To the Latimer–Macduffee theorem and beyond!

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Abstract

In this paper we discuss some of the mathematical contributions made by Olga Taussky Todd. We begin with fairly general considerations, commenting on her role as a thesis advisor and giving an overview of her research contributions. Then we consider a specific work of hers, Another look at a matrix of Mark Kac, her last published work, which was written jointly with John Todd. We make a series of observations that explain certain numerical phenomena that they prove in this paper and end with a generalization of their work. © 1998 Elsevier Science Inc. All rights reserved.

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1. Olga as mathematician and mentor

On 13 April 1996 the mathematics department at Caltech sponsored The Olga Day Celebration, a day of talks in tribute to the mathematical contributions of Olga Taussky Todd. I was honored to be one of the speakers along with Dick Gross, Bob Guralnik, Ken Ribet and Helene Shapiro. This article is an edited account of the remarks that I made at the Olga Day Celebration.

When I considered what things I should say in tribute to Olga’s work, my initial instinct was discuss her work on the Latimer–Macduffee Theorem. This

1 This work is partially supported by the NSF.
theorem gives a correspondence between classes of integral matrices (classes under unimodular similarity) and elements of the class groups of algebraic number fields. This result always held a special fascination for Olga. She studied the Latimer–Macduffee Correspondence over an extended period and in a series of papers ([11–17,19,20,23–28,30]) she established a number of its more interesting properties. Moreover, her work on the Latimer–Macduffee Theorem was closely connected to my thesis work and so seemed the ideal topic for this tribute. In preparing for this talk, I had occasion to look through many of Olga’s publications and I became more aware than before that her work on the Latimer–Macduffee Theorem was only a small piece of her total mathematical contribution. I decided to look beyond the Latimer–Macduffee Theorem to get a more global view of her contribution to the mathematical sciences.

Three significant ways in which a mathematician can contribute a lasting legacy are:
1. their influence on the profession;
2. their students;
3. the body of their research.

Of these three, probably Olga’s influence on the mathematical profession has been most often cited. Indeed, one cannot underestimate the leadership role she played by achieving great success in a professional climate that was unfriendly, if not hostile, to women. What accounts for Olga’s willingness to persist in a career track littered with gender-related obstacles? No doubt a significant contributing factor was her true devotion to mathematics. She was fascinated by and looked for significance in the most humble mathematical facts. She was particularly taken by numbers. She wore clothes decorated with numbers and wrote poems about numbers. Olga was interested in any mathematical theorem about numbers. Those who met her could not help but be struck and inspired by her devotion to our subject. She was someone who entered the mathematical profession out of pure love for the subject.

What contributions did Olga make as a thesis advisor? Certainly she was productive, having an average of almost one student per year graduate during her years at Caltech. Below is a list of her 17 Ph.D. students with their current affiliations.

Ed Bender, University of California at San Diego
Daniel Davis, Monterey Bay Aquarium Research Institute
Lorraine Foster, California State University at Northridge
Fergus Gaines, University College Dublin
Phil Hanlon, University of Michigan
Charles Hobby, University of Alabama
Charles Johnson, College of William and Mary
Raphael Loewy, The Technion
Don Maurer, John Hopkins University
Joseph Parker. Optivision, Inc.
Helene Shapiro, Swarthmore College
Robert C. Thompson, University of California at Santa Barbara
Frank Uhlig, Auburn University

Olga's advising style was marked by her attentiveness and care. When I worked with Olga, we met at least once a week. She had material prepared to discuss with me at each meeting. It was not always mathematics that was directly related to my research. Many times she had prepared a presentation of a theorem or result which she thought was important to my mathematical training.

It is difficult to quantify something like attentiveness – to give a concrete measure of how much care Olga took for her students as an advisor. I had an idea – a way to tell if her care and attentiveness were as strong as I remember. After graduating from Caltech, I took a post-doc at MIT. In the excitement of a new job and new location, I left Caltech without ever submitting my thesis for publication. Being a thesis advisor myself now, I realize how frustrated I would be if one of my students failed to submit his/her thesis for publication. So, if Olga was as attentive as I remember, this same frustration might show up in the correspondence I had with Olga during the period just following my graduation from Caltech in June of 1981. Indeed, in letters that Olga wrote in the subsequent eighteen months I found the following comments:

I would like to know how you spent your summer here, whether you are writing up parts of your thesis… – August 1981


I look forward to hearing about your thesis work and how the write-up is progressing… – February 1982.

I am anxious to discuss your thesis publication with you… but I do not dare to advise you – April 1982.

And finally, when all other pleas seem to be failing on deaf ears, Olga tried a different approach:

We send you our very best wishes and hope you will not overwork yourself, even if your results will appear a little later. – December 1982.

Regrettably, I never did return to my thesis work. Despite this, Olga's influence has been very much evident in my work. In more than half of my research publications the main result has been about matrices and in most of those, about spectral and algebraic properties of matrices. Matrix Theory was a subject that Olga emphasized in my training and I can directly relate her influence to much of the focus and success of my later research.
Certainly the most tangible contribution that Olga left us is her outstanding research. Scholarship was an important focus in Olga's life and she took great pride in her published work. Her first publication—the "Über eine Verschärfung des Hauptidealsatzes für algebraische Zahlkörper" ("Concerning a sharpening of the principal ideal theorem for algebraic numbers")—appeared in 1931 in the Journal für die reine und angewandte Mathematik [10]. Her final publication, Another Look at a Matrix of Mark Kac (joint with Jack Todd) appeared in LAA in 1991. In between these two, came over 170 research publications on a wide and diverse set of mathematical topics. The breadth of the work in these papers can be realized by noting that among them, her publications list 17 primary classifications from the current MR classifications scheme:

01 History and Biography
05 Combinatorics
10 Number Theory
12 Field Theory and Polynomials
14 Algebraic Geometry
15 Linear and Multilinear Algebra including matrix theory
16 Associative Rings and Algebras
18 Category Theory and Homological Algebra
20 Group Theory and generalizations
22 Topological Groups, Lie Groups
26 Real Functions
30 Functions of a complex variable
34 Ordinary differential equations
52 Convex and discrete geometry
65 Numerical analysis
68 Computer science
76 Fluid Mechanics

Which of these was Olga's favorite topic? The chart below divides up her publications by primary classification. Much of this information comes directly from the Math Reviews database. However, there are two complications. The first is that the MR Classification Scheme has changed several times since the start of MR. So, for publications that pre-date the current classification, I read through the abstracts and introductions and chose a classification that I thought most was most appropriate. Publications prior to 1940 were even more difficult. For these, I had to refer to the [37,5] to identify the publications. Then I again read the abstracts and introductions and assigned classifications that I felt were appropriate. My thanks to Jane Kister of Mathematical Reviews for her help with this project. I should also note that I have not included book reviews, research problems and reviews for MR (of which there are 177 included in the MR database!).

Those who knew Olga well will not be surprised to see that Number Theory and Linear Algebra (which includes Matrix Theory) are by far the largest...
slices. However, it was more surprising to me that History and Biography make up a full 10% of her publication record.

When did Olga do the bulk of her work? The bar chart below shows the number of Olga's publications by decade. You can see that she published in six different decades. The consistency of her publication rate over the sixty years from 1931 to 1992 is striking. Olga's publication rate was a bit depressed during the 1930s and 1940s when her life and the mathematical profession was in turmoil because of the war. During the four decades following World War II, her publication rate is nearly constant. This shows a remarkable commitment to research sustained over a very long period of time.

In addition, I considered length of publications. Olga often told me that she liked to write short papers. Indeed that is born out by the record which shows that almost half her papers were five pages or less in length and almost three-quarters were ten pages or less.

What are the general themes of Olga's work? Her devotion to Matrix Theory comes through clearly in her work. As we know, matrices can be viewed in many ways – as arrays of numbers, as linear transformations or as
representations of groups and algebras. The interplay between these multiple roles was an important point of for Olga. This is a recurrent theme in her article How I became a Torchbearer for Matrix Theory which appeared in The American Mathematical Monthly [38]. The algebraic concept of commutativity also fascinated Olga. She has a number of articles which explore the implications of commutativity in various situations and a number of others in which she explores what flexibility and power you get when the notion of commutativity is relaxed just a small amount. Indeed, two of her most celebrated research topics, her work on the Latimer–Macduffee Theorem and her work on Property $L$ are directed towards understanding what changes occur when marginal changes are made in the notion of commutativity.

What can be learned about the style of Olga's work from her papers? Suppose we think of a completed mathematical work as a process which begins with the discovery of a collection of related phenomena and ends with the development of a theory to unify and explain these discoveries. Much of Olga's work was at the front end of this process, exposing interesting facts which her intuition told her might have great significance. More than once she expressed to me her admiration of work in which new phenomena were discovered and her feeling that mathematical explorers were not given the credit they were due relative to mathematical developers. This focus on mathematical discovery can be seen clearly in her series of papers about $2$ by $2$ matrices ([18,20–29,31]).

These papers contain several instances of results about $2$ by $2$ matrices that are special cases of more general results proved by others. Not proving the
most general result did not seem to bother Olga. She took great pride in the role of her work as pointing the way to the eventual generalization.

2. The Kac matrices

To emphasize the point made in closing the last section, I want to take a closer look at an article in which Olga shows her knack for discovery. I chose her last publication, *Another look at a Matrix of Mark Kac* which appeared in 1991 in Linear Algebra and its Applications and which she wrote jointly with her husband John Todd. In this paper, Olga and Jack study a family of matrices that came up in the work of Mark Kac [7] on the Ehrenfest Urn Model. To describe this Markov chain, imagine than you have two urns placed side by side and \( n \) (distinguishable) balls. The states of the Markov chain correspond to the ways in which the \( n \) balls can be divided between the two urns. You move from one state to another by choosing one of the \( n \) balls at random and moving that ball from whatever urn it's in to the other urn.
The Kac matrix $K_n$ is the $(n + 1) \times (n + 1)$ matrix whose $i, j$ entry is $n$ times the probability that the left-hand urn goes from containing $j$ balls to $i$ balls in one step of the Ehrenfest Urn Markov chain. When I write this definition, I am adopting the convention that the rows and columns of the $K_n$ are indexed by the numbers $0, 1, \ldots, n$ rather than $1, 2, \ldots, (n + 1)$. Since you move exactly one ball at a time, it is easy to see that $K_n$ is the tridiagonal matrix with entries defined by

$$
(K_n)_{i,j} = \begin{cases} 
  j & \text{if } i = j - 1, \\
  n - j & \text{if } i = j + 1, \\
  0, & \text{otherwise.}
\end{cases}
$$

The matrix $K_4$ is given below

$$
K_4 = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
4 & 0 & 2 & 0 & 0 \\
0 & 3 & 0 & 3 & 0 \\
0 & 0 & 2 & 0 & 4 \\
0 & 0 & 0 & 1 & 0
\end{pmatrix}.
$$

The name "Kac matrix" is a bit misleading. As Olga and Jack point out in their work, these matrices appear in earlier work of Sylvester who stated that their characteristic polynomial is given by

$$
\det(\lambda I - K_n) = \prod_{i=0}^{n} (\lambda - (n - 2i)).
$$

These matrices are interesting if for no other reason than they come up in a variety of contexts. Besides appearing in the work of Kac and Sylvester, they have also been considered by Schrodinger [36], Rozsa [33], Hess [3] and they play a role in the theory of association schemes.

The starting point for Jack and Olga's work is a conjecture made by Olga, a conjecture based on an insight which in retrospect is quite remarkable. She had the idea that it would be interesting to consider a matrix that consisted of a direct sum of copies of these Kac matrices. This matrix, which she denoted $D_n$, consisted of a direct sum, over $j$ running from 0 to $n$, of $m_j$ copies of the matrix $K_{2j}$ where

$$
m_j = \frac{2j + 1}{n + j + 1} \binom{2n}{n + j}.
$$

Olga conjectured that the dimension of the matrix $D_{2n}$ is $2^{2n}$. In their paper, Olga and Jack prove this conjecture by means of combinatorial manipulations with binomial coefficients. They go on to show that $2j$ occurs with multiplicity
\[
\left( \begin{array}{c} \binom{2n}{n+j} \\
\end{array} \right)
\] as an eigenvalue of \(D_{2n}\) which shows that \(D_{2n}\) is similar to matrices \(H_{2n}\) considered by Hess.

In their paper, Jack and Olga give little indication of how Olga came up with the numbers \(m_j\), through it is possible that Olga’s original motivation was to construct matrices similar to the Hess matrices. In the remainder of this paper, I am going to give an algebraic interpretation of the multiplicities \(m_j\) and then discuss a remark made near the end of the Taussky-Todd paper that they attribute to the referee.

Hess defined his matrices in the process of analyzing a random walk which is equivalent to the Ehrenfest Urn Model. Hess visualizes his walk as one on the vertices of the \(2n\)-dimentional cube, \(C^{2n}\). The vertices of \(C^{2n}\) are all sequences \((e_1, \ldots, e_{2n})\) where each \(e_i\) is either \(0\) or \(1\). There is an edge in \(C^{2n}\) between \((e_1, \ldots, e_{2n})\) and \((\eta_1, \ldots, \eta_{2n})\) if they differ in exactly one coordinate. You can think of \(C^{2n}\) just as a graph with \(2^{2n}\) points and \(n2^{2n}\) edges. The random walk considered by Hess is the random walk on that graph. From a vertex \(x\) you choose a neighboring vertex at random and move to that neighbor. It is straightforward to see that this random walk coincides with the Ehrenfest Urn Model (with \(2n\) balls). Namely, the vertices of \(C^{2n}\) correspond to the distributions of balls into the two urns and the allowed steps in Hess’s random walk on \(C^{2n}\) amount to choosing a ball at random and moving it from one urn to the other.

Let \(V\) denote the complex vector space with basis consisting of the vertices of \(C^{2n}\) and let \(H_{2n}\) denote the adjacency matrix of the graph \(C^{2n}\). It is important to note that we are taking the vertices of \(C^{2n}\) as a basis for \(V\), so \(V\) consists of formal sums of vectors, NOT sums of basis vectors as elements in \(2n\)-dimensional space. We will take the point of view that \(H_{2n}\) is a linear transformation of \(V\). Note that the symmetric group \(G = S_{2n}\) acts as a group of automorphisms of the graph \(C^{2n}\) or equivalently the permutation action of \(G\) on \(V\) commutes with the linear transformation \(H_{2n}\). Next we are going to investigate what we learn about the matrices \(H_{2n}\) from representation-theoretic considerations.

Let \(V[j]\) denote the subspace of \(V\) spanned by all vertices which have exactly \(j\) ones. It is straightforward to see that each space \(V[j]\) is invariant under the action of \(G\). So we can ask how \(V[j]\) decomposes into irreducibles. The irreducible representations of \(G\) are indexed by partitions \(\lambda\) of \(2j\). For each \(\lambda\), we let \(S^\lambda\) denote the corresponding \(G\)-module. The following theorem gives the multiplicity of each \(S^\lambda\) in the invariant subspace \(V[j]\) (see [4] for a proof).

**Theorem 1.** Let notation be as above.
(a) For \(j \leq n\) the space \(V[j]\) decomposes as the direct sum of the irreducibles \(S^{2n-l,j}\) for \(l \leq j\).
(b) \(V[j]\) is isomorphic, as a \(G\)-module to \(V[2n-j]\).
Combining parts (a) and (b) of Theorem 1 above gives that the irreducible representation $S_{2n-l,l}$ occurs $2(n - l) + 1$ times in the vector space $V$. We are going to interpret the $m_j$ copies of $K_{2j}$ in terms of the action of $H_{2n}$ on the $S_{2n-l,l}$ - isotypic component.

To begin consider the case where $l = 0$, i.e., where $S_{2n-l,l} = S_{2n}$ is the trivial representation. The theorem above tells us that there is one occurrence of the trivial representation in each $V[j]$. It is easy to identify a vector $\tau_j$ that spans that copy of the trivial representation in $V[j]$, namely take the sum of all basis elements of $V[j]$. In other words, $\tau_j$ equals the sum of all basis vectors $(e_1, \ldots, e_{2n})$ in which exactly $j$ of the components are 1's. Let $V^{(2n)}$ denote the vector space spanned by these $2n + 1$ copies of the trivial representation. Since $H_{2n}$ commutes with the action of $G$, we know that $V^{(2n)}$ is invariant under $H_{2n}$.

We compute the matrix $M$ of $H_{2n}$ with respect to the ordered basis $\tau_0, \tau_1, \ldots, \tau_{2n}$. We know that $H_{2n}(\tau_j)$ is going to be a linear combination of the $\tau_i$. To compute the coefficient of $m_{ij}$ of $\tau_i$ we need only compute the coefficient of one coordinate. We will compute the coefficient of $v_i = (1, 1, \ldots, 1, 0, 0, \ldots, 0)$, the vector which begins with $i$ ones and ends up with $2n - i$ zeros. There are $2n$ basis vectors adjacent to $v_i$ in all of $C^{2n}$. The coefficient of $v_i$ in $H_{2n}(\tau_j)$ is equal to the number of basis vectors for $V[j]$ which are adjacent to $v_i$. Of these, $i$ are obtained by changing one of the first $i$ coordinates in $v_i$ from a 1 to a 0. These basis vectors are in $V[i - 1]$. The remaining $2n - i$ basis vectors adjacent to $v_i$ are obtained by changing one of the last $2n - i$ coordinates of $v_i$ from a 0 to a 1. These $2n - i$ basis vectors are in $V[i + 1]$. So,

$$m_{ij} = \begin{cases} 
  j & \text{if } j = i - 1, \\
  2n - j & \text{if } j = i + 1, \\
  0 & \text{otherwise}.
\end{cases}$$

This proves the following theorem.

**Theorem 2.** The restriction of $H_{2n}$ to the isotypic component of $G$ corresponding to the trivial representation is the Kac matrix $K_{2n}$.

Theorem 2 accounts for the single copy (note that $m_n = 1$) of $K_{2n}$ that occurs in the matrix $D_{2n}$ constructed by Olga in formulating her conjecture. The explanation of the other Kac matrices $K_{2j}$ will require a bit more work.

Before proceeding, let me recall some facts from the representation theory of finite groups. Let $\Gamma$ be a finite group acting on a vector space $\Omega$. It is well known that $\Omega$ can be written as a direct sum of $\Gamma$-invariant subspaces each of which is an irreducible representation of $\Gamma$. In general, this decomposition of $\Omega$ as a direct sum of irreducibles is not unique. However, for $Y$ an irreducible representation of $\Gamma$, the number $\mu$ of subspaces in such a decomposition which are isomorphic to $Y$ and the subspace $\Omega^Y$ of $\Omega$ that they span are both unique.
We call $\mu$ the \textit{multiplicity} of $Y$ in $\Omega$ and we call $\Omega^Y$ the \textit{$Y$-isotypic component} of $\Omega$.

Next, let $A$ be a linear transformation of $\Omega$ which commutes with the action of $\Gamma$ and let $Y$ be an $\Gamma$ irreducible with multiplicity $\mu$ in $\Omega$. It is not difficult to check that $\Omega^Y$ is invariant under $A$. But much more can be said. We can choose a basis for $\Omega^Y$ with respect to which $A$ is a direct sum of a $\mu$ by $\mu$ matrix $A^Y$ repeated $\dim(Y)$ times. The fact that $A$ commutes with the action of $\Gamma$ means that the entire action of $A$ on $\Omega^Y$ can be captured by a $\mu$ by $\mu$ matrix.

How can you find $A^Y$? One of the most useful methods for doing this is projection by primitive idempotents in the group algebra. Recall that the group algebra $\mathbb{C}^\Gamma$ of $\Gamma$ over $\mathbb{C}$ is a semisimple algebra which decomposes as a direct sum of matrix rings-one matrix ring $M_Y$ of size $\dim(Y)$ for each irreducible $Y$. Let $E_Y$ denote one of the primitive idempotents from the matrix ring $M_Y$ (for example, you can choose $E_Y$ to be the matrix in $M_Y$ with a one in the 1, 1 entry and zeros elsewhere). Because $M_Y$ is a subspace of $\mathbb{C}^\Gamma$, $E_Y$ is a linear combination of elements of $\Gamma$. Since each element of $\Gamma$ is a linear transformation of $\Omega$ so is $E_Y$. The following is a consequence of Schur's Lemma:

**Proposition.** \textit{The image of $E_Y$ is a $\mu$-dimensional subspace of $\Omega$ which is invariant under $A$. Moreover, the restriction of $A$ to is $A^Y$.}

This short diversion covers the elements of representation theory that we will use below. We will apply these ideas with the group $\Gamma$ being a symmetric group $S_N$. In this case, a great deal is known about the representation theory of $\Gamma$. For example, it is known that the irreducible representations $S^\lambda$ of $S_N$ are indexed by partitions $\lambda$ of $N$. Given a partition $\lambda$, it is known that the dimension of $S^\lambda$ is given, in terms of $\lambda$, by a simple combinatorial rule known as the \textit{hook-length formula}. It is also known how to write down a primitive idempotent $E = E^\lambda$ in the matrix ring indexed by $\lambda$ (see the proof of Theorem 3 below).

Those wishing a full treatment of the representation theory of finite groups can see [1]. Those interested in learning more about the representation theory of the symmetric groups in particular should consult [4,6,35].

We know that the $H_{2n}$-invariant subspace $V^{(n+j,n-j)}$ is a direct sum of $2j + 1$ copies of the irreducible $S^{(n+j,n-j)}$ and that the action of $H_{2n}$ commutes with the action of $G$. Let $E$ denote a primitive idempotent in the group algebra of $G$ corresponding to the irreducible $S^{(n+j,n-j)}$. The facts below are a specialization, to this situation, of some of the facts above:

1. The image of $E$ has dimension $2j + 1$.
2. The matrix $H_{2n}$ preserves the image of $E$. Let $H_{2n}[E]$ denote the restriction of $H_{2n}$ to the image of $E$.
3. With respect to a suitable basis, the restriction of $H_{2n}$ to $V^{(n+j,n-j)}$, the $S^{(n+j,n-j)}$ isotypic component, is a direct sum of $\dim(S^{(n+j,n-j)})$ copies of the matrix $H_{2n}[E]$. 
The next theorem generalizes Theorem 2 and gives a fuller understanding of Jack and Olga's observation that the matrices $D_{2n}$ and $H_{2n}$ are similar.

**Theorem 3.** Fix $j \leq n$. We can choose a suitable primitive idempotent for the irreducible $S^{n+j,n-j}$ so that:

(a) The matrix $H_{2n}[E]$ is the Kac matrix $K_{2j}$.

(b) The dimension of $S^{(n+j,n-j)}$ is $m_j$.

**Proof.** Part (b) is a well-known fact that follows easily from the “hook-length formula” for the dimensions of the irreducible representations of the symmetric groups. We now prove (a).

To start, we will identify a primitive idempotent $E$ corresponding to the irreducible $S^{n+j,n-j}$. Let $\tau$ be the standard Young tableau with shape $(n+j, n-j)$ which has the numbers $1, 2, \ldots, n+j$ in row one and the numbers $n+j+1, \ldots, 2n$ in row two. Let $R_{\tau}$ be the “row stabilizer” of $\tau$, i.e., the subgroup of $G$ consisting of all permutations which preserve the two rows of $\tau$ as sets. Similarly, define the column stabilizer $C_{\tau}$ of $\tau$. Next define two elements of the group algebra of $G$.

$$r_{\tau} = \sum_{\rho \in R_{\tau}} \rho,$$

$$c_{\tau} = \sum_{\gamma \in C_{\tau}} \text{sgn}(\gamma)\gamma.$$

Lastly, define $E$ to be

$$E = c_{\tau}r_{\tau},$$

where the product is in the group algebra.

It is well known (see [4]) that this particular element $E$ of the group algebra is (up to scalar multiple) a primitive idempotent corresponding to the irreducible $S^{n+j,n-j}$. This is the particular element we will use in defining $H_{2n}[E]$.

For $\ell$ with $j \leq \ell \leq n-j$, we know that $V[\ell]$ is an $S_{2n}$-invariant subspace which contains one copy of $S^{n+j,n-j}$. So when we apply the linear transformation $E$ to $V[\ell]$, the image will have dimension one. Our first step will be to identify the unique vector in $E(V[\ell])$ for each $\ell$ with $j \leq \ell \leq n-j$. Fix such an $\ell$ which we write as $\ell = n-j+s$. Recall that $v_{\ell}$ is the vector which has a 1 in the first $\ell$ positions and 0's elsewhere. We are going to compute the coefficient of each basis vector $u$ in $E(v_{\ell})$ by considering three cases:

**Case 1:** If $u$ has two identical entries in positions $i_1, i_2$ where $1 \leq i_1 \leq n-j$ and $n+j+i_1 = i_2$, then the transposition $(i_1, i_2)$ is in $C_{\tau}$ and so $(id - (i_1, i_2))c_{\tau} = 2c_{\tau}$. So $(id - (i_1, i_2))E(v_{\ell}) = 2E(v_{\ell})$. On the other hand, $(id - (i_1, i_2))u = 0$ so the coefficient of $u$ in $E(v_{\ell})$ must be 0.

**Case 2:** Next consider the case where $u_{i} = 1$ for $1 \leq i \leq n-j$ and $u_{i} = 0$ for $n+j+1 \leq i \leq 2n$. Note that if $\rho \in R_{\tau}$ and $\gamma \in C_{\tau}$ then $\rho(v_{\ell})$ has all entries equal
to 0 in coordinates \((n + j + 1) \ldots 2n\). So for \(\gamma\) not equal to the identity, \(\gamma(p(v))\) is not equal to \(\gamma\). Therefore, the coefficient of \(u\) in \(E(v)\) is equal to the coefficient of \(u\) in \(r_i(v)\) which is easily seen to be \(\ell!(n + j - \ell)!(n - j)!\).

**Case 3:** Lastly, suppose that the entry \(u_{n+j+i}\) is distinct from the corresponding entry \(u_i\) for all \(i \in 1, 2, \ldots, n - j\). There is a unique \(\gamma \in \mathbb{C}\), which has the property that \((\gamma u)_i = 1\) for \(1 \leq i \leq n - j\) and \((\gamma u)_i = 0\) for \(n + j + 1 \leq i \leq 2n\). Note that \(\gamma c = \text{sgn}(\gamma)c\), and so \(\gamma E(v) = \text{sgn}(\gamma)E(v)\). Hence the coefficient of \(u\) is \(\text{sgn}(\gamma)\) times the coefficient of \(\gamma(u)\). We computed the latter coefficient in Case 2 and so the coefficient of \(u\) in this case is \(\text{sgn}(\gamma)\ell!(n + j - \ell)!(n - j)!\).

One observation from the computation above is that every coefficient of \(E(v)\) is divisible by \(\ell!(n + j - \ell)(n - j)!\). Let \(w_i\) be the vector obtained from \(E(v)\) by dividing by \(\ell!(n + j - \ell)(n - j)!\). So every basis vector has coefficient \(+1, -1, 0\) in \(w_i\).

We now compute the coefficient \(m_{s,t}\) of \(w_{n-j+s}\) in \(H_{2n}W_{n-j+1}\). We know that the projection of \(H_{2n}W_{n-j+1}\) to \(V[n-j+s]\) is a scalar multiple of \(W_{n-j+s}\), and so \(m_{s,t}\) is just the coefficient of \(v_{n-j+s}\) in \(H_{2n}W_{n-j+1}\). Observe that \(H_{2n}\) maps \(V[i]\) into \(V[i] \oplus V[i+1]\) so \(m_{s,t}\) is 0 unless \(s = t \pm 1\).

Assume first that \(s = t - 1\). The coefficient of \(v_{t-1}\) in \(H_{2n}(E(w))\) is the sum of the coefficients \(c_\ell\) in \(E(w)\) of vectors \(z\) obtained from \(v_{t-1}\) by changing a single 0 to a 1. By the three case computation above, the coefficient \(c_\ell\) is zero unless the changed coordinate is in the range \(n - j + t\) to \(n + j\) in which case the coefficient is 1. So \(m_{t-1,t}\) is \(2j - t\).

Next assume that \(s = t + 1\). The coefficient of \(v_{t+1}\) in \(H_{2n}(E(w))\) is the sum of the coefficients \(c_\ell\) in \(E(w)\) of vectors \(z\) obtained from \(v_{t+1}\) by changing a single 1 to a 0. By the three case computation above, the coefficient \(c_\ell\) is zero unless the changed coordinate is in the range \(n - j + 1\) to \(n - j + t + 1\) in which case the coefficient is 1. So \(m_{t+1,t}\) is \(t + 1\).

These two computations show that the matrix \(M\) is in fact the \(2j + 1\) by \(2j + 1\) Kac matrix. Together with the representation-theoretic reasoning above, this shows that the \(m_j = \deg(S^{n+j-n-j})\) copies of the Kac matrix \(K_j\) can be interpreted as the restriction of \(H_{2n}\) on the \(S^{n+j-n-j}\)-isotypic component of the action of \(S_{2n}\) on the vertices of \(C^{2n}\). This gives a more conceptual understanding of the matrix \(D_{2n}\) that Olga wrote down when she formulated her conjectures.

To conclude, I will turn to a comment made in the final section of the Taussky and Todd paper. On page 359 they report “Attention to the physics of the problem suggests an intimate connection with Krawtchouk polynomials, a special case of those of Askey and Wilson” Taussky and Todd attribute this observation to the referee. I will briefly mention at least one conceptual framework that makes a direct connection between the work in their paper and the Krawtchouk polynomials.

Suppose you “twist” the Ehrenfest Urn Model so that you are \(x\) times as likely to move a ball from the left-hand urn than from the right-hand urn. It is not immediately clear what that means so let me be more precise. Define a
Markov chain in which the states are the distributions of the $2n$ balls in the two urns. As before, we will think of these states as identified with the vertices of $C^{2n}$, a vertex $(e_1, \ldots, e_{2n})$ corresponding to the distribution in which ball $i$ is put in the left-hand urn if and only if $e_i = 1$. Let the transition matrix $2nH_{2n}^{(x)}$ be given by the following rule: the $(e_1, \ldots, e_{2n})$, $(\eta_1, \ldots, \eta_{2n})$ entry of $H_{2n}^{(x)}$ is

\begin{align*}
&1, & \text{if } (\eta_1, \ldots, \eta_{2n}) \text{ is obtained from } (e_1, \ldots, e_{2n}) \\
&\alpha, & \text{by changing a } 0 \text{ to a } 1,
\end{align*}

\begin{align*}
&\alpha(1 - \alpha), & \text{if } (\eta_1, \ldots, \eta_{2n}) \text{ is obtained from } (e_1, \ldots, e_{2n}) \\
&\text{by changing a } 1 \text{ to a } 0,
\end{align*}

\begin{align*}
&\alpha(1 - \alpha), & \text{if } (\eta_1, \ldots, \eta_{2n}) = (e_1, \ldots, e_{2n})
\end{align*}

where $j$ is the number of 1's in $(e_1, \ldots, e_{2n})$.

Let $v$ be the vector with entries indexed by vertices of $C^{2n}$ which has $\alpha'$ in entry $x$ exactly when $x$ has $j$ entries equal to 1. It is straightforward to check that

\[ H_{2n}^x v = v \]

and so $v$ represents the stable distribution of the Markov chain with transition matrix $H_{2n}^x$. It is interesting to note that $H_{2n}^x$ is the transition matrix for the Markov chain that results when you apply the Metropolis algorithm (see [9]) to the original Ehrenfest Urn Model in order to obtain a deformation with stable distribution $v$.

It is easy to check that the matrix $H_{2n}^x$ commutes with the action of the symmetric group $S_{2n}$ on $C^{2n}$. So we can restrict $H_{2n}^x$ to the isotypic components of this action. Let $K_{2n}^x$ denote the restriction to the trivial isotypic component. As discussed above, there is a single copy of the trivial representation in $V[j]$ for each $j$ and we can find a natural basis element $\tau_j$ that spares the invariants in $V[j]$. With respect to this basis $K_{2n}^x$ is the $(n + 1)$ by $(n + 1)$ matrix whose $i, j$ entry is the probability that you go from a fixed vertex with $j$ 1's to any vertex with $i$ 1's. It is straightforward to check that the $i, j$ entry of $2nK_{2n}^x$ is given by:

\begin{align*}
&j, & \text{if } i = j - 1, \\
&(n - j)\alpha & \text{if } i = j + 1, \\
&(n - j)(1 - \alpha) & \text{if } i = j.
\end{align*}

For example, we see the matrix $4K_4^x$ below.

\[
4K_4^x = \begin{pmatrix}
4 - 4\alpha & 1 & 0 & 0 & 0 \\
4\alpha & 3 - 3\alpha & 2 & 0 & 0 \\
0 & 3\alpha & 2 - 2\alpha & 3 & 0 \\
0 & 0 & 2\alpha & 1 - \alpha & 4 \\
0 & 0 & 0 & \alpha & 0
\end{pmatrix}.
\]
The reader will note that the eigenvalues of $4K_4$ are $4$, $3 - \alpha$, $2 - 2\alpha$, $1 - 3\alpha$, $-4\alpha$. It is interesting to look at the eigenvectors associated with each of these eigenvalues:

- $4$ $[1, 4\alpha, 6\alpha^2, 4\alpha^3, \alpha^4]$,
- $3 - \alpha$ $[1, 3\alpha - 1, 3\alpha^2 - 3\alpha, \alpha^3 - 3\alpha^2, -\alpha^3]$,
- $2 - 2\alpha$ $[1, 2\alpha - 2, \alpha^2 - 4\alpha + 1, -2\alpha^2 + 2\alpha, \alpha^3]$
- $1 - 3\alpha$ $[1, \alpha - 3, 3 - 3\alpha, 3\alpha - 1, -\alpha]$
- $-4\alpha$ $[1, -4, 6 - 4, 1]$

The results seen here for $n = 4$ occur more generally. Recall the definition of the Krawtchouk polynomials $P_i(j, \alpha)$:

$$P_i(j, \alpha) = \left( \frac{\alpha^j}{i} \right)^{1/2} \sum_{k=0}^{i} (-1)^k \binom{j}{k} \binom{i-j}{i-k} \alpha^k.$$

These polynomials were introduced by Krawtchouk in a 1929 paper (see [8]) and have been studied by numerous authors since that time. The following result is observed in [2] though it follows easily from known properties of the Krawtchouk polynomials. We speculate that the referee had this result in mind when he or she suggested to Olga and Jack a connection between their work and the Krawtchouk polynomials.

**Theorem 4** (see [2]). The matrix $K_{2n}^2$ has eigenvalues

$$\lambda_j = n - j(1 + \alpha), \quad 0 \leq j \leq n.$$

The corresponding right eigenvector is the Krawtchouk polynomial $P_i(j, \alpha)$.

### 3. Conclusion

Throughout her long and productive career, Olga Taussky Todd contributed to the mathematical profession in a variety of ways. Her scholarship constitutes a major contribution made over a span of six decades. Olga possessed remarkable mathematical instincts which emerge as a clear feature in much of her research. She is one who valued interesting examples and great ideas as much as general theories and technical details. Thus, in addition to her general mathematical skills and knowledge, she had an intuition and feel for mathematics which set her apart as truly exceptional. In this paper we have discussed just one amazing insight which appears in Olga’s work. But there are many others which make Olga Taussky’s scholarly record well worth exploring.
References

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