Generalized Vector Version of Minty’s Lemma and Applications

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Abstract—In this paper, we consider a generalized vector version of Minty’s lemma, and then show the existence of solutions to generalized vector variational-type inequality problems for set-valued mappings as an application. © 2003 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

Minty [1] showed the linearization lemma for the scalar case, which have played useful roles in variational inequalities. In fact, the classical Minty’s inequality and Minty’s lemma have been shown to be important tools in the regularity results of the solution for a generalized nonhomogeneous boundary value problem [2] and, when the operator is a gradient, also a minimum principle for convex optimization problems [3]. And Behera and Panda [4] obtained a nonlinear generalization of Minty’s lemma. Furthermore, they applied the result to obtain a solution of a certain variational-like inequality. Kassay and Kolumban [5] considered the following Minty-type problem for set-valued mappings with two variables for the scalar case; find an element \( z \in K \) such that:

\[
\sup_{z \in T(x,z)} \langle z, y - x \rangle \geq 0, \quad \text{for } y \in K,
\]

where \( K \) is a nonempty convex subset of a dual space \( X^* \) of a Banach space \( X \) and \( T: K \times K \to 2^X \) is a set-valued mapping.
For the vector-valued case, some extensions of Minty's lemma were obtained by many authors [6-15]. In particular, recently Konnov and Yao [10] obtained a generalized linearization lemma for set-valued mappings and considered the existence of solutions to the following generalized vector variational inequality problems; find \( x \in K \) such that for \( y \in K \), there exists \( z \in T(x) \) such that:
\[
\langle z, y - x \rangle \notin -\text{int} C(x),
\]
introduced by Lin, Yang and Yao [16], where \( T : K \to 2^{L(X,Y)} \) is a set-valued mapping and \( C : K \to 2^Y \) is a set-valued mapping such that \( C(x) \) is a proper closed convex cone of \( Y \) with \( \text{int} C(x) \neq \emptyset \) for each \( x \in K \). In 1999, Lee et al. [11] obtained a vector version of Minty's lemma using Nadler's result [17], and with their result they considered the existence of solutions to two kinds of vector variational-like inequalities for multifunctions under certain new pseudomonotonicity condition and certain new hemicontinuity condition, respectively, different from conditions in [18,19]. On the other hand, Ansari et al. [6] obtained the nonlinear case of Minty's lemma and showed the existence of solutions to some vector variational-like inequality problems using their nonlinear case of Minty's lemma. In their paper, Lee et al. [12] considered a generalized result of Behera and Panda for Minty's lemma by extending it to the vector case under traditional pseudomonotonicity and traditional hemicontinuity conditions. And then with the result, they considered the existence of solutions to some vector variational-type inequalities.

On the other hand, Stampacchia vector variational inequality is defined in [20] as; find \( x \in K \) such that
\[
F(x) (y - x) \notin C \setminus \{0\}, \quad \text{for } y \in K,
\]
while Minty vector variational inequality is defined in [9] as; find \( x \in K \) such that
\[
F(y) (x - y) \notin C \setminus \{0\}, \quad \text{for } x \in K,
\]
where \( F : X(\subset \mathbb{R}^n) \to \mathbb{R}^{\times n} \) is a set-valued mapping and \( C \) is a convex cone in \( \mathbb{R}^l \).


In this paper, we introduce the \( M\Phi\)-pseudomonotone and \( M\Phi\)-pseudomonotone-type set-valued mappings on topological vector spaces. And we consider the existence of solution to the following generalized vector variational-type inequality problems \( GVVTIP(1) \) and \( GVVTIP(II) \) for \( M\Phi\)-pseudomonotone-type set-valued mappings to be called the generalized vector version of Minty's lemma, and show their equivalence.

**GVVTIP(1).** Find \( x_0 \in K \) such that for all \( y \in K \) there exists an \( u_0 \in T(x_0) \) satisfying
\[
\langle M(x_0, u_0), \theta(y, x_0) \rangle + \eta(x_0, y) - \eta(x_0, x_0) \notin -\text{int} C(x_0).
\]

**GVVTIP(II).** Find \( x_0 \in K \) such that for all \( y \in K \) there exists a \( v \in T(y) \) satisfying
\[
\langle M(y, v), \theta(y, x_0) \rangle + \eta(x_0, y) - \eta(x_0, x_0) \notin -\text{int} C(x_0),
\]
where \( \langle l, x \rangle \) denotes the value of \( l \in L(X,Y) \) at \( x \).

By putting \( \eta = 0, C(y) = -C(y) \) for \( y \in K \) and \( L(X,Y) = X^* \), we obtain the following vector variational-like inequality introduced by Ansari et al. [6]; find \( x_0 \in K \) such that for all \( y \in K \) there exists an \( u_0 \in T(x_0) \) satisfying
\[
\langle M(x_0, u_0), \theta(y, x_0) \rangle \notin -\text{int} C(x_0) 0,
\]
which generalizes some kinds of vector variational inequalities considered by many authors [10, 16,18,21–27].
2. PRELIMINARIES

Let $X, Y$ be topological vector spaces, $K$ a nonempty subset of $X$, and $N$ a nonempty subset of $L(X, Y)$, where $L(X, Y)$ is the space of all linear continuous operators from $X$ to $Y$. Let $M : K \times N \rightarrow L(X, Y), \theta : K \times K \rightarrow X$ and $\eta : K \times K \rightarrow Y$ be mappings, and $\{C(x) : x \in K\}$ a family of closed convex cones in $Y$. A partial order $\leq_{C(x)}$ in $Y$ with the closed convex cone $C(x)$ is defined as for $y_1, y_2 \in Y$,

$$y_1 \leq_{C(x)} y_2, \quad \text{if and only if } y_2 - y_1 \in C(x).$$

Now, we introduce the $M$-$\theta$-pseudomonotonicity, $M$-$\theta$-pseudomonotone-type, and $M$-$\theta$-hemicontinuity.

**DEFINITION 2.1.** A set-valued mapping $T : K \rightarrow 2^N$ is called

1. **$M$-$\theta$-pseudomonotone** if for every pair of points $x, y \in K$ and for all $u \in T(x), v \in T(y)$, we have

$$\langle M(x, u), \theta(y, x) \rangle + \eta(x, y) - \eta(x, x) \not\in - \text{int } C(x), \quad \text{implies} \quad \langle M(y, v), \theta(y, x) \rangle + \eta(x, y) - \eta(x, x) \not\in - \text{int } C(x).$$

2. **$M$-$\theta$-pseudomonotone-type** if for every pair of points $x, y \in K$ and for all $u \in T(x)$, we have

$$\langle M(x, u), \theta(y, x) \rangle + \eta(x, y) - \eta(x, x) \not\in - \text{int } C(x), \quad \text{implies} \quad \langle M(y, v), \theta(y, x) \rangle + \eta(x, y) - \eta(x, x) \not\in - \text{int } C(x), \quad \text{for some } v \in T(y).$$

**REMARK 2.1.**

1. (1) implies (2), but not conversely.
2. If $M(x, u) = u, \theta(y, x) = y - x$, and $\eta$ is a zero mapping for all $x, y \in K$, then we obtain Definition 2.1 (iii) and (vi) in [10], respectively.
3. If $L(X, Y) = X^*$ and $C(x) = \mathbb{R}^+$, then we obtain Definition 2.1 (i) in [28].

**DEFINITION 2.2.** Let $\theta : K \times K \rightarrow K$ be a mapping. For $x, y \in K$ and $\alpha \in [0, 1]$, set $x_\alpha \in K$ be defined by $\theta(x_\alpha, x) = \alpha \cdot \theta(y, x)$, then $x_\alpha$ is called a point in the curve segment $l = \theta(y, x)$. Then we say $\theta$ satisfies a condition (*).

**REMARK 2.2.** If $\theta(y, x) = y - x$, then $x_\alpha = x + \alpha(y - x)$ is a point in the line segment $[x, y]$.

**DEFINITION 2.3.** Let $M : K \times N \rightarrow L(X, Y)$ be a mapping and $\theta : K \times K \rightarrow X$ a mapping satisfying condition (*). A set-valued mapping $T : K \rightarrow 2^N$ is said to be $M$-$\theta$-hemicontinuous on $K$ if for every $x, y \in K$, for $x_\alpha \in K$ such that $\theta(x_\alpha, x) = \alpha \cdot \theta(y, x)$, the multifunction

$$\alpha \in [0, 1] \mapsto \langle M(x_\alpha, T(x_\alpha)), \theta(y, x) \rangle$$

is upper semicontinuous at $0^+$, where $\langle M(x_\alpha, T(x_\alpha)), \theta(y, x) \rangle = \{\langle M(x_\alpha, u_\alpha), \theta(y, x) \rangle : u_\alpha \in T(x_\alpha)\}$.

**REMARK 2.3.** Putting $M(x, T(x_\alpha)) = T(x_\alpha)$ and $\theta(y, x) = y - x$, we obtain the usual $u$-hemicontinuity in [10].

3. MAIN RESULTS

We first consider a generalized vector version of Minty's lemma. And then, we show the existence of solutions to generalized vector variational type inequalities for $M$-$\theta$-pseudomonotone-type set-valued mappings.
Theorem 3.1. Let $X$ and $Y$ be topological vector spaces. Let $K$ be a nonempty convex subset of $X$, $N$ a nonempty subset of $L(X, Y)$ and $\{C(x) : x \in K\}$ a family of closed convex cones in $Y$. Let $M : K \times N \to L(X, Y)$, $\theta : K \times K \to X$ and $\eta : K \times K \to Y$ be mappings. Assume that $T : K \to 2^N$ is a $\Theta$-pseudo-monotone-type, $\Theta$-hemicontinuous mapping. Then GVVTIP(I) and GVVTIP(II) are equivalent.

Proof. Suppose that there exists an $x_0 \in K$ such that for all $y \in K$ there exists $u_0 \in T(x_0)$ satisfying $(M(x_0, u_0), \theta(y, x_0)) + \eta(x_0, y) - \eta(x_0, x_0) \notin -\text{int} C(x_0)$, then by the definition of $\Theta$-pseudo-monotone-type, we have an $x_0 \in K$ such that for all $y \in K$ there exists $u \in T(y)$ satisfying $(M(y, u), \theta(y, x_0)) + \eta(x_0, y) - \eta(x_0, x_0) \notin -\text{int} C(x_0)$.

Conversely, suppose that we can find an $x_0 \in K$ such that for all $y \in K$ there exists $v \in T(v)$ satisfying $(M(y, v), \theta(y, x_0)) + \eta(x_0, y) - \eta(x_0, x_0) \notin -\text{int} C(x_0)$. Assume to the contrary that for some $y_0 \in K$, for all $z \in K$ and for all $u \in T(z)$, we have

$$\langle M(z, u), \theta(y_0, z) \rangle + \eta(z, y_0) - \eta(z, z) \in -\text{int} C(z).$$

Set $x_\alpha \in K$ be a point on the curve segment $t = \theta(y_0, x)$ such that $\theta(x_\alpha, x) = \alpha \cdot \theta(y_0, x)$ for $\alpha \in (0, 1)$. Then by the $\Theta$-hemicontinuity of $T$, there exists a $\delta > 0$ such that

$$\langle M(z, u'), \theta(x_\alpha, x) \rangle + \eta(z, y_0) - \eta(z, z) \in -\text{int} C(z)$$

for $u' \in T(x_\alpha), \alpha \in (0, \delta)$. But this is a contradiction to our hypothesis.

Hence, there exists an $x_0 \in K$ such that for all $y \in K$ there exists an $u_0 \in T(x_0)$ satisfying

$$\langle M(x_0, u_0), \theta(y, x_0) \rangle + \eta(x_0, y) - \eta(x_0, x_0) \notin -\text{int} C(x_0).$$

Corollary 3.2. If we take $M(x, u) = u, \theta(y, x) = y - x$, and a zero mapping $\eta$, then we obtain Lemma 2.1 in [10] as a corollary.

Let $K$ be a subset of a Hausdorff topological vector space $X$. Then a mapping $T : K \to 2^X$ is called a KKM mapping if for each nonempty finite subset $N$ of $K$, $\text{co} N \subseteq T(N)$, where $\text{co}$ denotes the convex hull and $T(N) = \bigcup\{T(x) : x \in N\}$.

The following theorem will play a crucial role in proving the existence of solutions to generalized vector variational-type inequalities for $\Theta$-pseudo-monotone-type set-valued mappings.

Theorem 3.3. (See [29].) Let $K$ be an arbitrary nonempty subset of a Hausdorff topological vector spaces $X$. Let a set-valued mapping $T : K \to 2^X$ be a KKM mapping such that $T(x)$ is closed for all $x \in K$ and compact for at least one $x \in K$. Then

$$\bigcap_{x \in K} T(x) \neq \emptyset.$$  

Now, we consider the existence of solutions to generalized vector variational type inequality for $\Theta$-pseudo-monotone-type, $\Theta$-hemicontinuous set-valued mappings.

Theorem 3.4. Let $X$ be a Banach space, $Y$ a topological vector space. Let $K$ be a nonempty weakly compact convex subset of $X$, $N$ a nonempty subset of $L(X, Y)$ and $\{C(x) : x \in K\}$ a family of closed convex cones in $Y$. Let a set-valued mapping $W : K \to 2^Y$ be defined by $W(x) = Y \setminus \{-\text{int} C(x)\}$ such that $\text{Gr}(W) = \{(x, y) \in X \times Y : x \in X, y \in W(x)\}$ is weakly closed in $X \times Y$. Assume that $M : K \times N \to L(X, Y)$ is a mapping such that $x \mapsto M(\cdot, x)$ is continuous, $\theta : K \times K \to X$ is a mapping such that $x \mapsto \theta(x, \cdot)$ is affine, $x \mapsto \theta(\cdot, x)$ is continuous and $\theta(x, x) = 0$, and $\eta : K \times K \to Y$ is a continuous mapping such that $x \mapsto \eta(x, \cdot)$ is convex for all $x \in K$. Let $T : K \to 2^N$ be a $\Theta$-pseudo-monotone-type $\Theta$-hemicontinuous mapping with compact values. Then GVVTIP(I) holds.
PROOF. For each $y \in K$, we define a set-valued mapping $F_1 : K \to 2^K$ by

$$F_1(y) := \{ x \in K : \text{there exists an } u \in T(x) \text{ satisfying} \}
$$

$$\langle M(x,u), \theta(y,x) \rangle + \eta(x,y) - \eta(x,x) \notin \text{int } C(x) \rangle.$$

Then $F_1$ is a KKM mapping. In fact, suppose that $F_1$ is not a KKM mapping, then there exists $\{x_1, x_2, \ldots, x_n\} \subset K, \alpha_i \geq 0, i = 1, 2, \ldots, n$ with $\sum_{i=1}^{n} \alpha_i = 1$ such that $x = \sum_{i=1}^{n} \alpha_i x_i \notin \bigcup_{i=1}^{n} F_1(x_i)$ for any $j = 1, 2, \ldots, n$. Thus, for $u \in T(x)$,

$$A_j := \langle M(x,u), \theta(x_j, x) \rangle + \eta(x,x) - \eta(x,x) \notin \text{int } C(x).$$

for all $j = 1, 2, \ldots, n$. Hence,

$$\sum_{j=1}^{n} \alpha_j A_j \in -\text{int } C(x), \quad (3.1)$$

where $\sum_{j=1}^{n} \alpha_j = 1, \alpha_j \geq 0$, for all $j = 1, 2, \ldots, n$.

On the other hand, by the affinity of $y \mapsto \theta(y,x)$ and the convexity of $y \mapsto \eta(x,y)$, for $u \in T(x)$,

$$B := \left\{ \left( M(x,u), \theta \left( \sum_{j=1}^{n} \alpha_j x_j, x \right) \right) + \eta \left( x, \sum_{j=1}^{n} \alpha_j x_j \right) - \eta(x,x) \right\} \subseteq C(x).$$

Hence,

$$-\sum_{j=1}^{n} \alpha_j A_j + B \in -C(x). \quad (3.2)$$

By adding (3.1) and (3.2), we have $0 \in -\text{int } C(x)$, which is a contradiction. Hence, $F_1$ is a KKM mapping. Define another set-valued mapping $F_2 : K \to 2^K$ by for each $y \in K$,

$$F_2(y) := \{ x \in K : \text{there exists an } u \in T(y) \text{ satisfying} \}
$$

$$\langle M(y,u), \theta(y,x) \rangle + \eta(x,y) - \eta(x,x) \notin -\text{int } C(x) \rangle.$$

then $F_1(y) \subset F_2(y)$, by the definition of M-$\theta$-pseudomonotone-type. Therefore, $F_2$ is also a KKM mapping. Now, we claim that $F_2(y)$ is a weakly closed subset of $K$ for each $y \in K$. Indeed, let $\{x_n\}$ be a sequence in $F_2(y)$ such that $x_n \to x_0 \in K$. Since $x_n \in F_2(y)$, there exists an $v_n \in T(y)$ satisfying

$$\langle M(y,v_n), \theta(y,x_n) \rangle + \eta(x_n,y) - \eta(x_n,x_n) \notin -\text{int } C(x_n).$$

So,

$$\langle M(y,v_n), \theta(y,x_n) \rangle + \eta(x_n,y) - \eta(x_n,x_n) \in W(x_n).$$

Since $T(y)$ is compact, $\{v_n\}$ has a convergent subsequence in $T(y)$, without loss of generality, $v_0 = \lim_{n \to \infty} v_n \in T(y)$. Since $Gr(W)$ is a weakly closed and $M, \theta, \eta$ are continuous, we have

$$\langle M(y,v_0), \theta(y,x_0) \rangle + \eta(x_0,y) - \eta(x_0,x_0) \in W(x_0).$$

Hence, $x_0 \in F_2(y)$, $F_2(y)$ is a weakly closed subset of $K$. Since $K$ is a weakly compact, $F_2(y)$ is a weakly compact subset of $K$, for each $y \in K$. Thus, by Theorem 3.3,

$$\bigcap_{y \in K} F_2(y) \neq \emptyset.$$

Thus, GVVTIP(II) holds. Hence, by Theorem 3.1, GVVTIP(I) holds.
COROLLARY 3.5. If we take $M(x, u) = u$, $\theta(y, x) = y - x$, and a zero mapping $\eta$, then we obtain Theorem 3.3 in [10] as a corollary.

REMARK 3.1. We can obtain the same result as Theorem 3.4 for $M$-$\theta$-pseudomonotone mappings, which generalizes the following theorem.

COROLLARY 3.6. If we take $M(x, u) = u$, $\theta(y, x) = y - x$, and a zero mapping $\eta$, then we obtain Theorem 3.1 in [10] as a corollary.

COROLLARY 3.7. When $X$ is a reflexive Banach space, $L(X, Y) = X^*$, $Y = \mathbb{R}$ and $C(x) = \mathbb{R}^+$, then we obtain Theorem 2.1 in [28] as a corollary.

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