j \ddot{r}_j = \sum_{i \neq j} \frac{m_i m_j - q_i q_j}{r_{ij}^3} (r_i - r_j) = \frac{\partial U}{\partial r_j},
\]

where \(r_{ij} = |r_i - r_j|\), and the potential \(U\) is given by

\[
U = \sum_{i<j} \lambda_{ij} \frac{\lambda_{ij}}{r_{ij}}, \quad \lambda_{ij} = m_i m_j - q_i q_j.
\]

In this paper we assume that the center of mass is fixed at the origin, that is \(\sum_{i=1}^{n} m_i r_i = 0\), and we denote \(\mathbf{r} = (r_1, \ldots, r_n) \in \mathbb{R}^{3n}\). Let \(\Delta_{ij} = \{r \in \mathbb{R}^{3n} \mid r_i = r_j\}\) and \(\Delta = \bigcup_{i<j} \Delta_{ij}\), then the configuration space is given by \(\Omega = \mathbb{R}^{3n} \setminus \Delta\). Let \(\mathbf{p} = M^{-1} \mathbf{r}\) be the linear momentum of the system of particles, where \(M\) is the diagonal matrix \(M = \text{diag}\{m_1, m_1, m_1, \ldots, m_n, m_n, m_n\}\). Then Eqs. (1) can be written as a Hamiltonian system

\[
\dot{\mathbf{r}} = \frac{\partial H}{\partial \mathbf{p}}, \quad \dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{r}},
\]

where \(H : \Omega \times \mathbb{R}^{3n} \to \mathbb{R}\) is the Hamiltonian function given by

\[
H(\mathbf{r}, \mathbf{p}) = \frac{1}{2} \mathbf{p}' M^{-1} \mathbf{p} - U(\mathbf{r}).
\]

Here \(T = \frac{1}{2} \mathbf{p}' M^{-1} \mathbf{p}\) is the kinetic energy. The total energy \(H\) is a first integral for the system (2), then along any orbit, \(T - U = h\) is always constant.

Let us observe that if all charges are zero, the problem reduces to the classical Newtonian one. In other words, the classical Newtonian \(n\)-body problem and the charged problem have the same Hamiltonian structure. They differ just in the number of parameters, which in the second case implies a more complicated dynamical behavior. Central configurations and relative equilibria are basic topics which will help us understand the complexity of the charged problem.

Definition 1. A point \(r_0 \in \Omega\) is a central configuration (C.C.), if there is some scalar \(\alpha \in \mathbb{R}\) so that \(M^{-1} \nabla U(r_0) - \alpha r_0 = 0\).
In the planar Newtonian $n$-body problem, it is well known that if we rotate rigidly any C.C. around its center of mass, we obtain a special periodic solution called a relative equilibrium [24]. The history of relative equilibria starts in 1767 with the work of Euler on the three-body problem, in which he finds the collinear relative equilibria. Some years later Lagrange rediscovered Euler’s collinear relative equilibria and found a new class of relative equilibria, those formed by equilateral triangles [20]. In rotating coordinates, these special solutions become fixed points—from here the name of relative equilibria. A famous conjecture about the finiteness of relative equilibria stated by Wintner in 1941 [24] was labeled by Smale [21] as one of the main mathematical questions for the 21st century [22]; concerning this famous conjecture known today as Wintner–Smale conjecture, the name Chazy is also being associated to this famous problem [7], very few is known about this conjecture, recently Hampton and Moeckel [9] showed that the number of central configurations in the Newtonian planar four-body problem is finite, whereas for $n \geq 5$ is still an open problem. In a previous work [2], we proved the existence of continuum of C.C. for $n \geq 4$ for the charged problem. Previously, G. Roberts gave an example of a continuum of C.C. in the five-body problem but with one negative mass [17]. Of course, these examples do not answer Wintner–Smale conjecture, but they show that the conjecture is not true for some non-Newtonian potentials.

In 2002, G. Roberts [18] gave a nice description of the linear stability for the Lagrangian relative equilibria in the classical Newtonian three-body problem. In the same way, in this paper we are interested in the study of the linear stability of relative equilibria for charged problems. Since for $n \geq 4$ there exists a continuum of C.C. [2], we restrict our analysis to the study of linear stability of relative equilibria for the charged 3-body problem.

The study of the stability of relative equilibria is done in the rotating system. In this way, we obtain the eigenvalues of the linearized vector field. The ideas of Moeckel [13] and [14], to obtain a nice factorization of the characteristic polynomial have been used in this part of the paper.

In [15], the authors found the total number of C.C. for the charged three-body problem. Here we reconsider this problem in order to study the stability of the corresponding periodic orbits obtained from C.C. Our main result concerns the complete classification of the stability of relative equilibria for the charged three-body problem. As a corollary, we show the existence of stable relative equilibria with equal masses. This property does not take place in the classical Newtonian three-body problem. Actually an important conjecture (Moeckel [14]) for the Newtonian $n$-body problem is that a relative equilibrium can be stable only if it contains a mass significantly larger than the others; then the mentioned corollary shows that this result is not true for charged problems, where if choosing charges in a convenient way, a relative equilibrium with three equal masses can be stable. We also get some necessary conditions on the parameters to obtain stable collinear relative equilibria. This provides another difference from the Newtonian case, in which collinear relative equilibria are unstable.

2. General aspects

As in the classical Newtonian $n$-body problem, relative equilibria in charged problems form a special class of periodic orbits, which rotate like a rigid body around their center of mass. In the Newtonian case each planar C.C. determines a relative equilibrium.

In this paper we first answer two questions: 1. In the charged $n$-body problem does any C.C. produce relative equilibria? 2. Is the corresponding C.C. planar? In other words, we look for C.C. which make possible a perfect balance between the Newtonian and Coulombian forces with the centrifugal ones. We consider the problem both in the plane, $\mathbb{R}^2$, and in space, $\mathbb{R}^3$. 

Since a relative equilibrium is a configuration \( r_0 \in \Omega \) which becomes an equilibrium point, we can fix the axis of rotation without loss of generality. So we define

\[
\begin{align*}
x &= Rr, \\
y &= Rp,
\end{align*}
\]

where \( R \) is the \( 3n \times 3n \) block-diagonal matrix, with blocks given by the rotation matrix

\[
A(wt) = \begin{pmatrix} \cos wt & -\sin wt & 0 \\ \sin wt & \cos wt & 0 \\ 0 & 0 & 1 \end{pmatrix},
\]

here \( w \) is the constant angular velocity. In these coordinates, Eqs. (2) can be written as

\[
\begin{align*}
\dot{x} &= Kx + M^{-1}y, \\
\dot{y} &= \nabla U(x) + Ky,
\end{align*}
\]

(4)

where \( K \) is a \( 3n \times 3n \) block diagonal matrix, with blocks of the form

\[
\begin{pmatrix} 0 & -w & 0 \\ w & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

As usual, the equilibrium points of (4) are solutions of the system

\[
Kx + M^{-1}y = 0, \quad \nabla U(x) + Ky = 0.
\]

(5)

From here

\[
y = -MKx \quad \text{and} \quad \nabla U(x) - KMKx = 0.
\]

Let us observe that \( KM = MK \) and \( K^2 = -w^2E \), where \( E \) is the block-diagonal matrix, with blocks

\[
I_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

Then the equilibrium points of (4) are given by

\[
\{(x, y) \mid M^{-1}\nabla U(x) + w^2Ex = 0, \quad y = -MKx\}.
\]

(6)

**Definition 2.** A configuration \( x \in \Omega \) is called a relative equilibrium if it satisfies the equation

\[
M^{-1}\nabla U(x) + w^2Ex = 0.
\]

(7)

The first result about relative equilibria is the following:
Proposition 1. In the charged $n$-body problem, if every $\lambda_{ij}$ is positive, then all relative equilibria are planar.

Proof. Let $x \in \Omega$ be a relative equilibrium. From (7) and the definition of $E$ it is clear that
\[
\nabla_j U(x) \in \mathbb{R}^2 \times \{0\}, \quad j = 1, \ldots, n, \quad (8)
\]
where $\nabla_j U(x)$ denotes the $j$th partial derivative of $U$, and $x = (x_1, \ldots, x_n)$. Let the $j$th coordinate of $x$ be given by $x_j = (x_{j1}, x_{j2}, x_{j3})$; then (8) can be written as
\[
\sum_{i \neq j} \lambda_{ij} r_{ij}^{-3} (x_{i3} - x_{j3}) = 0. \quad (9)
\]
Suppose $x_j$ is at a maximum distance from the plane of rotation, then $x_{i3} - x_{j3}$ in last equation cannot have different signs for all $i \neq j$. Since every parameter $\lambda_{ij}$ is positive, (9) implies $x_{13} = \cdots = x_{n3}$. Then $x$ is a planar configuration. \(\square\)

As another curiosity of relative equilibria in the charged $n$-body problem, let us observe that multiplying Eq. (7) by the factor $x^t M$, we get the scalar equation
\[
x^t \nabla U(x) + w^2 x^t M Ex = 0. \quad (10)
\]
Then considering the fact that $U$ is a homogeneous function with degree of homogeneity $-1$, Euler’s theorem implies
\[
w^2 = \frac{U(x)}{x^t M Ex}. \quad (11)
\]
So, in order to generate a relative equilibrium from a C.C., it is enough to compare the equations given in Definitions 1 and 2, using (11). Both equations have the same solutions if and only if $U(x) > 0$ and $Ex = x$; the last one implies that $x$ is planar. We have proved the following result, which answers the first question at the beginning of this section.

Proposition 2. All relative equilibria $x \in \Omega$ satisfy the condition $U(x) > 0$. Moreover, a central configuration generates a relative equilibrium if it is planar and the respective potential is positive.

The kinetic energy $T$ along relative equilibria takes the form
\[
T = \frac{1}{2} \sum_{i=1}^{n} m_i^{-1} |p_i|^2 = \frac{1}{2} \sum_{i=1}^{n} m_i^{-1} y_i' y_i,
\]
but (5) implies $y_i' y_i = w^2 m_i^2 x_i' I_0 x_i$. Then using (11) we get $T = \frac{1}{2} U$ along relative equilibria, and from the energy relation (3), $U = -2h$. So, by the above proposition we obtain the following result.

Proposition 3. The periodic orbits of relative equilibria $\varphi(\tau) = R(w\tau)x$ lie on negative energy levels.
We finish this section by showing an example of a spatial relative equilibrium for a particular charged problem in $\mathbb{R}^3$. This was a surprise for us, since this phenomena never happens in the classical Newtonian problem, where all relative equilibria are planar, that is all them are in $\mathbb{R}^2$, [24] and [12]. In charged problems, with the existence of more forces among particles it is possible to get the perfect balance with centrifugal forces even in the space $\mathbb{R}^3$. This fact answers the second question posed at the beginning of this section.

We consider six point particles in the space, four of them are located at the vertices of a square in a plane, and the other two are symmetrically located on the orthogonal axis passing through the center of the square (see Fig. 1).

The configuration $x = (x_1, \ldots, x_6)$ is given by

\[
x_1 = -x_2 = (1, 0, 0), \quad x_3 = -x_4 = (0, 1, 0), \quad x_5 = -x_6 = (0, 0, c),
\]

where $c \in \mathbb{R}^+$. Let masses and charges be such that

\[
m_1 = m_2 = 1, \quad m_3 = m_4 = m, \quad m_5 = m_6 = \mu,
\]

\[
q_1 = q_2 = q, \quad q_3 = q_4 = m + 1 - q, \quad q_5 = q_6 = \mu,
\]

(12)

where $m \in \mathbb{R}^+$, $\mu \in \mathbb{R}^+$, and $q \in \mathbb{R}$. Then the parameters $\lambda_{ij} = m_i m_j - q_i q_j$ are given by

\[
\lambda_{12} = 1 - q^2, \quad \lambda_{34} = (1 - q)(q - 2m - 1), \quad \lambda_{56} = 0,
\]

\[
\lambda_{ik} = \mu(1 - q), \quad \lambda_{ij} = (1 - q)(m - q), \quad \lambda_{jk} = -\mu(1 - q),
\]

(13)

for $i = 1, 2$, $j = 3, 4$, and $k = 5, 6$. Then $\nabla U(x)$ takes a simple form given by

\[
\nabla_1 U = -\nabla_2 U = -\left(\frac{\lambda_{12}}{4} + \frac{\lambda_{13}}{\sqrt{2}} + \frac{2\lambda_{15}}{(c^2 + 1)^{3/2}}\right)x_1,
\]

\[
\nabla_3 U = -\nabla_4 U = -\left(\frac{\lambda_{34}}{4} + \frac{\lambda_{13}}{\sqrt{2}} + \frac{2\lambda_{35}}{(c^2 + 1)^{3/2}}\right)x_3,
\]

\[
\nabla_5 U = -\nabla_6 U = 0.
\]

(14)

This shows that $\nabla_j U(x) \in \mathbb{R}^2 \times \{0\}$ for $j = 1, \ldots, 6$, which is essential to get non-planar relative equilibria, because the product $Ex$ in (7) projects any configuration to the plane. Using again
Eq. (7) and the expressions for the partial derivatives of $U$ given in (14), in order to find the relative equilibria in this example we must solve the system:

\[
\begin{align*}
\frac{1-q^2}{4} + \frac{(1-q)(m-q)}{\sqrt{2}} + \frac{2\mu(m-q)}{(1-c^2)^{3/2}} - \omega^2 &= 0, \\
\frac{(1-q)(q-2m-1)}{4} + \frac{(1-q)(m-q)}{\sqrt{2}} + \frac{2\mu(1-q)}{(1-c^2)^{3/2}} - m\omega^2 &= 0.
\end{align*}
\tag{15}
\]

Solving for $\omega^2$ in the first equation of (15) and substituting in the second, we reduce system (15) to solve just one scalar equation given by

\[
\frac{(m-1)(q+1) + 2(m+1)}{4} + \frac{(m-1)(m-q)}{\sqrt{2}} + \frac{2\mu(m+1)}{(c^2+1)^{3/2}} = 0.
\tag{16}
\]

Finally, as the third part in the left-hand side of Eq. (16) is always positive, the sum of the first two parts must be negative, then $m$ and $q$ must satisfy

\[
qu > \frac{\sqrt{8}m^2 + (3 - \sqrt{8})m + 1}{(m-1)(\sqrt{8} - 1)}.
\tag{17}
\]

Choosing $m$ and $q$ such that (17) holds, we obtain a negative value for the sum of the first two parts in (16), then with this value we choose the values of $\mu$ and $c$. For example fixing $m = 1/2$ and $q = -100$, the sum of the first two parts is $-9.46$, then we can take $c^2 = 3$ and $\mu = 8(9.46)/3$ to obtain a spatial relative equilibrium.

Let us observe that the existence of relative equilibria in the above example is independent of the value of the angular velocity $\omega$.

**Remark 1.** The trick to obtain a relative equilibrium in the above example is to use many symmetries and choose the value of the parameters such that $\lambda_{56} = 0$, because in this case we can annul the interaction between particles 5 and 6. We do not know if there is an example of relative equilibrium with $\lambda_{56} \neq 0$. Should be nice to give a classification of the linear stability in this example in terms of the parameters, we let this question for a further paper.

### 3. Stability of relative equilibria

In this section we follow the Moeckel’s ideas given in [14] to obtain a nice factorization of the characteristic polynomial of the linear part of (4). The linearized vector field of (4) around an equilibrium point determines a linear Hamiltonian system $\dot{z} = Lz$, $z \in \mathbb{R}^6n$, where

\[
L = \begin{pmatrix} K & M^{-1} \\ D\nabla U(x) & K \end{pmatrix}.
\tag{18}
\]

The Hamiltonian function is given by the product $G = \hat{J}L$, where $\hat{J}$ is the $6n \times 6n$ matrix

\[
\hat{J} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}.
\]
Let \( p(\lambda) \) be the characteristic polynomial of \( L \). Then using the fact that \( \hat{J}^{-1} = -\hat{J} \) and that \( \hat{J}^2 = -I \), we obtain

\[
p(\lambda) = |L - \lambda I| = | -\hat{J}G + \lambda \hat{J}^2 | = | -\hat{J}(G - \lambda \hat{J}) |
\]

that is, \( p(\lambda) \) is an even function. Therefore, if \( \lambda \) is a root of \( p(\lambda) \), then \( -\lambda, \bar{\lambda} \) and \( -\bar{\lambda} \) also are roots of \( p(\lambda) \), where as usual \( \bar{\lambda} \) denotes the complex conjugate of \( \lambda \). Therefore, if \( L \) has at least one eigenvalue at an equilibrium point with non-zero real part, then necessarily \( L \) has one eigenvalue with positive real part, and therefore the equilibrium point is unstable. Then a necessary condition to get linear stability is that all the corresponding eigenvalues are either zero or purely imaginary.

**Definition 3.** A relative equilibrium is spectrally stable if all roots of the corresponding characteristic polynomial \( p(\lambda) = 0 \) satisfy \( \lambda^2 \leq 0 \).

The equation \( Lu = \lambda u, \, u = (u_1, u_2) \in \mathbb{R}^{6n} \), can be written as

\[
Ku_1 + M^{-1} u_2 - \lambda u_1 = 0,
\]

\[
D\nabla U(x)u_1 + Ku_2 - \lambda u_2 = 0.
\]

Since \( u = (u_1, u_2) \neq 0 \), the eigenvalues \( \lambda \) must annul the determinant of the \( 3n \times 3n \) matrix

\[
A = M^{-1} D\nabla U(x) + 2\lambda K + w^2 E - \lambda^2 I.
\]

Therefore the polynomial

\[
p_1(\lambda) = \det A(\lambda) = 0
\]

has the same roots that \( p(\lambda) \). So, a relative equilibrium in the charged \( n \)-body problem is spectrally stable if the roots of \( p_1(\lambda) \) are either zero or purely imaginary, that is, if \( \lambda^2 \leq 0 \).

In order to solve (22), we use linear algebra to find vectorial subspaces \( A \)-invariant, and get a factorization of \( p_1(\lambda) \). In (21), the blocks \( B_{jk} \) of the matrix \( B = M^{-1} D\nabla U(x) \) are given by the \( 3 \times 3 \) matrices

\[
B_{jk} = m_j^{-1} \lambda_{jk} r_{jk}^{-3} (I - 3u_{jk}u_{jk}^t) \quad \text{if} \quad j \neq k,
\]

\[
B_{jj} = -\sum_{k \neq j} B_{jk},
\]

where \( u_{jk} \in \mathbb{R}^3 \) is the unit vector \( u_{jk} = (x_j - x_k)r_{jk}^{-1} \).

From here on, since we are particularly interested on relative equilibria of the charged three-body problem, we will restrict our analysis to planar configurations, i.e., we will assume \( x_k \in \mathbb{R}^2 \setminus \{0\} \), for \( k = 1, \ldots, n \). In this case, the \( B_{jk} \)-blocks take the form

\[
B_{jk} = \begin{pmatrix} L_{jk} & 0 \\ 0 & c_{jk} \end{pmatrix},
\]
where $L_{jk}$ is a $2 \times 2$ matrix given by
\[
L_{jk} = m_j^{-1} \lambda_{jk} r_{jk}^{-3} (I - 3u_j u'_k) \quad \text{if} \quad j \neq k,
\]
\[
L_{jj} = -\sum_{k \neq j} L_{jk}.
\]
(23)

The corresponding scalar $c_{jk}$ is
\[
c_{jk} = m_j^{-1} \lambda_{jk} r_{jk}^{-3} \quad \text{if} \quad j \neq k,
\]
\[
c_{jj} = -\sum_{k \neq j} c_{jk}.
\]
(24)

Reordering the coordinates as $(x_{11}, x_{12}, \ldots, x_{n1}, x_{n2}, x_{13}, \ldots, x_{n3})$, we obtain a block-diagonal form for the matrix $A$,
\[
A = \text{diag}(A_1, A_2),
\]
where the $2n \times 2n$ matrix $A_1$ and the $n \times n$ matrix $A_2$ are given by
\[
A_1 = L + 2\lambda J + (w^2 - \lambda^2) I,
\]
\[
A_2 = C - \lambda^2 I.
\]
(25)

The block matrices $L = (L_{jk})$ and $C = (c_{jk})$ have been defined in (23) and (24); $J$ is the usual block matrix,
\[
\begin{pmatrix}
0 & -wI \\
wI & 0
\end{pmatrix}.
\]

Since the determinant of a matrix is invariant under change of coordinates, we get the following factorization for the polynomial $p_1(\lambda)$:
\[
p_1(\lambda) = \det A_1(\lambda) \cdot \det A_2(\lambda) = 0.
\]
(26)

Then spectral stability is divided in two components: planar spectral stability if $\lambda^2 \leq 0$ for every root $\lambda$ of $\det A_1(\lambda) = 0$, and normal spectral stability if $\lambda^2 \leq 0$ for every root $\lambda$ of $\det A_2(\lambda) = 0$. If one of them fails, then the relative equilibrium is unstable.

3.1. Normal spectral stability

By (25), the condition $\lambda^2 \leq 0$ in $\det A_2 = 0$ is equivalent to showing that all the eigenvalues $\mu$ of $C$ are real and non-positive. Let us observe that $C$ is a symmetric matrix with respect to the inner product $\langle u, v \rangle = u' M v$, therefore all eigenvalues $\mu$ of $C$ are real. Another observation is that $\tilde{w} = (1, \ldots, 1)' \in \text{Ker}(C) \subset \mathbb{R}^n$, that is, $\mu = 0$ is always an eigenvalue for any relative equilibrium. This eigenvalue corresponds to first integral of the center of mass. If the remainder $n - 1$ eigenvalues are negative, we say that normal spectral stability is non-degenerate.
For the charged \( n \)-body problem with \( \lambda_{jk} > 0 \) for \( j \neq k \), the attractive case, we know from the equation \( Cu = \mu u, u \in \mathbb{R}^n \), that

\[
\sum_{k \neq j} c_{jk} (u_k - u_j) = \mu u_j .
\] (27)

Then the choice of \( u_j \) as the component of \( u \) with larger absolute value (\( u_j > 0 \) without loss of generality) implies \( u_k - u_j \leq 0 \) for all \( k \neq j \), but the coefficients \( c_{jk} \) are positive, so in this case the remainder \( n - 1 \) eigenvalues \( \mu \) are negative. We have thus proved the following result.

**Proposition 4.** In the charged \( n \)-body problem, if \( \lambda_{jk} > 0 \) for \( k \neq j \), then relative equilibria are normal spectrally stable and non-degenerate.

### 3.2. Planar spectral stability

As before, the idea here is factorize the polynomial \( \det A_1 = 0 \), where \( A_1 \) is a linear transformation \( A_1 : \mathbb{R}^{2n} \to \mathbb{R}^{2n} \). The main idea is to decompose \( \mathbb{R}^{2n} \) along invariant subspaces \( A_1 \)-invariant. By (25) we know that a subspace \( S \) is \( A_1 \)-invariant, if \( S \) is invariant under \( L \) and \( J \) simultaneously. Since the polynomial \( \det A_1(\lambda) \) is an even function, the dimension of \( S \) is necessarily even. The simplest case would be dimension two. Moreover, as we will see, \( S \) can be generated by two eigenvectors of \( L \).

It is easy to check that \( L \) is symmetric and \( J \) is antisymmetric with respect to the inner product \( \langle v, w \rangle = v^T M w \), so we obtain \( \langle u, Ju \rangle = 0 \) for any vector \( u \in \mathbb{R}^{2n} \), i.e., \( u \) and \( Ju \) are \( M \)-orthogonal. The symmetry of \( L \) also implies that the eigenvalues of \( L \) must be real numbers.

Since \( S \), is an \( L \)-invariant space, it contains an eigenvector. So \( Lu = au \) with \( u \in S \) and \( a \in \mathbb{R} \) (\( L \) eigenvalues are real). But \( S \) is also \( J \)-invariant, then \( Ju \in S \), and both vectors, \( u \) and \( Ju \), generate \( S \). Of course, the product \( LJ u \) is a linear combination of them, but \( L \)-properties let us to conclude that it only depends of the second one, that is, \( LJ u = bJu \) with \( b \in \mathbb{R} \).

The subspace \( S \) generated by \( u \) and \( Ju \) is \( A_1 \)-invariant, and the restriction of \( A_1 \) to \( S \) in the basis \( \{ u, Ju \} \), is given by

\[
\begin{pmatrix}
a & 0 \\
0 & b
\end{pmatrix} + 2\lambda \begin{pmatrix}
0 & -w^2 \\
1 & 0
\end{pmatrix} + (w^2 - \lambda^2) \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix} .
\] (28)

The determinant \( Q \) of the above matrix is a fourth-order factor for the polynomial \( \det A_1 = 0 \), that is,

\[
Q = \lambda^4 + \alpha \lambda^2 + \beta ,
\] (29)

where \( \alpha = 2w^2 - a - b \) and \( \beta = (a + w^2)(b + w^2) \); the constants \( a \) and \( b \) are eigenvalues of \( L \) associated to the eigenvectors \( u \) and \( Ju \), respectively. The roots of \( Q(\lambda) = 0 \) satisfy the condition for spectral stability \( \lambda^2 \leq 0 \), in the following cases:

\[
\alpha > 0, \beta > 0 \text{ and } \alpha^2 - 4\beta \geq 0 \implies \lambda^2 \leq 0 ,
\]
\[
\alpha > 0 \text{ and } \beta = 0 \implies \lambda_1^2 < 0, \lambda_2^2 = 0 ,
\]
\[
\alpha = 0 \text{ and } \beta = 0 \implies \lambda = 0 .
\] (30)
In order to factorize the polynomial $\det A_1 = 0$, we apply the above result several times. Thus, observe that the particular vectors $u = (1, 0, \ldots, 1, 0) \in \mathbb{R}^{2n}$ and $Ju$ are both in $\ker(L)$, that is, $u$ and $Ju$ are eigenvectors of $L$ with eigenvalues $a = b = 0$. Then (29) is reduced to

$$Q_1 = (\lambda^2 + w^2)^2.$$  

Further let $x$ be a relative equilibrium. Then Euler theorem implies

$$Lx = M^{-1}D\nabla U(x)x = M^{-1}(-2)\nabla U(x) = (-2)(-w^2x) = 2w^2x,$$

that is, $x$ is an eigenvector of $L$ with eigenvalue $2w^2$. Now if $R(\theta)$ is the rotation matrix around the center of mass, then the distances among particles do not change, so $\nabla U(R(\theta)x) = R(\theta)\nabla U(x)$, and

$$L(Jx) = M^{-1}J\nabla U(x) = JM^{-1}\nabla U(x) = J(-w^2x) = -w^2(Jx).$$

Substituting in (29), the eigenvalues $a = 2w^2$ and $b = -w^2$ corresponding to the eigenvectors $x$ and $Jx$, we obtain a second factor $Q_2$ for the polynomial $\det A_1(\lambda)$ given by

$$Q_2 = \lambda^2(\lambda^2 + w^2).$$

We have thus proved the following result.

**Proposition 5.** For every planar relative equilibrium, the polynomial $\det A_1 = 0$ can be factorized as

$$\det A_1 = Q_1Q_2Q_3 = \lambda^2(\lambda^2 + w^2)^3Q_3(\lambda) = 0,$$

where $Q_3$ is a polynomial of degree $4(n - 2)$.

If all roots of $Q_3$ satisfy $\lambda^2 < 0$, we say that the planar relative equilibrium is spectrally stable and non-degenerate.

4. The charged 3-body problem

In the previous paper [15], we did a total classification of collinear and planar C.C. in the charged 3-body problem. This section comes as a natural continuation of that paper. Here we study linear stability of the corresponding relative equilibria.

We introduce the rate charge-mass $\delta_i = q_i/m_i$, $i = 1, 2, 3$, to define $\delta_{jk}$ as

$$\delta_{jk} = \frac{\lambda_{jk}}{m_jm_k} = 1 - \delta_j\delta_k, \quad \text{for } j \neq k.$$

In order to get non-collinear C.C., all $\delta_{jk}$ must have the same sign [15], and by Proposition 2, the potential $U$ must be positive, therefore non-collinear relative equilibrium corresponds to choice the parameters as

$$\delta_{12} > 0, \quad \delta_{13} > 0, \quad \delta_{23} > 0.$$  

(31)
In other words, planar relative equilibria exist just for the attractive charged problem, and they are given by

\[ r_{jk} = \sqrt{\frac{\delta_{jk}}{\delta_{12}}}, \quad j \neq k. \]  \hfill (32)

The above formula is obtained by using Jacobi coordinates and then computing \( \nabla (\tilde{I}U^2) = 0 \), where \( \tilde{I} \) is the moment of inertia of the system, see [15] for a complete deduction of formula (32). Let us observe that if all charges are zero, then \( r_{jk} = 1, \forall j \neq k \), obtaining the Lagrange relative equilibrium of the classical 3-body problem.

As we will see in the next proposition, using formula (32) it is easy to verify that we can have triangular relative equilibrium, where the triangle has any shape. This is very different from the Newtonian three-body problem, which yields only equilateral triangles [12].

**Proposition 6.** In the charged three-body problem, given a triangle of any shape \( \Gamma \), there exist masses and charges, such that \( \Gamma \) represents a relative equilibrium.

**Proof.** Without lost of generality, let \( \Gamma \) be a triangle such that \( r_{12} = 1, r_{13} > 1 \) and \( r_{23} > 1 \). Then we will prove the existence of parameters \( \delta_1, \delta_2 \) and \( \delta_3 \) that solve the system (31)–(32).

Let \( k_0 \) be a constant defined as

\[ k_0 = \min \{ 1 - r_{13}^{-3}, 1 - r_{23}^{-3} \}, \]

then \( 0 < k_0 < 1 \). Now, let \( B : (0, k_0) \to \mathbb{R} \) and \( C : (0, k_0) \to \mathbb{R} \) be functions defined as

\[ B(k) = r_{12}^3 (1 - k) - 1, \quad C(k) = (r_{23}^3 (1 - k) - 1)k^{-1}. \]

Then

\[ k < k_0 \quad \Rightarrow \quad k < 1 - r_{13}^{-3} \quad \Rightarrow \quad B > 0, \]
\[ k < k_0 \quad \Rightarrow \quad k < 1 - r_{23}^{-3} \quad \Rightarrow \quad C > 0. \]

From here, considering the relation \( \delta_{ij} = 1 - \delta_i \delta_j \), it follows that the arc \( \gamma : (0, k_0) \to \mathbb{R}^3 \) given by

\[ \delta_1(k) = \sqrt{B/C}, \quad \delta_2(k) = k \sqrt{C/B}, \quad \delta_3(k) = -\sqrt{BC}, \]

satisfies the system (31)–(32). \( \Box \)

Now, in order to study normal spectral stability for relative equilibria in the charged three-body problem, we analyze the matrix \( C \) given in (24). By straightforward computations we obtain the characteristic polynomial of \( C \)

\[ q(\mu) = \mu^3 + \alpha \mu^2 + \beta \mu = 0, \]  \hfill (33)

where
\[ \alpha = c_{12} + c_{21} + c_{13} + c_{31} + c_{23} + c_{32}, \]
\[ \beta = c_{12}c_{31} + c_{13}c_{21} + c_{21}c_{32} + c_{23}c_{13} + c_{12}c_{23} + c_{13}c_{21} + c_{13}c_{23}. \]

Therefore, the two non-zero roots in (33) are negative, if \( \alpha > 0 \) and \( \beta > 0 \). We have thus proved the following result.

**Proposition 7.** If the parameters \( \alpha \) and \( \beta \) in (33) are both positive, then a collinear relative equilibrium of the charged three-body problem is normal spectrally stable.

**Remark 2.** The non-collinear relative equilibria only exist if \( \lambda_{12}, \lambda_{13} \) and \( \lambda_{23} \) are positive, then by Proposition 2 all of them are normally spectrally stable.

From here on, we focus our attention on planar spectral stability, which we will simply call spectral stability. According to Proposition 7, determining spectral stability suffices to obtain the linear stability of the respective relative equilibria. In the classical Newtonian three-body problem, it is well known that any collinear relative equilibrium is unstable, whereas the triangular or Lagrangian relative equilibria are linearly stable if the masses satisfy the Routh’s condition \([19]\) (see also \([8]\) for an older reference to this inequality),

\[ 27(m_1m_2 + m_1m_3 + m_2m_3) < (m_1 + m_2 + m_3)^2. \]

In the charged 3-body problem, according to Proposition 5, spectral stability depends of the quartic polynomial \( Q_3 \). To obtain \( Q_3 \), we start getting the eigenvalues of the matrix \( L \) defined in (23). Since \( n = 3 \), the matrix \( L \) has six eigenvalues (all of them real numbers), given by

\[ \{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6\} = \{0, 0, 2w^2, -w^2, a, b\}, \quad (34) \]

where \( a \) and \( b \) are real numbers to be determined. We know that

\[ \text{tr} L = \sum_{i=1}^{6} \lambda_i = w^2 + a + b, \quad (35) \]

therefore

\[ \text{tr}^2 L = \sum_{k=1}^{6} \lambda_k^2 + 2 \sum_{j<k} \lambda_j \lambda_k = \text{tr} L^2 + 2 \sum_{j<k} \lambda_j \lambda_k, \quad (36) \]

and using (34)

\[ \text{tr}^2 L = \text{tr} L^2 - 4w^4 + 2w^2(a + b) + 2ab. \quad (37) \]

By (35), \( a + b = \text{tr} L - w^2 \), so

\[ 2ab = (\text{tr} L - w^2)^2 + 5w^4 - \text{tr} L^2. \quad (38) \]
Finally, substituting these values in (29), we get
\[ Q_3 = \lambda^4 + (3w^2 - \text{tr } L)\lambda^2 + \left( 3w^4 + \frac{1}{2} \text{tr}^2 L - \frac{1}{2} \text{tr } L^2 \right). \] (39)

4.1. Collinear relative equilibria

Since we have obtained the polynomial \( Q_3(\lambda) \), we divide our analysis in two parts: collinear and non-collinear relative equilibria. The first one is studied in this section. We start with one of the main results of this paper.

**Theorem 1.** In the charged three-body problem, a collinear relative equilibrium is linearly stable and non-degenerate if
\[ w - 2 \text{tr } C \in (-2, -17/9] \cup [-1, -1/2). \] (40)

For \( w - 2 \text{tr } C = -2 \) or \( w - 2 \text{tr } C = -1/2 \), the relative equilibrium is degenerate.

**Proof.** Let \((s_1, 0), (s_2, 0), (s_3, 0) \in \mathbb{R}^3 \times \{0\}\) be a collinear configuration, where we assume \( s_1 < s_2 < s_3 \). Since \( u_{jk} = (\pm 1, 0) \) by (23), the blocks \( L_{jk} \) are diagonal. Therefore, applying the change of coordinates \((s_{11}, s_{12}, s_{21}, s_{22}, s_{31}, s_{32}) \rightarrow (s_{11}, s_{21}, s_{31}, s_{12}, s_{22}, s_{32})\), the matrix \( L \) converts to block-diagonal
\[ L = \text{diag}\{-2C, C\}, \] (41)
where \( C = (c_{jk}) \) has been defined in (24). Then by (41), it follows that \( Cu = \lambda u \) which implies that \( Lx = \lambda x \), where \( x = (0, u')' \). We have seen previously that if \( x \) is an eigenvector of \( L \), then \( Jx \) also is an eigenvector of \( L \), where \( J \) is the \( 6 \times 6 \) matrix
\[ J = w \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}, \]
then \( L(Jx) = -2\lambda(Jx) \). Using this relationship \(-2 : 1\) between the eigenvalues corresponding to the eigenvectors \( Jx \) and \( x \), and the fact that \( \text{tr } L = -\text{tr } C \), we obtain
\[ a = w^2 + \text{tr } C, \quad b = -2a. \] (42)
Then by (29), the polynomial \( Q_3 \) becomes
\[ Q_3(\lambda) = \lambda^4 + (3w^2 + \text{tr } C)\lambda^2 - (2w^2 + \text{tr } C)(w^2 + 2\text{tr } C). \] (43)
Since the roots \( \lambda \) of \( Q_3 \) must satisfy \( \lambda^2 \leq 0 \) to have linear stability, applying the inequalities (30), we obtain the theorem. \( \square \)

Now, to study the expression \( w^{-2} \text{tr } C \) in order to find linearly stable relative equilibria, we consider a collinear configuration \( (s_1, s_2, s_3) \in \mathbb{R}^3 \), such that \( s_1 < s_2 < s_3 \). Without loss of generality, let \( s_3 - s_1 = 1 \). If we define \( x = (s_2 - s_1)^{-1} \) and \( y = (s_3 - s_2)^{-1} \) (see Fig. 2), Eq. (7) for relative equilibria takes the form
where

\[
x^{-1} + y^{-1} = 1, \quad x > 1, \quad y > 1.
\] (45)

Adding the equations in (44) and using (45), we obtain

\[
w^2 = m_2 \delta_{12} x^2 + m_2 \delta_{23} y^2 + m_1 \delta_{13} + m_3 \delta_{13}.
\] (46)

The trace of the matrix \( C \), (24) is

\[
\text{tr} C = c_{11} + c_{22} + c_{33} = -c_{12} - c_{13} - c_{21} - c_{23} - c_{31} - c_{32},
\]

\[-\text{tr} C = \delta_{12} (m_1 + m_2) x^3 + \delta_{23} (m_2 + m_3) y^3 + \delta_{13} (m_1 + m_3),
\] (47)

and using (44), we obtain

\[-\text{tr} C = 2w^2 + m_1 (\delta_{13} - \delta_{13} y + \delta_{12} x^2 y) + m_3 (\delta_{13} - \delta_{13} x + \delta_{23} x y^2).
\]

By (45), we have \( x^2 y = x^2 + x + y \), and \( xy^2 = y^2 + y + x \), therefore

\[
w^{-2} \text{tr} C = -2 - \frac{c_1 p(x) + c_2 p(y) + c_3 q(x) + c_4 q(y)}{m_2 \delta_{12} x^2 + m_2 \delta_{23} y^2 + m_1 \delta_{13} + m_3 \delta_{13}},
\] (48)

where

\[
p(x) = x^2 + x + 1, \quad q(x) = x - 1,
\]

and

\[
c_1 = m_1 \delta_{12}, \quad c_2 = m_3 \delta_{23}, \quad c_3 = m_3 (\delta_{23} - \delta_{13}), \quad c_4 = m_1 (\delta_{12} - \delta_{13}).
\]

We end this subsection with analyzing the conditions for linear stability in two particular examples.
Example 1. Let masses and charges be such that $\delta_{12} = \delta_{13} = \delta_{23} > 0$. Observe that this case contains the classical Newtonian three-body problem, where all the parameters $\delta_{jk}$ are equal to 1. Then (48) implies

$$w^{-2} \text{tr } C = -2 - \frac{m_1 p(x) + m_3 p(y)}{m_2 x^2 + m_2 y^2 + m_1 + m_3}.$$ 

Since $p$ and $q$ are positive functions, we have

$$w^{-2} \text{tr } C < -2,$$ 

then by Theorem 1, we conclude that collinear relative equilibria are unstable for $\delta_{12} = \delta_{13} = \delta_{23} > 0$, in particular, as it is well known, collinear relative equilibria in the Newtonian 3-body problem are unstable.

Example 2. Suppose a symmetric distribution of masses and charges

$$m_1 = m_3 = 1, \quad q_1 = q_3 \in \mathbb{R}, \quad m_2 = m \in \mathbb{R}^+, \quad q_2 \in \mathbb{R},$$ 

then $\delta_{12} = \delta_{23}$. After eliminating $w$ in (44), collinear relative equilibria are given by

$$(y - x)\left[\delta_{12}(m + 1)(x^2 + xy + y^2) + \delta_{12}xy + \delta_{13}\right] = 0,$$ 

$$U = m\delta_{12}(x + y) + \delta_{13} > 0,$$ 

where we assume that $x$ and $y$ satisfy (45). A natural solution of the above system is the symmetric collinear relative equilibrium $x = y = 2$, where masses and charges satisfy $U = 4m\delta_{12} + \delta_{13} > 0$. In this case, Eq. (48) becomes

$$w^{-2} \text{tr } C = -\frac{8(m + 1)\delta_{12} + \delta_{13}}{4m\delta_{12} + \delta_{13}}.$$ 

Applying Theorem 1 to this relative equilibrium, it follows that it is linearly stable and non-degenerate if $w^{-2} \text{tr } C \in [-1, -1/2)$, and this holds provided that

$$\delta_{12} < 0, \quad \delta_{13} > 0, \quad -\frac{\delta_{13}}{\delta_{12}} > 12m + 16;$$

another possibility is $w^{-2} \text{tr } C \in (-2, -17/9]$, which holds if masses and charges satisfy

$$\delta_{12} > 0, \quad \delta_{13} > 0, \quad 8 < -\frac{\delta_{13}}{\delta_{12}} \leq 9 + \frac{m}{2}.$$ 

To complete the above, it is necessary to analyze normal spectral stability. Applying Proposition 7, we conclude that only for the second group of inequalities the symmetric collinear relative equilibria is normal and planar non-degenerate spectrally stable. Observe that this is impossible in the classical Newtonian $n$-body problem, where all collinear relative equilibria are unstable [12].
The non-symmetric solutions of (51) correspond to \( x \neq y \) and parameters \( \delta_{12} \) and \( \delta_{13} \) of opposite sign. In this case, (51) implies that

\[
x + y = xy = P > 4,
\]

where

\[
P = \frac{m + \sqrt{m^2 + 4(m + 1)Q}}{2(m + 1)}, \quad Q = -\frac{\delta_{13}}{\delta_{12}} > 0.
\]  

(52)

Then masses and charges for the non-symmetric collinear relative equilibrium must satisfy the inequalities \( P > 4 \) and \( U > 0 \), that is,

\[
\delta_{12} < 0, \quad \delta_{13} > 0, \quad P > 4.
\]

In order to analyze the spectral stability, observe that

\[
p(x) + p(y) = P^2 - P + 2, \quad q(x) + q(y) = P - 2, \quad x^2 + y^2 = P^2 - 2P,
\]

then Eq. (48) takes the form

\[
-w^{-2} \text{tr } C = -2 + \frac{P^2 - P + 2 + (Q + 1)(P - 2)}{2Q - m(P^2 - 2P)}.
\]

By (52), we get \( Q = P^2(m + 1) - mP \), then

\[
-w^{-2} \text{tr } C = -\frac{1}{2} + \frac{(P - 4)[m(P - \frac{1}{2}) + P]}{m(P + 2)} > -\frac{1}{2}.
\]

Therefore, applying Theorem 1, we conclude that every non-symmetric collinear relative equilibrium in this example is unstable.

4.2. Non-collinear relative equilibria

We have seen in Proposition 6 that for any triangular shape \( \Gamma \), it is possible to find masses and charges such that \( \Gamma \) represents a relative equilibrium. In this subsection we are interested in studying its stability.

Since non-collinear relative equilibria of the three-body problem exist only if \( \lambda_{12}, \lambda_{13} \) and \( \lambda_{23} \) are positive, Proposition 4 implies that normal spectral stability always holds. So we just need to prove planar spectral stability. Let us start with the following definition.

**Definition 4.** Let \( \theta_i \) be the interior angle corresponding to the vertex containing the \( i \)th particle in the triangle formed by three particles \( i, j, k \) (see Fig. 3).

As done previously, the idea is to analyze the polynomial \( Q_3(\lambda) \) given in (39), and the nature of its roots. Since we are interested in comparing our results with the classical ones for the Newtonian problem, we state our main result as follows:
Theorem 2. In the charged three-body problem a non-collinear relative equilibrium is linearly stable and non-degenerate if and only if the masses and charges satisfy the condition

\[ 36(m_1m_2 \sin^2 \theta_3 + m_1m_3 \sin^2 \theta_2 + m_2m_3 \sin^2 \theta_1) < (m_1 + m_2 + m_3)^2. \]

Proof. We start by computing \( w^2 \) for a non-collinear relative equilibrium \( x \). Since \( x \) is planar, (11) implies \( w^2 = U(x)/x^tMx \). The potential \( U \) is given by

\[ U(x) = m_1m_2 \delta_{12} + m_1m_3 \delta_{1/3} \delta_{1/3} + m_2m_3 \delta_{1/3} \delta_{1/3}. \]

Since the center of mass is fixed at the origin, we have

\[ x^tMx = \frac{m_1m_2 r_{12}^2 + m_1m_3 r_{13}^2 + m_2m_3 r_{23}^2}{m_1 + m_2 + m_3}, \]

and using (32)

\[ x^tMx = \frac{m_1m_2 \delta_{12} + m_1m_3 \delta_{1/3} \delta_{1/3} + m_2m_3 \delta_{1/3} \delta_{1/3}}{\delta_{12}(m_1 + m_2 + m_3)}. \]

Therefore

\[ w^2 = \delta_{12}(m_1 + m_2 + m_3). \] (53)

Now, coming back to the matrix \( L \) defined in (23), we have

\[ \text{tr} L = \text{tr}(L_{11} + L_{22} + L_{33}) = -\text{tr}(L_{12} + L_{13} + L_{21} + L_{23} + L_{31} + L_{32}), \]

where \( L_{jk} = m_k D_{jk} \) and \( D_{jk} = \delta_{jk} r_{jk}^{-3}(I - 3u_{jk}u_{jk}^t) \) for \( j \neq k \), then by (32)

\[ D_{jk} = \delta_{12}(I - 3u_{jk}u_{jk}^t), \quad j \neq k. \] (54)

Since \( u_{jk} \) is a unit vector in \( \mathbb{R}^2 \), and \( \text{tr}(uu^t) = |u|^2 \) for any \( u \in \mathbb{R}^2 \), we have

\[ \text{tr}(u_{jk}u_{jk}^t) = 1, \quad j \neq k. \] (55)
and from here \( \text{tr} D_{jk} = -\delta_{12}, \text{tr} L_{jk} = -m_k \delta_{12} \). Therefore
\[
\text{tr} L = 2(m_1 + m_2 + m_3) \delta_{12}.
\]
(56)

Using the fact that \( L_{jk} \) and \( L_{kj} \) commute, we obtain
\[
\text{tr} L^2 = \text{tr} P,
\]
where \( P \) is a \( 2 \times 2 \) matrix given by
\[
P = (L_{12} + L_{21})^2 + (L_{13} + L_{31})^2 + (L_{23} + L_{32})^2 + 2(L_{12}L_{13} + L_{21}L_{23} + L_{31}L_{32}).
\]
Since \( L_{jk} = m_k D_{jk} \), we get
\[
P = (m_1 + m_2)^2 D_{12}^2 + (m_1 + m_3)^2 D_{13}^2 + (m_2 + m_3)^2 D_{23}^2 + 2m_2m_3 D_{12} D_{13} + 2m_1m_3 D_{21} D_{23} + 2m_1m_2 D_{31} D_{32}.
\]
(57)

By (54) and (55) we have
\[
\text{tr} D_{jk} D_{jl} = \delta_{12}^2(-4 + 9 \text{tr}(u_{jk} u_{jk}')(u_{ji} u_{ji}')).
\]
now, using the fact that \( \text{tr}(uu')(vv') = (u \cdot v)^2 \) for all \( u, v \in \mathbb{R}^2 \), we obtain
\[
\delta_{12}^2 \text{tr} D_{jk} D_{jl} = -4 + 9(u_{jk} \cdot u_{jl})^2.
\]
Let \( \theta_j = \text{ang}\{u_{jk}, u_{jl}\} \), then \( u_{jk} \cdot u_{jl} = \cos \theta_j \) and
\[
\delta_{12}^2 \text{tr} D_{jk} D_{jl} = 5 - 9 \sin^2 \theta_j,
\]
and for \( k = l \)
\[
\text{tr} D_{jk}^2 = 5 \delta_{12}^2.
\]
Therefore
\[
\text{tr} L^2 = \text{tr} P = 10 \delta_{12}^2 \left( \sum_{i=1}^{3} m_i \right)^2 - 18 \delta_{12}^2 \sum_{i \neq j \neq k} m_i m_j \sin^2 \theta_k.
\]
(58)

Finally, we substitute (53), (56) and (58) in (39) to get
\[
Q_3(\lambda) = \lambda^4 + \left( \delta_{12} \sum m_i \right) \lambda^2 + \left( 9 \delta_{12} \sum m_i m_j \sin^2 \theta_k \right).
\]
To conclude the proof, it is enough to apply the inequalities given in (30). \( \square \)

In order to provide some remarks and implications of the above theorem, we start with an example.
Example 3. This is a continuation of Example 1, but here we are interested in non-collinear relative equilibria. Let masses and charges be such that $\delta_{12} = \delta_{13} = \delta_{23} > 0$. By (32), it follows that the triangular shape of the relative equilibrium is equilateral, just as in the Newtonian case, where $\delta_{12} = \delta_{13} = \delta_{23} = 1$. Applying Theorem 2, we get that non-collinear relative equilibria are stable if the inequality

$$27(m_1m_2 + m_1m_3 + m_2m_3) < (m_1 + m_2 + m_3)^2$$

holds.

This is the same inequality of the Newtonian case, but here we are considering masses and charges, regardless of the lack of the charges in (59). Therefore, as in the Newtonian case, we can establish the same conclusions: If the three masses are equal and $\delta_{12} = \delta_{13} = \delta_{23} > 0$, then the relative equilibrium is unstable, but if one of the masses is dominant, then it is stable.

What happens in general for any triangular shape? We know from (32), that triangular shapes depend on the ratios $q_1/m_1$, $q_2/m_2$ and $q_3/m_3$, and by Proposition 6, given any triangular shape, there exist masses and charges such that this configuration leads to a relative equilibrium. Then, as a direct consequence of Theorem 2, we have the following result.

Proposition 8. In the charged three-body problem, given any triangular configuration $\Gamma$, there exist masses and charges such that $\Gamma$ represents a linearly stable and non-degenerate relative equilibrium.

Proof. It is enough to take a dominant mass in Theorem 2. $\square$

In 1995, R. Moeckel conjectured that in order to have linear stability for a relative equilibrium in the Newtonian $n$-body problem, the system must have a particle with a mass significantly larger than the others [14]. We will show that this conjecture is not true for charged problems, even when $n = 3$.

Suppose $m_1 = m_2 = m_3$. Then in order to have linear stability, Theorem 2 requires that

$$\sin^2 \theta_1 + \sin^2 \theta_2 + \sin^2 \theta_3 \leq 1/4.$$ 

Therefore, if we choose the charges such that the interior angles of the triangle satisfy the above inequality, the respective relative equilibrium is linearly stable. Let us observe that since $\sin(\pi - \theta) = \sin \theta$, the above inequality holds if two of the angles are close to zero and the third one close to $\pi$; in other words, a triangular relative equilibrium with three equal masses is stable if the configuration is almost collinear (see Fig. 4).
Suppose now that we have found masses and charges that lead to a triangular linearly stable relative equilibrium. We construct the high passing by the particle 2 in the above triangle; then by Proposition 8, for each point on the segment of the high contained in the triangle, there exist masses and charges for which the triangle formed by the particles 1, 3 and this new point is a linearly stable relative equilibrium. Taking the limit of the above triangular relative equilibrium we obtain a collinear configuration. We end this paper with answering the following two questions: Is the asymptotic collinear configuration a relative equilibrium? Is it linearly stable?

We assume without lost of generality that the limiting collinear configuration satisfies \( s_1 < s_2 < s_3 \), with \( s_3 - s_1 = 1 \). Then the distances among particles are given by

\[
r_{jk} = \sqrt[3]{\frac{\delta_{jk}}{\delta_{13}}}, \quad \text{where} \quad \delta_{12}^{1/3} + \delta_{23}^{1/3} = \delta_{13}^{1/3}.
\]

In coordinates \( x = (s_2 - s_1)^{-1}, y = (s_3 - s_2)^{-1} \), used in the proof of Theorem 1, we can verify by straightforward computations that (60) satisfy (44). This means that the limiting collinear configuration is a relative equilibrium. It is not difficult to check that \( w^{-2} \text{tr} C = -2 \). Therefore, by Theorem 1, we have that the respective collinear relative equilibrium is linearly stable but degenerate.

**Remark 3.** The above result does not imply the existence of a continuum of central configurations in the charged three-body problem since any configuration in the previous limit depends on the parameters. In other words, the values of the masses and charges in any configuration are different.

**References**