The space of scalarly integrable functions for a Fréchet-space-valued measure

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\begin{abstract}
The space $L^1_w(\nu)$ of all scalarly integrable functions with respect to a Fréchet-space-valued vector measure $\nu$ is shown to be a complete Fréchet lattice with the $\sigma$-Fatou property which contains the (traditional) space $L^1(\nu)$, of all $\nu$-integrable functions. Indeed, $L^1(\nu)$ is the $\sigma$-order continuous part of $L^1_w(\nu)$. Every Fréchet lattice with the $\sigma$-Fatou property and containing a weak unit in its $\sigma$-order continuous part is Fréchet lattice isomorphic to a space of the kind $L^1_w(\nu)$.
\end{abstract}

\section{Introduction}

Let $X$ be a Fréchet space and $\nu : \Sigma \to X$ be a vector measure (i.e. $\nu$ is $\sigma$-additive), where $(\Omega, \Sigma)$ is a measurable space. A measurable function $f : \Omega \to \mathbb{R}$ is said to be $\nu$-integrable if

1. $f$ is scalarly $\nu$-integrable, that is, $f$ is integrable with respect to the scalar measure $\langle \nu, x^* \rangle : A \mapsto \langle \nu(A), x^* \rangle$, for each $x^* \in X^*$ (the continuous dual space of $X$), and
2. for each $A \in \Sigma$ there exists an element $\int_A f d\nu \in X$ such that

$$\int_A f d\nu, x^* = \int_A f d\langle \nu, x^* \rangle, \quad x^* \in X^*.$$ 

For $X$ a Banach space, the space $L^1(\nu)$, consisting of all (equivalence classes of) $\nu$-integrable functions equipped with the norm

$$\|f\|_{\nu} := \sup_{x^* \in B_{X^*}} \int_{\Omega} |f| d|\langle \nu, x^* \rangle|, \quad f \in L^1(\nu),$$

is also a Banach space. Here $B_{X^*}$ is the closed unit ball of $X^*$ and $|\langle \nu, x^* \rangle|$ is the variation measure of $\langle \nu, x^* \rangle$, for $x^* \in X^*$. The space $L^1(\nu)$ was introduced and investigated in [14], even for vector measures taking values in a locally convex Hausdorff space.
space (briefly, lcHs). Since then, \( L^1(\nu) \) has been intensively studied in the Banach setting by many authors; see, e.g. \([4,25]\) and the references therein.

G. Stefansson introduced the space \( L^1_{\nu}(\nu) \) of all (classes of) scalarly \( \nu \)-integrable functions and showed that the same functional \((1)\) above is a norm in \( L^1_{\nu}(\nu) \) \([28]\). This is to be interpreted to mean that if \( f \) is scalarly \( \nu \)-integrable, then \( \| f \|_\nu < \infty \) and that \((1)\) specifies a norm in \( L^1_{\nu}(\nu) \). Actually, Stefansson also showed that \( L^1_{\nu}(\nu) \) is complete for \( \| \cdot \|_\nu \). It is then clear that \( L^1(\nu) \) is a closed subspace of \( L^1_{\nu}(\nu) \).

In contrast to \( L^1(\nu) \), the Banach space \( L^1_{\nu}(\nu) \) has received relatively little attention up to now; see, for example, \([12,25,\text{Chapter 3}], [28]\). Recently however, new features of \( L^1_{\nu}(\nu) \) have emerged, based on the theory of Banach function spaces, which indicate the importance of this space in its own right \([3,5]\). In particular, it is closely related to the “optimal domain property” for certain kernel operators.

For a Fréchet space \( X \) and a vector measure \( \nu : \Sigma \to X \), the corresponding (Fréchet) space \( L^1(\nu) \) is well understood; see, e.g. \([8-11,14,21,23,24,26]\) and the references therein. Although the notion of an individual scalarly \( \nu \)-integrable function already occurs in the Fréchet and even more general lcHs setting \([14,16,17]\), there is no study made there or elsewhere (as far as we aware) of what the corresponding property and possessing a weak unit in its \( \sigma \)-order continuous part is Fréchet lattice isomorphic to \( L^1_{\nu}(\nu) \) for a suitable vector measure \( \nu \). This shows how extensive the family of all spaces of the type \( L^1_{\nu}(\nu) \) is within the class of all Fréchet lattices.

2. The space \( L^1_{\nu}(\nu) \)

Let \( X \) be a metrizable lcHs generated by a fundamental sequence of increasing seminorms \( \{ \| \cdot \|^{(n)} \}_{n \in \mathbb{N}} \). The sets \( B_n := \{ x \in X : \| x \|^{(n)} \leq 1 \} \) form a fundamental sequence of zero neighbourhoods for \( X \) and their polars \( B^n_0 := \{ x^* \in X^* : \| x^* \| \leq 1, \forall x \in B_n \} \), for \( n \in \mathbb{N} \), are absolutely convex \([19, \text{Theorem 23.5}]\). Moreover, \( \{ B^n_0 : n \in \mathbb{N} \} \) is a fundamental sequence of bounded sets in the strong dual \( X_n^* \) (i.e., each bounded set in \( X_n^* \) is contained in a multiple of \( B^n_0 \) for some \( n \in \mathbb{N} \)). In addition, each set \( B^n_0 \) for \( n \in \mathbb{N} \) is a Banach disc \([19, \text{Lemma 25.5}]\), that is, the linear hull \( L(B^n_0) = \bigcup_{t > 0} tB^n_0 \) of \( B^n_0 \) (formed in \( X^* \)) is a Banach space when equipped with its Minkowski functional

\[
\| x^* \|_{B^n_0} := \inf \{ s > 0 : x^* \in sB^n_0 \}, \quad x^* \in L(B^n_0).
\]

For each \( n \in \mathbb{N} \), the local Banach space \( X_n \) is the completion of the quotient \( X/M_n \) endowed with the quotient norm induced by \( \| \cdot \|^{(n)} \), where \( M_n := \{ x \in X : \| x \|^{(n)} = 0 \} \). Let \( \pi_n : X \to X_n \) be the quotient map. Then its dual map \( \pi_n^* \) is an isometric bijection between the Banach spaces \( X_n^* \) and \( L(B^n_0) \) \([19, \text{Remark 24.5(b)}]\).

Given a vector measure \( \nu : \Sigma \to X \), defined on a measurable space \((\Omega, \Sigma)\), a set \( A \in \Sigma \) is \( \nu \)-null if \( \nu(A) = 0 \) for all \( B \in \Sigma \) with \( B \subseteq A \). Let \( L^0(\nu) \) denote the \( \sigma \)-Dedekind complete Riesz space of all (classes, modulo \( \nu \)-a.e., of) scalar-valued, \( \Sigma \)-measurable functions defined on \( \Omega \), with respect to the \( \nu \)-a.e. pointwise order \([18, \text{pp. 126-127}]\). For each \( n \in \mathbb{N} \), define a \([0, \infty] \)-valued seminorm \( \| \cdot \|^{(n)} \) in \( L^0(\nu) \) by

\[
\| f \|^{(n)} := \sup \int \Omega | f | d\nu, \quad f \in L^0(\nu).
\]

The map \( \nu_n : \Sigma \to X_n \) given by

\[
\nu_n(A) := \pi_n(\nu(A)), \quad A \in \Sigma,
\]

is a Banach-space-valued measure. Observe that \( A \in \Sigma \) is \( \nu \)-null if and only if it is \( \nu_n \)-null for all \( n \in \mathbb{N} \).

**Proposition 2.1.** Let \( X \) be a metrizable lcHs, \( \nu \) an \( X \)-valued measure and \( f \in L^0(\nu) \). Then \( f \in L^1_{\nu}(\nu) \) if and only if \( \| f \|^{(n)} < \infty \), for all \( n \in \mathbb{N} \).

**Proof.** Let \( f \) satisfy \( \| f \|^{(n)} < \infty \), for all \( n \in \mathbb{N} \). Given \( x^* \in X^* \), there exist \( m \in \mathbb{N} \) and \( C > 0 \) such that \( |\langle x^*, x^* \rangle| \leq C \| x \|^{(m)} \) for all \( x \in X \), i.e. \( C^{-1}x^* \in B_m^0 \), and so

\[
\int_\Omega | f | d\nu, \langle x^* \rangle = C \int_\Omega | f | d\nu, C^{-1}x^* \leq C \| f \|^{(m)} < \infty.
\]

Conversely, if \( f \in L^1_{\nu}(\nu) \), then \( f \) is scalarly \( \nu_n \)-integrable, for \( n \in \mathbb{N} \), since \( \langle x^*, \pi_n^*(\xi^*) \rangle = \langle \nu_n, \xi^* \rangle \) as measures, for \( \xi^* \in X_n^* \). Furthermore,

\[
\| f \|_{\nu_n} := \sup \int_\Omega | f | d\nu_n, \xi^* = \sup \int_\Omega | f | d\nu, \pi_n^*(\xi^*)
\]

where
and hence,
\[ \|f\|_{v}^{(n)} := \sup_{x' \in B^n} \int_{\Omega} |f| \, d|\langle v, x' \rangle| = \|f\|_{v_n} \quad (4) \]
where, for the last equality, we use the fact that \( \pi_n^* \) is an isometry and so \( \pi_n^* (B_{X^*}) = B_n^o \). But, \( X_n \) is a Banach space and so \( \|f\|_{v_n} < \infty \). \( \square \)

**Corollary 2.2.** Let \( X \) be a metrizable lcHs and \( \nu \) be an \( X \)-valued vector measure. Then, \( L^1_w(\nu) \) is an ideal in \( L^0(\nu) \) and the restricted functionals \( \| \, \cdot \, \|_{v}^{(n)} : L^1_w(\nu) \to [0, \infty) \) given by (2), for each \( n \in \mathbb{N} \), are an increasing sequence of Riesz seminorms which turn \( L^1_w(\nu) \) into a metrizable, locally convex-solid Riesz space.

**Proof.** As \( B_{n+1} \subseteq B_n \), the seminorms \( (\| \, \cdot \, \|_{v}^{(n)})_{n \in \mathbb{N}} \) are increasing in \( L^1_w(\nu) \). Also, if \( g \in L^0(\nu) \) and \( f \in L^1_w(\nu) \) with \( |g| < |f| \), then \( g \in L^1_w(\nu) \) and \( \|g\|_{v}^{(n)} \leq \|f\|_{v}^{(n)} \), for \( n \in \mathbb{N} \). This shows that \( L^1_w(\nu) \) is an ideal in \( L^0(\nu) \) and each \( \| \, \cdot \, \|_{v}^{(n)} \) is a Riesz seminorm in \( L^1_w(\nu) \).

Suppose that \( f \in L^1_w(\nu) \) satisfies \( \|f\|_{v}^{(n)} = 0 \) for all \( n \in \mathbb{N} \). According to (4) we have \( f \in L^1_w(\nu_n) \) with \( \|f\|_{v_n} = \|f\|_{v}^{(n)} = 0 \) for all \( n \in \mathbb{N} \). That is, the set \( A := \{ w \in \Omega : |f(w)| > 0 \} \) is \( \nu_n \)-null for each \( n \in \mathbb{N} \). Hence, \( A \) is a \( \nu \)-null set and so \( f = 0 \) in \( L^1_w(\nu) \). Now general theory can be invoked to conclude that \( L^1_w(\nu) \) becomes a metrizable, locally convex-solid Riesz space when equipped with the topology induced by the seminorms \( (\| \, \cdot \, \|_{v}^{(n)})_{n \in \mathbb{N}} \); see, for example, [1, Theorem 6.1], [19, Lemma 22.5]. \( \square \)

Functions \( f \) from \( L^1_w(\nu) \) differ from those of \( L^1(\nu) \) in that not all of their “integrals” belong to \( X \). For \( X \) a Banach space, given any \( A \in \Sigma \) there always exists a “generalized integral” \( x_A^* \in X^{**} \) satisfying
\[ \langle x_A^*, x_A^* \rangle = \int_A f \, d|\langle v, x_A^* \rangle|, \quad x_A^* \in X^*; \]
see [5,16,28], for example. For \( X \) a metrizable lcHs, we now show that a similar phenomenon occurs. First some notation: given \( f \in L^1_w(\nu) \) and \( A \in \Sigma \), define a linear functional \( (w)f_A f \, dv : X^* \to \mathbb{R} \) by
\[ \langle w \rangle \int_A f \, dv : x^* \mapsto \int_A f \, d|\langle v, x^* \rangle|, \quad x^* \in X^*. \]
(5)
The continuous dual space \( (X_A^*)^* \) of \( X_A^* \) is denoted simply by \( X^{**} \).

**Proposition 2.3.** Let \( X \) be a metrizable lcHs and \( \nu \) be an \( X \)-valued vector measure. For each \( f \in L^1_w(\nu) \) and \( A \in \Sigma \), the linear functional \( (w)f_A f \, dv \) given by (5) belongs to \( X^{**} \).

**Proof.** Since \( L^1_w(\nu) \) is an ideal in \( L^0(\nu) \) (see Corollary 2.2), it suffices to consider \( 0 \leq f \in L^1_w(\nu) \). Fix \( A \in \Sigma \). Select \( \Sigma \)-simple functions \( 0 \leq f_k \uparrow f \) pointwise on \( \Omega \). Given \( x^* \in X^* \), we have \( f \in L^1(|\langle v, x^* \rangle|) \) and so the Dominated convergence theorem for scalar measures yields
\[ \lim_{k \to \infty} \int_A f_k \, d|\langle v, x^* \rangle| = \int_A f \, d|\langle v, x^* \rangle| = \langle x^*, (w)f_A f \, dv \rangle. \]
Hence, \( C := \{ f_A f \, dv \} \) is a bounded set in \( X \). From the previous formula, we have \( |\langle x^*, (w)f_A f \, dv \rangle| \leq 1 \) for all \( x^* \in C^0 \), which is a neighbourhood of zero in \( (X_A^*)^* \). So, \( (w)f_A f \, dv \in X^{**} \). \( \square \)

**Remark 2.4.** If \( X \) is weakly sequentially complete, then it follows that \( (w)f_A f \, dv \in X \), for every \( A \in \Sigma \) and every \( f \in L^1_w(\nu) \). That is, \( L^1_w(\nu) = L^1(\nu) \) in this case. Actually, whenever \( X \) does not contain an isomorphic copy of the Banach space \( c_0 \), it is known that \( L^1_w(\nu) = L^1(\nu) \); see, for example, [14, p. 31], [17, Theorem 5.1].

If \( X \) is a Fréchet space, then it is known that \( L^1(\nu) \) is also a Fréchet space for every \( X \)-valued vector measure \( \nu \) ([10], [14, Chapter 4, Theorems 4.1 and 7.1]). The same is true of \( L^1_w(\nu) \), even without completeness of \( X \! \). \( \square \)

**Theorem 2.5.** Let \( X \) be a metrizable lcHs and \( \nu \) be an \( X \)-valued vector measure. Then \( L^1_w(\nu) \) is complete and, in particular, is a Fréchet lattice. If, in addition, \( X \) is a Fréchet space, then the Fréchet lattice \( L^1_w(\nu) \) contains \( L^1(\nu) \) as a closed subspace.
Proof. Fix $n \in \mathbb{N}$. Let $\nu_n : \Sigma \to X_n$ be the Banach-space-valued vector measure given by (3). Rybakov’s theorem states that there exists $\xi_n^* \in X_n^*$ such that $\{(\nu_n, \xi_n^*)\}$ is a control measure for $\nu_n$ (i.e. $\nu_n$ and $\{(\nu_n, \xi_n^*)\}$ have the same null sets) [6, p. 268]. In particular, $\|(\nu_n, \xi_n^*)\| \ll \|(\nu_n, \xi_n^*)\|$, for every $\xi_n^* \in X_n^*$. Since $\pi_n^*$ is a bijection from $X_n^*$ onto $\text{Lin}(B_n^0)$, it follows that the linear functional $x_n^* := \pi_n^*(\xi_n^*) \in \text{Lin}(B_n^0)$ satisfies $\|(v, x_n^*)\| \ll \|(v, x_n^*)\|$. Let $\mu_n := \|(v, x_n^*)\|$, for $n \in \mathbb{N}$. Then

$$\mu := \sum_{n=1}^{\infty} \frac{\mu_n}{2^n(1 + \mu_n(\Omega))}$$

is a positive, finite control measure for $v$; this follows from (6) and the fact that $(B_n^0)_{n\in\mathbb{N}}$ is a fundamental sequence of bounded sets in $X_n^*$.

Let $\tau_0$ and $\tau$ denote the topology of convergence in measure in $L^0(\mu)$ and the topology in $L^0(\mu)$ defined (in a standard way) by the (extended valued) seminorms (2), respectively. Then $(L^0(\mu), \tau_0)$ is a complete metrizable topological vector space and $(L^0(\mu), \tau)$ is a Hausdorff topological vector group. Let $f_k \xrightarrow{\tau_0} 0$. Since $\int_{\Omega} |f_k|d\mu_n \leq \|f_k\|_n$ for all $n \in \mathbb{N}$, we get $f_k \xrightarrow{\tau_0} 0$ in $\mu_n$-measure for each $n \in \mathbb{N}$. Consequently, $f_k \xrightarrow{\tau_0} 0$ and so $\nu_0 \subseteq \tau$. On the other hand, it follows from the (classical) Fatou lemma that the closed $\|\cdot\|_n$-balls centred at 0 are $\tau_n$-closed. Therefore, $(L^0(\mu), \tau)$ is complete [15, Section 18.4(4)].

Noting that $L^1_n(\nu)$ is the same null sets) [6, p. 268].

Remark 2.6. Let $\tilde{X}$ denote the completion of the metrizable lCHs $X$ and let $\tilde{\nu}$ denote the $X$-valued vector measure $\nu$ when interpreted as taking its values in $\tilde{X}$. Then $L^1_\nu(v) = L^1_\nu(\tilde{\nu})$ as vector spaces with $\|\cdot\|_n = \|\cdot\|_n$, for each $n \in \mathbb{N}$. This explains why $L^1_\nu(v)$ is always complete, independent of whether $X$ is complete or not.

For $L^1(v)$ the situation is different. Indeed, $L^1(v)$ is always complete but, $L^1(v)$ may fail to be complete if $X$ is not complete; explicit examples can be found in [22,27], for instance. Since $L^1(v)$ has the relative topology from the complete space $L^1_\nu(v)$, we see that the completion of $L^1(v)$ is the closure of $L^1(\nu)$ in the Fréchet space $L^1_\nu(v)$. On the other hand, $L^1(v)$ is always complete in itself and has the relative topology from $L^1_\nu(v)$. Since the $\Sigma$-simple functions are dense in both $L^1(v)$ and $L^1(\nu)$, by the Dominated convergence theorem [16, Theorem 2.2], we see that $L^1(v)$ is also the closure of $L^1(\nu)$ in $L^1_\nu(v) = L^1_\nu(\nu)$. So, $(L^1(v))$ can also be identified with $L^1(\nu)$.

Let $F$ be a Fréchet lattice with topology generated by a fundamental sequence of Riesz seminorms $\{q_n\}_{n=1}^{\infty}$. We say that $F$ has the $\sigma$-Fatou property if, for every increasing sequence $(u_k)_{k=1}^{\infty}$ contained in the positive cone $F^+$ (of $F$) which is topologically bounded in $F$, the element $u := \sup_{k=1}^{\infty} u_k$ exists in $F^+$ and $q_n(u_k) \uparrow q_n(u)$, for each $n \in \mathbb{N}$. This terminology is not "standard"; e.g. in [1, p. 94] such an $F$ is called a $\sigma$-Nakano space.

Theorem 2.7. Let $v$ be a vector measure taking its values in a metrizable lCHs. Then $L^1_\nu(v)$ is a Fréchet lattice with the $\sigma$-Fatou property.

Proof. Let $\mu$ be any control measure for $v$. Fix $n \in \mathbb{N}$. Observe that $\|\cdot\|_n$, as given by (2), is a classical function seminorm in $L^0(\mu)$ in the sense of [29, §63]. Since the norm of the $L^1$-space of any positive measure has the Fatou property and $\|\cdot\|_n$ is the supremum of such norms (see (2)), it is known that $\|\cdot\|_n$ also has the Fatou property (in the sense of [29, §65]); see [29, §65, Theorem 4].

Now, let $(u_k)_{k=1}^{\infty}$ be any positive, increasing, topologically bounded sequence in $L^1_\nu(v)$. Then, Theorem 3 of [29, §65] implies that $\|u\|_n \leq \sup_{k=1}^{\infty} \|u_k\|_n < \infty$, for $n \in \mathbb{N}$, where $u = \sup_{k=1}^{\infty} u_k = \lim_{k=1}^{\infty} u_k$ (pointwise). Hence, $u \in L^1_\nu(v)$ by Proposition 2.1. Moreover, $\|u_k\|_n \uparrow \|u\|_n$ because $\|\cdot\|_n$ has the Fatou property as a function seminorm.

3. A representation theorem for Fréchet lattices

Let us begin with a summary of some fundamental properties of spaces of the kind $L^1(v)$.

Let $(F, \tau)$ be a Fréchet lattice. A positive element $e$ in $F$ is called a weak unit if, for every $u \in F$ we have $u \leq (ne) \uparrow u$ [13, 14]. Note, for any vector measure $v$ with values in a metrizable lCHs, that the constant function $\chi_D$ is a weak unit for both $L^1_\nu(v)$ and $L^1(v)$.

We say that $F$ has a Lebesgue (resp. $\sigma$-Lebesgue) topology, if $u_n \downarrow 0$ implies $u_n \xrightarrow{\tau} 0$ in $F$ (resp. $u_n \downarrow 0$ implies $u_n \xrightarrow{\tau} 0$ in $F$) [1, Definition 8.1]. It is a direct consequence of the Dominated convergence theorem for vector measures [14, p. 30], [16], that if $v$ is any vector measure with values in a Fréchet space, then $L^1(v)$ always has a $\sigma$-Lebesgue topology. Actually, $L^1(v)$ even has a Lebesgue topology. To see this, let $\mu$ be given by (7) and recall that the classical Riesz space $L^0(\mu) = L^0(v)$ is always Dedekind complete [18, Example 23.3(iv)]. Since $L^1(v)$ is an ideal in $L^0(v)$, it follows that $L^1(v)$ is also Dedekind complete [18, Theorem 25.2]. It is well known that this property of $L^1(v)$, together with a $\sigma$-Lebesgue topology, imply that $L^1(v)$ has a Lebesgue topology [1, Theorem 17.9].
The above three properties of $L^1(\nu)$, namely, Dedekind completeness, having a Lebesgue topology and possessing a weak unit, are known to characterize a large class of Fréchet lattices.

**Proposition 3.1.** Let $(F, \{q_n\}_{n \in \mathbb{N}})$ be a Dedekind complete Fréchet lattice with a Lebesgue topology and possessing a weak unit $e \in F^+$. Then there is a vector measure $\nu : \Sigma \to F^+$ such that the integration map $I_\nu : L^1(\nu) \to F$, defined by $f \mapsto \int_\Omega f \, d\nu$, for $f \in L^1(\nu)$, is a Fréchet lattice isomorphism of $L^1(\nu)$ onto $F$ satisfying $I_\nu(\chi_\Omega) = e$ and

$$q_n(I_\nu(f)) = \|f\|_0^{(n)}, \quad f \in L^1(\nu), \quad n \in \mathbb{N}.$$  

(8)

For $F$ a Banach lattice, Proposition 3.1 occurs in [2]. In the setting of a Fréchet lattice $F$ we refer to [7, Proposition 2.4(vi)], after reading its proof carefully, together with p. 364 of [7]. A similar but, somewhat different proof of Proposition 3.1 occurs in [21, Theorem 1.22]. Unlike in [7], the proof given in [21] does not rely on the theory of band projections.

An element $u$ of a Fréchet lattice $(F, \tau)$ is $\sigma$-order continuous if it has the property that $u_k \xrightarrow{\tau} 0$ as $k \to \infty$ for every sequence $(u_k)_k \subseteq F^+$ satisfying $|u| \geq u_k \downarrow 0$. The $\sigma$-order continuous part $F_\sigma$ of $F$ is the collection of all $\sigma$-order continuous elements of $F$; it is a closed ideal in $F$ [30, pp. 331–332], and clearly has a $\sigma$-Lebesgue topology.

**Theorem 3.2.** For any vector measure $\nu$ taking values in a Fréchet space, $(L^1_\nu(\nu))_\nu = L^1(\nu)$.

**Proof.** As already noted, $L^1(\nu)$ has a $\sigma$-Lebesgue topology. Since $L^1(\nu)$ has the relative topology from $L^1_\nu(\nu)$, we have $L^1(\nu) \subseteq (L^1_\nu(\nu))_\nu$. On the other hand, let $f \in (L^1_\nu(\nu))_\nu$ and assume (without loss of generality) that $f > 0$. Select $\Sigma$-simple functions $(s_k)_k$ such that $0 \leq s_k \uparrow f$ ($\nu$-a.e.). Then $0 \leq f - s_k \leq [f]$ for all $k$ and $(f - s_k) \downarrow 0$. Hence, $(f - s_k)_k$ converges to $0$ in $L^1_\nu(\nu)$, that is, $(s_k)_k \subseteq L^1(\nu)$ with $L^1(\nu)$ closed in $L^1_\nu(\nu)$. Hence, $f \in L^1(\nu)$ and so $(L^1_\nu(\nu))_\nu \subseteq L^1(\nu)$. \[\square\]

It is known that every Banach lattice $E$ having the $\sigma$-Fatou property and a weak unit which belongs to $E_\sigma$, is Banach lattice isomorphic to $L^1_\sigma(\nu)$ for some vector measure $\nu$ taking values in $E^+_\sigma$ [3, Theorem 2.5]. Our final result extends this fact to the Fréchet lattice setting. The proof proceeds along the lines of that of Theorem 2.5 in [3] but, with various differences due to the more general setting.

**Theorem 3.3.** Let $(F, \{q_n\}_{n \in \mathbb{N}})$ be any Fréchet lattice with the $\sigma$-Fatou property and possessing a weak unit $e$ which belongs to $F_\sigma$. Then there exists an $F_\sigma^+$-valued vector measure $\nu$ such that $F$ is Fréchet lattice isomorphic to $L^1_\nu(\nu)$ via an isomorphism $T : L^1_\nu(\nu) \to F$ which satisfies $T(\chi_\Omega) = e$ and

$$q_n(Tf) = \|f\|_0^{(n)}, \quad f \in L^1(\nu), \quad n \in \mathbb{N}.$$  

Moreover, the restriction map $T|_{L^1(\nu)} = I_\nu$.

**Proof.** The proof proceeds via a series of steps.

(i) Since $F$ satisfies the $\sigma$-Fatou property, $F$ is $\sigma$-Dedekind complete and hence, so is $F_\sigma$. Therefore, $F_\sigma$ is a $\sigma$-Dedekind complete Fréchet lattice with a $\sigma$-Lebesgue topology. Theorem 17.9 in [1] then guarantees that $F_\sigma$ has a Lebesgue topology and is Dedekind complete. Since $e$ is also a weak unit of $F_\sigma$, Proposition 3.1 ensures that there exists a measurable space $(\Omega, \Sigma)$ and a positive vector measure $\nu : \Sigma \to F_\sigma^+$ such that $F_\sigma$ is Fréchet lattice isomorphic to $L^1(\nu)$ via the integration map $T := I_\nu$. Moreover, (8) is also satisfied.

(ii) We extend $T$ to $L^1_\nu(\nu)^+$. Given $0 \leq f \in L^1_\nu(\nu)$, choose $\Sigma$-simple functions $0 \leq s_k \uparrow f$. Since $s_k \in L^1(\nu)$, we have $0 \leq x_k := T f_k \in F_\sigma$ and $q_n(x_k) = \|f_k\|_0^{(n)} \leq \|f\|_0^{(n)}$ for all $k \in \mathbb{N}$ and all $n \in \mathbb{N}$. Hence, $(x_k)_k$ is an increasing, topologically bounded sequence in $F_\sigma \subseteq F$. By the $\sigma$-Fatou property of $F$ the element $x := \sup_k x_k$ exists in $F^+$ and $q_n(x) = \lim_k q_n(x_k)$. Define $Tf := x \geq 0$. Fix $n \in \mathbb{N}$. By the $\sigma$-Fatou property of $L^1_\nu(\nu)$ and (8) we have

$$q_n(Tf) = \lim_k q_n(x_k) = \lim_k q_n(T f_k) = \lim_k \|f_k\|_0^{(n)} = \|f\|_0^{(n)}.$$  

Let us see that this extension of $T$ is well defined. First note that if $0 \leq h_k \uparrow h$ in $L^1(\nu)$, then $h_k \to h$ in $L^1(\nu)$. Hence, by continuity, $Th_k \to Th$ in $F$ with $(Th_k)_k$ increasing and, consequently, $Th = \sup_k Th_k$ in $F$ [1, Theorem 5.6(iii)]. Now, let $f \in L^1_\nu(\nu)$ and suppose that $f_k$ and $g_k$ are $\Sigma$-simple functions such that $0 \leq f_k \uparrow f$ and $0 \leq g_k \uparrow g$. Then $f_k \wedge g_m \uparrow k g_m$ for all $m \in \mathbb{N}$ and so $T g_m = \sup_k T(f_k \wedge g_m)$. Likewise, $T f_k = \sup_m T(f_k \wedge g_m)$. Therefore, $T f_k = \sup_m T(f_k \wedge g_m) = \sup_m T f_k \wedge T g_m = \sup_m T g_m$.

(iii) $T$ is positive, additive and positively homogeneous on $L^1_\nu(\nu)^+$ and so, can be uniquely extended to a positive linear map $T : L^1_\nu(\nu) \to F$ in a standard way. Indeed, $T$ is clearly positive and $T(\alpha f) = \alpha Tf$ for all $\alpha \in [0, \infty)$ and $f \in L^1_\nu(\nu)^+$. To check additivity, let $0 \leq f, g \in L^1_\nu(\nu)$ and choose $\Sigma$-simple functions $0 \leq f_k \uparrow f$ and $0 \leq g_j \uparrow g$. Define $x_k := Tf_k$ and $y_j := Tg_j$ for all $k, j \in \mathbb{N}$. By the definition of $T(f + g)$, $T(f)$ and $T(g)$ and [1, Theorem 1.6] we have

$$T(f + g) = \sup(x_k + y_j) = \sup x_k + \sup y_j = Tf + Tg.$$
(iv) $T$ is a Riesz space homomorphism on $L^1_w(v)$ (equivalently, $|Tf| = |f|$ for all $f \in L^1_w(v)$) [1, Theorem 1.17]). Suppose first that $0 \leq f, g \in L^1_w(v)$ satisfy $f \wedge g = 0$. Choose simple functions $0 \leq f_k \uparrow f$ and $0 \leq g_k \uparrow g$. Then $f_k \wedge g_k = 0$ for all $k \in \mathbb{N}$. Since $T$ is a Riesz space isomorphism of $L^1(v)$ onto $F_0$, it follows that $T[f_k \wedge g_k] = T(f_k \wedge g_k) = T(0) = 0$ for all $k \in \mathbb{N}$ and hence, via [18, Theorem 15.3], that $Tf \wedge Tg = (\sup Tfk) \wedge (\sup Tgk) = \sup(Tfk \wedge Tgk) = 0$. For arbitrary $f \in L^1_w(v)$ we have that $f^+ \wedge f^- = 0$ and so the previous argument yields $Tf^+ \wedge Tf^- = 0$. Hence, $T$ is a Riesz space homomorphism on $L^1_w(v)$ [20, Proposition 1.3.11].

(v) Fix $n \in \mathbb{N}$. Let $f \in L^1_w(v)$. From (iv) it follows that $q_n(Tf) = q_n(Tf) = q_n(Tf)$. But, in (ii) it was shown that $T$ satisfies $q_n(Tf) = \|f\|^{(n)}_p = \|f\|^{(n)}_p$ (because $f \in L^1_w(v)$). Therefore,

$$q_n(Tf) = \|f\|^{(n)}_p, \quad f \in L^1_w(v).$$

Since this holds for every $n \in \mathbb{N}$, we see that $T$ is also injective.

(vi) $T$ is surjective. Fix $x \in F^+$. Since $e$ is a weak unit of $F$, we have $x_k \uparrow x$, where $x_k = x \wedge (ke) \geq 0$, for $k \in \mathbb{N}$. Moreover, $q_n(x_k) \uparrow q_n(x)$, for $n \in \mathbb{N}$, as $F$ has the σ-Fatou property. Since $e \in F_0$ and $F_0$ is an ideal, it is clear that $(x_k)_k \subseteq F_0$. But, $T$ is a Riesz space isomorphism of $L^1(v)$ onto $F_0$ and so there is an increasing sequence $(f_k)_k \subseteq L^1(v)$ such that $x_k = Tf_k$ for $k \in \mathbb{N}$. Fix $n \in \mathbb{N}$. By (v) it follows that $\|f_k\|^{(n)}_n = q_n(Tf_k) = q_n(x_k)$, for $k \in \mathbb{N}$, and so $\sup_k \|f_k\|^{(n)}_n \leq q_n(x) < \infty$. Hence, the σ-Fatou property of $L^1_w(v)$ ensures that $f = \sup f_k \in L^1_w(v)$ and $\|f\|^{(n)}_n \uparrow \|f\|^{(n)}_n$. From the definition of the extension we have $Tf = x$. For arbitrary $x \in F$ there exist $f, g \in L^1_w(v)$ such that $x = Tf$ and $x = Tg$. So, $x = T(f - g)$. □

Example 3.4. Any increasing sequence $A = (a_n)_n$ of functions $a_n : \mathbb{N} \to (0, \infty)$ is called a Köthe matrix on $\mathbb{N}$, where increasing means $0 < a_0 \leq a_{n+1}$ pointwise on $\mathbb{N}$, for each $n \in \mathbb{N}$. The Köthe echelon space $\lambda_\infty(A)$ is the vector space

$$\lambda_\infty(A) := \{x \in \mathbb{R}^\mathbb{N} : a_n x \in \ell_\infty \text{ for all } n = 1, 2, \ldots \},$$

equipped with the increasing sequence of solid Riesz seminorms

$$\|x\|_k := \sup_{m \geq n} a_n(m)|x_m|, \quad x = (x_m) \in \lambda_\infty(A).$$

Of course, the order in $\lambda_\infty(A)$ is the pointwise order on $\mathbb{N}$. Then $\lambda_\infty(A)$ is a Fréchet lattice and $(\lambda_\infty(A), a)$ is the proper closed ideal

$$\lambda_0(A) := \{x \in \lambda_\infty(A) : a_n x \in c_0 \text{ for all } n = 1, 2, \ldots \}.$$ 

It is routine to check that $\lambda_\infty(A)$ has the σ-Fatou property and contains a weak unit $e \in \lambda_\infty(A)^+$. Indeed, any $e \in \lambda_\infty(A)^+$ satisfying $e_n > 0$ for all $m \in \mathbb{N}$ suffices. According to Theorem 3.3, the space $\lambda_\infty(A)$ is Fréchet lattice isomorphic to $L^1_w(v)$ for some vector measure $v$. However, since $\lambda_\infty(A)$ does not have a Lebesgue topology, it cannot be Fréchet lattice isomorphic to $L^1(\eta)$ for any vector measure $\eta$.

References