Journal of Applied Logic 7 (2009) 388–402

Contents lists available at ScienceDirect

Journal of Applied Logic

www.elsevier.com/locate/jal

Blueprint for a dynamic deontic logic

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ARTICLE INFO

Article history:
Received 16 January 2008
Accepted 9 April 2009
Available online 23 May 2009

Keywords:
Deontic logic
Dynamic logic
Events
Real actions
Deontic actions
Norms
Chisholm's Paradox
Forrester's Paradox
Ross's Paradox

ABSTRACT

Extending the idiom of dynamic logic we outline a deontic logic in which deontic operators operate on terms rather than on formulæ. In a second step we distinguish between what we call real and deontic actions.

A famous paper by Alchourrón, Gärdenfors and Makinson opened up a new avenue of research into the logic of belief and belief change [1]. One of the later extensions is dynamic doxastic logic (DDL), which develops the AGM approach as a modal logic [7,12]. Work in this area continues.

It is noteworthy that the erstwhile interest in theory change of at least one of the founding fathers of AGM was not in belief change but in normative change. What the late Carlos Alchourrón, professor of jurisprudence, had originally wanted was, it seems, a logic of norms and norm change. Many years later it makes sense to ask whether there is a dynamic deontic logic (DΔL) that pursues Alchourrón's ambition. In this note we outline a blueprint of an answer. (For previous efforts by this author to provide such an answer, see in particular [11,15] and [16].)

We proceed in three steps. Already Georg Henrik von Wright, the founder of modern deontic logic, came to the conclusion that deontic logic must be built on a logic of action. In accordance with this insight we outline in Section 1 a logic of action, admittedly fairly meagre: it avoids a number of important but difficult topics, such as agency, causality and intentionality. In Section 2 we outline a deontic logic which is dynamic in the sense of allowing for what we call real actions. In Section 3 we suggest how one can also provide for what we call deontic actions.

In Appendix A three well-known paradoxes of deontic logic are discussed in the light of the theory developed in Section 2.

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[1] Some of the work reported in this note was carried out when the author was a fellow-in-residence in 2006 at N.I.A.S., the Netherlands Institute of Advanced Study at Wassenaar. An earlier version of the paper was presented in 2007 at a workshop on dynamic logic at the Université du Québec à Montréal as well as at a Dagstuhl Seminar. The paper was revised during the tenure of a Killam Visiting Fellowship at the University of Calgary 2008-9. The author wishes to thank two anonymous referees for helpful comments, and students and faculty in the Calgary philosophy department for stimulating discussions. Special thanks are also due to Olivier Roy.

1570-8683/$ – see front matter
doi:10.1016/j.jal.2009.04.001

1. A temporal logic of action

1.1. Model theory

Without giving rigorous explanations, let us outline some key concepts. The base of any model will be a set (universe) U of points called the *environment*. Nonempty sequences of points will be called *paths*; they can be either finite or infinite in one or two directions. One-element sequences are regarded as *trivial*. A path p is a *subpath* of a path q if there are paths r and s such that $q = rps$; if r is trivial, p is a an *initial subpath* (is an *initial*) of q, while if s is trivial, q is a *final subpath* (is *final* in) q. Two paths p and q can be combined into one path, denoted by pq, if p has a last element $p(\#)$ and q has a first element $q(*)$ and the two are the same (if not, we regard the notation pq as meaningless). Notice that if $p(*) = u$ and $p(\#) = v$, then $\langle u, u\rangle p = p = p\langle v, v\rangle$.

Another fundamental model theoretical ingredient is that of a given set E of *actions* or *events* in U.[2] An *event* in U is a set of finite paths in U. If a is an event and $p \in a$, we say that *p realizes a* (p is a *realization* of a). Two special events are the *trivial* event, which is realized by every one-element sequence, and the *impossible* event, which is never realized. One can think of a number of set-theoretical operations on events under which E should be closed, for example, the sum $a \cup b$, the relative product $a \mid b$ and the difference $a - b$. Other set-theoretical operations on events, more difficult to relate to informal intuitions, are the product, the universal complement and Kleene's star operation, and they are not considered here. (But see [2] for one effort to accommodate the complement of events.)

Yet another fundamental concept is a given set H of *(complete) histories* in U—paths in U that are complete in the sense that if f is a proper subpath of a history h, then f is not itself a history. If a history is of the form hg, where thus the last element $h(\#)$ of h is also the first element $g(*)$ of g, then we will refer to (h, g) as an *articulated history*. One may say that (h, g) represents a particular way of looking at hg with h as the past, g as the future and the point $h(\#) = g(*)$ as the present.

We say that h is a *(possible) past* if $hg \in H$ for some g, while g is a *(possible) future* if $hg \in H$ for some h. If h is a past, then we write $\mathrm{cont}(h)$ for the set $\{g\colon\ hg \in H\}$ of possible continuations (possible futures) of h. Our pasts and futures are nonstrict in the sense that they include the present. In particular, if h is a past and g is a future that is trivial in the sense that $hg = h$, then the present is represented by a unique element u such that $u = h(\#)$ and $g = \{\langle u, u\rangle\}$.

If $S \subseteq \mathrm{cont}(h)$ we refer to (h, S) as a *possible situation* (with S the set of possible futures). In the special case that $S = \mathrm{cont}(h)$ we say that $(h, S) = (h, \mathrm{cont}(h))$ is an *actual situation*. Furthermore, if (h, S) is a possible situation, and if p is a finite path such that $h(\#) = p(*)$, then we write $S^p = \{f\colon\ pf \in S\}$. And if (h, S) is a possible situation and $g \in S$ we say that (h, g, S) is a *possible scenario* (note that in this case hg is a complete history).

If f is a history or a past or a future we say that f *includes* an event a if f contains a subpath that realizes a, and that *f excludes a* if there is no such subpath.

1.2. Syntax and meaning conditions

Our object languages must contain a denumerable set of propositional letters (primitive formulæ) $\mathsf{P}_0, \mathsf{P}_1, \ldots, \mathsf{P}_n, \ldots$ and a disjoint denumerable set $\mathsf{e}_0, \mathsf{e}_1, \ldots, \mathsf{e}_n, \ldots$ of event letters (primitive terms). In addition there has to be an adequate supply of Boolean (truth-functional) connectives as well as special operators; the latter will include at least the sum operator (+), and we will very briefly consider also two concatenation operators (; and ;;). Whatever the details, our language will contain both formulæ and terms.

A *basic frame* is a triple (U, E, H) such that U is a universe, E is a set of events (with certain closure conditions) and H is a set of complete histories. A *valuation* is a function V from the set of propositional letters into the power set of U and from the set of event letters into E. This function is extended in a natural way to all pure Boolean formulæ and to all terms. We will write $[\![\phi]\!]$ for the value assigned to a pure Boolean formula ϕ and $[\![\alpha]\!]$ for the value assigned to a term α. Thus, for all n,

$$[\![\mathsf{e}_n]\!] = V(\mathsf{e}_n),$$

$$[\![\mathsf{P}_n]\!] = V(\mathsf{P}_n).$$

Examples of meaning conditions (where ϕ and ψ are pure Boolean formulæ, and α and β are terms):

$$[\![\neg\phi]\!] = U - [\![\phi]\!],$$

$$[\![\phi \wedge \psi]\!] = [\![\phi]\!] \cap [\![\psi]\!],$$

$$[\![\phi \vee \psi]\!] = [\![\phi]\!] \cup [\![\psi]\!],$$

$$[\![\alpha + \beta]\!] = [\![\alpha]\!] \cup [\![\beta]\!],$$

[2] Many philosophers distinguish between actions and events, as they should. However, in this paper we do not.

$$[\![\alpha \,;\, \beta]\!] = [\![\alpha]\!] \mid [\![\beta]\!],$$
$$[\![\alpha \,;\, \beta]\!] = [\![\alpha]\!] \mid\mid [\![\beta]\!].$$[3]

Relative to such a model it is straight-forward to give meaning conditions also for temporal operators. As an example, consider our language with three new proposition-forming propositional operators added: [F], [P], and [H]. We wish to define the notion that a formula ϕ in this language is *true* with respect to an articulated history (h, g), in symbols, $(h, g) \vDash \phi$.

If ϕ is a pure Boolean formula, the defining condition is simply

$$(h, g) \vDash \phi \text{ iff } u \in [\![\phi]\!], \text{ where } u = h(\#) = g(*).$$

(Note that the notation $[\![\phi]\!]$ is meaningful only if ϕ is a pure Boolean formula.)

The conditions for the Boolean connectives are obvious. For the new operators they are:

$$(h, g) \vDash [\mathsf{F}]\theta \text{ iff}$$
$$(h', g') \vDash \theta, \text{ for all } p, h', g' \text{ such that } hp = h' \text{ and } pg' = g$$
$$(\text{and therefore } h'g' = hg),$$

$$(h, g) \vDash [\mathsf{P}]\theta \text{ iff}$$
$$(h', g') \vDash \theta, \text{ for all } p, h', g' \text{ such that } h'p = h \text{ and } g' = pg$$
$$(\text{and therefore } h'g' = hg),$$

$$(h, g) \vDash [\mathsf{H}]\theta \text{ iff } (h, g') \vDash \theta, \text{ for all } g' \in \text{cont}(h).$$[4]

Here [F] and [P]—F for "future" and P for "past"—correspond to Prior's G and H, respectively, while [H] is an operator that has been read variously as "historically necessary", "unavoidably", and "settled true". Each of these temporal "box" operators is of the form $[\cdots]$, and for each of them there is a corresponding "diamond" operator $\langle\cdots\rangle$, in general defined by the condition that the schema $\langle\cdots\rangle\phi \leftrightarrow \neg[\cdots]\neg\phi$ be valid.

In a similar way we define

$$(h, g) \vDash [\mathsf{UNTIL}\,\phi]\theta \text{ iff either}$$
$$\text{(i) } (h', g') \nvDash \phi, \text{ for all } h' \text{ and } g' \text{ such that } h'g' = hg, \text{ or else}$$
$$\text{(iia) there is a shortest path } p \text{ such that } (h', g') \vDash \phi,$$
$$\text{where } h' = hp \text{ and } pg' = g, \text{ and}$$
$$\text{(iib) for all initial subpaths } q \text{ of } p, (h'', g'') \vDash \theta,$$
$$\text{where } h'' = hq \text{ and } qg'' = g.$$

The dynamic operators that we need are less common. First there are the three proposition-forming term operators occurs, occurring and occurred, the informal meaning of which is "is just about to occur", "is occurring" and "has just occurred":

$$(h, g) \vDash \mathsf{occurs}\,\alpha \text{ iff } g = pg', \text{ for some finite path } p \in [\![\alpha]\!] \text{ and (unique) future } g',$$

$$(h, g) \vDash \mathsf{occurring}\,\alpha \text{ iff } h = h'p \text{ and } g = qg', \text{ for some finite nonempty paths } p \text{ and } q \text{ such that}$$
$$pq \in [\![\alpha]\!], \text{ (unique) past } h' \text{ and (unique) future } g',$$

$$(h, g) \vDash \mathsf{occurred}\,\alpha \text{ iff } h = h'p, \text{ for some finite path } p \in [\![\alpha]\!] \text{ and (unique) past } h'.$$

Then there are the following five complex formula-making operators $[\mathsf{after}\,\alpha]$, $[\mathsf{after}^\star\,\alpha]$, $[\mathsf{during}\,\alpha]$, $[\mathsf{before}\,\alpha]$ and $[\mathsf{before}^\star\,\alpha]$, where α must be a real term:

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[3] If a and b are sets of finite paths, then $a \mid b = \{pq\colon\ p \in a \ \&\ q \in q \ \&\ p(\#) = q(*)\}$, while $a \mid\mid b = \{prq\colon\ r \ \&\ p \in a \ \&\ q \in q \ \&\ p(\#) = r(*) \ \&\ r(\#) = q(*)\}$.

[4] Note that it would be natural to regard the notation $h \vDash \phi$ as meaningful whenever h is a past and ϕ does not contain the operator [P], and similarly to regard the notation $g \vDash \phi$ as meaningful if g is a past and ϕ does not contain either [F] or [H].

$(h, g) \vDash [\mathsf{after}\,\alpha]\,\phi$ iff $(h', g') \vDash \phi$, for all $p \in [\![\alpha]\!]$ and h' and g' such that $h' = hp$ and $pg' = g$,

$(h, g) \vDash [\mathsf{after}^\star\,\alpha]\phi$ iff $(h', g') \vDash \phi$, for all paths $p \in [\![\alpha]\!]$ and paths q, r (where q may be the null path) such that $p = qr$ and $hr = h'$ and $g = rg'$.

$(h, g) \vDash [\mathsf{during}\,\alpha]\phi$ iff $(h', g') \vDash \phi$, for all h' and g' such that for some paths p, q, r, x and y (where either q or r may be null) and the conditions $h = xq$ and $h' = xq'$ and $g = ry$ and $g' = r'y$ and $p = qr = q'r'$.

$(h, g) \vDash [\mathsf{before}\,\alpha]\,\phi$ iff $(h', g') \vDash \phi$, for all $p \in [\![\alpha]\!]$ and h' and g' such that $h'p = h$ and $g' = pg$,

$(h, g) \vDash [\mathsf{before}^\star\,\alpha]\phi$ iff $(h', g') \vDash \phi$, for all paths $p \in [\![\alpha]\!]$ and paths q, r (where r may be the null path) such that $p = qr$ and $h = h'q$ and $qg = g'$.

Note that our "after"-operators are different from Pratt's well-known operator in [9]: his $[\alpha]\phi$ is rendered by our $[\mathsf{H}](\mathsf{occurs}\,\alpha \to [\mathsf{after}\,\alpha]\phi)$.

Truth in a model and *validity* in a frame are defined along traditional lines.[5]

As examples of valid schemas we select the following:

$\mathsf{occurs}\,\alpha \leftrightarrow \langle \mathsf{after}\,\alpha \rangle \top$,

$\mathsf{occurring}\,\alpha \leftrightarrow \langle \mathsf{during}\,\alpha \rangle \top$,

$\mathsf{occurred}\,\alpha \leftrightarrow \langle \mathsf{before}\,\alpha \rangle \top$;

$\mathsf{occurs}\,\alpha \to \big([\mathsf{after}^\star\,\alpha]\phi \leftrightarrow [\mathsf{after}\,\alpha]\phi\big)$,

$\mathsf{occurred}\,\alpha \to \big([\mathsf{before}^\star\,\alpha]\phi \leftrightarrow [\mathsf{before}\,\alpha]\phi\big)$.

$\mathsf{occurs}\,\alpha \leftrightarrow \langle \mathsf{after}\,\alpha \rangle \top$,

$\mathsf{occurred}\,\alpha \leftrightarrow \langle \mathsf{before}\,\alpha \rangle \top$.

1.3. The result operator

Minimality is a concept that surfaces in connexion with concepts such as conditionals and belief revision. Ramsey was happy to accept a conditional "if A then B" if B would be true in a situation in which things had been changed just enough to make A true. In AGM type belief revision, a new piece of information is incorporated into one's set of beliefs by making a certain minimal adjustment. (Makinson has given other examples of conceptual analysis where minimality comes up [8]).

One such example is offered by what may be called *resultative* actions or events. In everyday life there may be many ways in which a certain state of affairs can result, but talking about them we automatically filter out from consideration ways that are extraordinary or inappropriate. Say we wish to capture the notion of "bringing it about that P" or "the coming about that P", where P is a proposition; that is, we wish to characterize the "paradigmatic" or "standard" event resulting in its being the case that P. Then minimality comes in.

To proceed more formally, say that f is a *selection function* for U if f is defined on the set of subsets of U and the following three conditions are satisfied: for all P and Q,

$fP \subseteq P$ (INCLUSION);

if $P \subseteq Q$, if $fP \neq \varnothing$ then $fQ \neq \varnothing$ (MONEYS); [6]

if $P \subseteq Q$ and $P \cap fQ \neq \varnothing$ then $fP = P \cap fQ$ (ARROW).

Let (U, E, H) be a basic frame and F a function defined on U such that, for each $u \in U$, F_u is a selection function for U. Assume that E is closed under F in the sense that $F_u P \in E$, for each point $u \in U$ and proposition P. Then F is called the *result functor* while (U, E, F) is called a *result frame*.

[5] Our notation for the truth of a formula ϕ is of the format $\ldots \vDash \phi$, where in lieu of the three dots we have what is sometimes called an *index*. The index is supposed to contain enough information to let you evaluate the formula that comes after the double turnstile. Thus it would be enough to write $u \vDash \phi$ if u is a point of the universe and ϕ is a pure Boolean formula, enough to write $h \vDash \phi$ if ϕ is made up of pure Boolean formulæ and the operator [P], and so on.

[6] "MOnotonicity for NonEmptY Segments."

On the syntactic side we add a new term-forming propositional operator ∂, the *result operator* (or the *delta operator*) with the following meaning-condition:

$$[\![\partial\phi]\!] = \{(u, v)\colon\ v \in F_u [\![\phi]\!]\}.$$

(We consider ϕ to be the "result" of the action $\partial\phi$.)

Consider a propositional dynamic logic in which the atomic terms are all of the form $\partial\phi$, where ϕ is a formula. Such a logic would be closely related to a David Lewis type conditional logic: a dynamic formula $[\partial\phi]\theta$ would correspond to a conditional formula $\phi \mathrel{\Box\!\!\rightarrow} \theta$. (Cf. Chellas's version of Lewis's conditional logic in [3]. For discussions of the delta operator, see also [13] and [14].)

2. Deontic logic with real actions

2.1. Pre-theoretical remarks

As indicated by the heading, in this section we consider the deontic of logic of what we call *real* actions; in the following section we also consider *deontic* actions. A yet further kind of action, not considered here, is what may be called *legislative* action, by which one norm is replaced by another. Real and deontic actions are performed by agents, legislative actions by a norm-giver.

A norm draws a distinction between what is acceptable and what is not—what is in accordance with the norm and what is not. Legal codes separate legal from illegal; moralities, right from wrong and good from bad; conventions, correct from incorrect; fashion, what is "in" from what is "out"; and so on. In real life norms may not sharp enough or complete enough to settle all questions, but the norms that appear in this paper are supposed to be both sharp and complete.

So what is a complete norm? Here we think of a complete norm as a norm-giver (legislator, moral genius, arbiter, God) who can answer all questions as to what, in any given situation, is normal—that is, in accordance with the norm. With respect to any given past, however irregular from a normative point of view, the norm-giver should be able to delineate a subset consisting of exactly those futures that are still possible at that time and that are in accordance with the norm.[7]

Traditionally the major deontic notions are obligation, permission and prohibition. In the dominant *Seinsollen* ("ought-to-be") tradition they are treated as concepts applying to propositions. In our modelling, as presented so far, it seems more natural to follow the *Tunsollen* ("ought-to-do") tradition, in which they are treated as concepts applying to actions or events. Thus here we classify the deontic status of an action or event according to whether it is must or may be done or must or may be omitted:

a is obligatory—a must be done,

a is permissible—a may be done,

a is forbidden—a must be omitted,

a is omissible—a may be omitted.[8]

Pre-theoretical intuitions dictate that no action be both forbidden and permitted at the same time, nor both obligatory and omissible. Hence if we were to limit ourselves to so-called closed systems—systems in which every action is either permitted or forbidden, and either obligatory or omissible—then we would have

a is permissible iff a is not forbidden,

a is omissible iff a is not obligatory.

In other words, in closed systems we could begin with the notions of obligation and forbiddance (prohibition), or with the notions of obligation and permission, and then define the remaining two notions. But in general we need all four.

In addition to these unconditional concepts there are numerous conditional ones: a is obligatory/permissible/forbidden/omissible relative to certain condition.

[7] As logicians, we should leave open the possibility that the subset of normal futures is empty. A situation in which there are no normal futures could perhaps be called "tragic". This terminology would fit those moral philosophers who believe in the possibility of moral dilemmas—situations in which no action is right. See further comments are the end of Section 2.2.

[8] While the terms *obligatory, permissible* (or *permitted*), and *forbidden* (or *prohibited*) are standard; *omissible* is not. However, "omissible" *is* an English word, at least according to the O.E.D., which credits Bentham with using it in 1816. Furthermore, finding a better term is a problem. The term *non-obligatory* is a possibility, but doesn't also the fourth member of the quartet deserve a name of its own? The Stanford Encyclopedia of Philosophy suggests *gratuitous*, but since "must be omitted" implies "may be omitted" we would then have "forbidden" implying "gratuitous", and that does not sound right.

2.2. Simple norms

In this paper we present two notions of norms, first, in this section, a simpler one and then, later, a more inclusive one. A *(simple) norm* is a rule that assigns to each past a set of normal futures. Thus if h is any past and $\mathrm{cont}(h)$ is the set of futures that are possible after h, then a norm N will single out a subset $N(h)$ of $\mathrm{cont}(h)$—the set of futures after h that are normal (legal) according to N. (Thus if the future is trivial in the sense that $hg = h$, then the present is represented by the element u, where $u = h(\#)$ and $g = \{\langle u, u\rangle\}$. Consequently, in this degenerate case either $N(h) = \{\langle u, u\rangle\}$ or $N(h) = \varnothing$.)

We impose the following coherence condition—a kind of independence condition:

if $g = pg'$, for any finite path p,

then $g \in N(h)$ only if $g' \in N(hp)$ (COHERENCE).

In social choice theory independence conditions are controversial, but in the present context (COHERENCE) seems acceptable.

It would also be possible to impose the following condition:

$N(h) \neq \varnothing$ (NORMAL OPTIMISM).

Are there "tragic dilemmas"—situations in which no action is right and every alternative is wrong? Some would answer this question in the affirmative; in our terminology, to the question whether there are past histories h such that the set $N(h)$ of normal continuations is empty they would say, yes. Here, though, in contradiction with such views, but in accordance with traditional deontic logic, we will accept the postulate (NORMAL OPTIMISM). But nothing of theoretical significance would seem to depend on this assumption.

2.3. Two sets of deontic term-to-term operators

We now give formal definitions of two sets of deontic term operators, the "local" operators ob, pm, fb and ex for, respectively, (locally) obligatory, (locally) permissible, (locally) forbidden and (locally) omissible, and the "normal" operators ob*, pm*, fb* and ex* for, respectively, (normally) obligatory, (normally) permissible, (normally) forbidden and (normally) omissible.[9]

Given a basic frame (U, E, H) and a norm N, here are truth-conditions for the local operators: if (h, g) is an articulated history, then

$(h, g) \vDash \mathsf{ob}\,\alpha$ iff

(1) $\forall g' \in N(h) \exists q \in [\![\alpha]\!]$ (q is a subpath of g'),

(A) $\forall q \in [\![\alpha]\!] \exists g' \in N(h)$ (q is a subpath of g');

$(h, g) \vDash \mathsf{pm}\,\alpha$ iff

(2) $\exists g' \in N(h) \exists q \in [\![\alpha]\!]$ (q is a subpath of g'),

(A) $\forall q \in [\![\alpha]\!] \exists g' \in N(h)$ (q is a subpath of g');

$(h, g) \vDash \mathsf{fb}\,\alpha$ iff

(3) $\forall g' \in N(h) \forall q \in [\![\alpha]\!] \sim$ (q is a subpath of g'),

(B) $\forall q \in [\![\alpha]\!] \exists g' \in N(h) \sim$ (q is a subpath of g');

$(h, g) \vDash \mathsf{ex}\,\alpha$ iff

(4) $\exists g' \in N(h) \forall q \in [\![\alpha]\!] \sim$ (q is a subpath of g'),

(B) $\forall q \in [\![\alpha]\!] \exists g' \in N(h) \sim$ (q is a subpath of g').

Thus for each of the four term operators, two conditions are defined which might be called, respectively, the *substantive condition* ((1), (2), (3), (4)) and the *homogeneity condition* ((A), (B)). The motivation for the substantive condition is obvious enough: if an action is obligatory, it will be performed in every legal future; if permitted, it will be performed in some legal future; if forbidden, it will be remain unperformed in every legal future; if omissible, it will remain unperformed in some legal future. The motivation for the homogeneity condition is less obvious; it is there in order to model Ross's concept of obligation in [10] and Kamp's concept of free-choice permission in [6]. Thus in a situation in which an action is obligatory or

[9] In order to distinguish the omissibility operator from the obligation operator, the symbol ex has been chosen over the symbol om. One may imagine that ex stands for something like "extra" or "exotic" or perhaps even "eccentric".

permissible, any way of performing the action will discharge the obligation or exercise the permission. Similarly, if an action is forbidden or omissible, then any way of performing it will violate the prohibition or exhaust the omissibility. (Remember that the concepts we are considering here are one-time concepts.)

It is worth noting that, of these conditions, (A) implies (2), and each of (3) and (4) implies (B). Thus a more economic set of definitions would be the following:

$$(h, g) \vDash \mathsf{ob}\,\alpha \text{ iff } (1)\ \&\ (\mathrm{A}),$$

$$(h, g) \vDash \mathsf{pm}\,\alpha \text{ iff } (\mathrm{A}),$$

$$(h, g) \vDash \mathsf{fb}\,\alpha \text{ iff } (3),$$

$$(h, g) \vDash \mathsf{ex}\,\alpha \text{ iff } (4).$$

We note that the following schemas are valid:

$$\mathsf{ob}\,\alpha \to \mathsf{pm}\,\alpha,$$

$$\mathsf{fb}\,\alpha \to \mathsf{ex}\,\alpha,$$

$$\neg(\mathsf{ob}\,\alpha \wedge \mathsf{fb}\,\alpha)$$

and that the following schemas are not generally valid:

$$\neg(\mathsf{pm}\,\alpha \wedge \mathsf{ex}\,\alpha),$$

$$\mathsf{ob}\,\alpha \vee \mathsf{fb}\,\alpha,$$

$$\mathsf{pm}\,\alpha \vee \mathsf{ex}\,\alpha.$$

We also list the following validities[10]:

$$\mathsf{ob}(\alpha + \beta) \leftarrow (\mathsf{ob}\,\alpha \wedge \mathsf{ob}\,\beta),$$

$$\mathsf{pm}(\alpha + \beta) \leftrightarrow (\mathsf{pm}\,\alpha \wedge \mathsf{pm}\,\beta),$$

$$\mathsf{fb}(\alpha + \beta) \leftrightarrow (\mathsf{fb}\,\alpha \wedge \mathsf{fb}\,\beta),$$

$$\mathsf{ex}(\alpha + \beta) \to (\mathsf{ex}\,\alpha \wedge \mathsf{ex}\,\beta).$$

Now truth-conditions for the normal operators:

$$(h, g) \vDash \mathsf{ob}^*\,\alpha \text{ iff, for all finite paths } p \text{ such that } h(\#) = p(*),$$
$$\text{if } \sim \exists q \in [\![\alpha]\!]\ (q \text{ is a subpath of } p) \text{ then}$$

$$(1^*)\ \forall g' \in N(hp) \exists q \in [\![\alpha]\!]\ (q \text{ is a subpath of } g'),$$
$$(\mathrm{A}^*)\ \forall q \in [\![\alpha]\!] \exists g' \in N(hp)\ (q \text{ is a subpath of } g').$$

$$(h, g) \vDash \mathsf{pm}^*\,\alpha \text{ iff, for all finite paths } p \text{ such that } h(\#) = p(*),$$
$$\text{if } \sim \exists q \in [\![\alpha]\!]\ (q \text{ is a subpath of } p) \text{ then}$$

$$(2^*)\ \exists g' \in N(hp) \exists q \in [\![\alpha]\!]\ (q \text{ is a subpath of } g'),$$
$$(\mathrm{A}^*)\ \forall q \in [\![\alpha]\!] \exists g' \in N(hp)\ (q \text{ is a subpath of } g').$$

$$(h, g) \vDash \mathsf{fb}^*\,\alpha \text{ iff, for all finite paths } p \text{ such that } h(\#) = p(*),$$
$$\text{if } \sim \exists q \in [\![\alpha]\!]\ (q \text{ is a subpath of } p) \text{ then}$$

$$(3^*)\ \forall g' \in N(hp) \forall q \in [\![\alpha]\!] \sim (q \text{ is a subpath of } g'),$$
$$(\mathrm{B}^*)\ \forall q \in [\![\alpha]\!] \exists g' \in N(hp) \sim (q \text{ is a subpath of } g').$$

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[10] We write $\to$ for material implication, $\leftrightarrow$ for material equivalence, and $\leftarrow$ for the converse of material implication.

$(h, g) \vDash \mathsf{ex}^* \alpha$ iff, for all finite paths p such that $h(\#) = p(*)$,

if $\sim \exists q \in [\![\alpha]\!]$ (q is a subpath of p) then

(4*) $\exists g' \in N(hp) \forall q \in [\![\alpha]\!] \sim$ (q is a subpath of g'),

(B*) $\forall q \in [\![\alpha]\!] \exists g' \in N(hp) \sim$ (q is a subpath of g').

As in the case of the local operators, since (A*) implies (2*), and each of (3*) and (4*) implies (B*), also these definitions can be simplified:

$(h, g) \vDash \mathsf{ob}^* \alpha$ iff, for all finite paths p such that $h(\#) = p(*)$,

if $\sim \exists q \in [\![\alpha]\!]$ (q is a subpath of p) then (1*) & (A*),

$(h, g) \vDash \mathsf{pm}^* \alpha$ iff, for all finite paths p such that $h(\#) = p(*)$,

if $\sim \exists q \in [\![\alpha]\!]$ (q is a subpath of p) then (A*),

$(h, g) \vDash \mathsf{fb}^* \alpha$ iff, for all finite paths p such that $h(\#) = p(*)$,

if $\sim \exists q \in [\![\alpha]\!]$ (q is a subpath of p) then (3*),

$(h, g) \vDash \mathsf{ex}^* \alpha$ iff, for all finite paths p such that $h(\#) = p(*)$,

if $\sim \exists q \in [\![\alpha]\!]$ (q is a subpath of p) then (4*).

The following schemas are valid:

$\mathsf{ob}^* \alpha \leftrightarrow [\mathsf{H}][\mathsf{UNTIL\ occurs}\,\alpha]\mathsf{ob}\,\alpha,$

$\mathsf{pm}^* \alpha \leftrightarrow [\mathsf{H}][\mathsf{UNTIL\ occurs}\,\alpha]\mathsf{pm}\,\alpha,$

$\mathsf{fb}^* \alpha \leftrightarrow [\mathsf{H}][\mathsf{UNTIL\ occurs}\,\alpha]\mathsf{fb}\,\alpha,$

$\mathsf{ex}^* \alpha \leftrightarrow [\mathsf{H}][\mathsf{UNTIL\ occurs}\,\alpha]\mathsf{ex}\,\alpha.$

2.4. Complete norms

The deontic logic generated by these definitions is meagre; simple norms are not in general enough. A simple norm tells us, for each given past, what the normal continuations are. But in general more is needed: given a past we often wish to give special consideration to some particular possible development—or, as we shall say, *focus* on a particular set of possible futures. This is a wish that need not involve normative considerations: given that such and such were to happen, then what? But this way of thinking is particularly relevant in a normative context: in the best of all possible worlds, such-and-such would be done; but, given that we don't live in the best of all possible worlds, what should we do?

In the technical jargon used in the paper, if after a past history h we focus on a particular set S of possible futures (so S is a subset of cont(h)), then what is the set $N(h, S)$ of possible futures in S that is singled out (recommended by, prescribed by) the norm N? Simple norms cannot answer this question. What is needed is what we will call "complete" norms: norms that assign, to every possible situation, a normative (legal) position.

In a more general context one would consider hierarchies of norms, but in this paper there is but one norm, and it is unchanging. By contrast, the normative position changes with each real action. Thus if (U, E, H) is a basic frame, a *(complete) norm* N is a function assigning, to each situation (h, S) (where $S \subseteq \mathrm{cont}(h)$), a set $N(h, S)$ of possible futures of h—the set of futures that, according to N, are normal (legal) in that situation. There are three familiar conditions on N:

(i) $N(h, S) \subseteq S$ (INCLUSION),
(ii) if $S \subseteq T$, then $N(h, S) \neq \varnothing$ only if $N(h, T) \neq \varnothing$ (MONEYS),
(iii) if $S \subseteq T$ and $S \cap fT \neq \varnothing$, then $N(h, S) = S \cap N(h, T)$ (ARROW).

To these we add a fourth condition already familiar to readers of this paper:

if $g = pg'$, for any finite path p,

then $g \in N(h)$ only if $g' \in N(hp)$ (COHERENCE).

It is important to notice that a norm-giver must be able to handle not only situations in which $S = \mathrm{cont}(h(\#))$; in order to be complete, the norm must govern every possible situation.

The meaning conditions for the four basic deontic term operators go through with "(h, g, S)" replacing "(h, g)" throughout (S being a subset of cont(h)). Thus for the operator ob we have

$(h, g, S) \vDash \mathsf{ob}\,\alpha$ iff

$\forall g' \in N(h, S)\exists q \in [\![\alpha]\!]$ (q is a subpath of g'),

$\forall q \in [\![\alpha]\!]\exists g' \in N(h, S)$ (q is a subpath of g').

Similarly, the generalized truth-condition for the operator ob^* is

$(h, g, S) \vDash \mathsf{ob}^*\,\alpha$ iff,

for all finite paths p such that $h(\#) = p(*)$,

$\forall g' \in N(hp, S^p)\exists q \in [\![\alpha]\!]$ (q is a subpath of g'),

$\forall q \in [\![\alpha]\!]\exists g' \in N(hp, S^p)$ (q is a subpath of g'),

where $S^p = \{f\colon\ pf \in S\}$.

The conditions for the other three operators are analogous.

It should again be emphasized that the operators defined here are the members of only one family among many. In particular there are also other families of obligations, permissions, prohibitions and omissibilities that may be described as one-time, standing, conditional, and so on. Although we don't do so here, it is possible to give formal definitions of such concepts within the system just defined.

2.5. The focus operator

As explained above, there are situations in which we wish to consider, not all the futures that are possible, but to focus on a restricted set of possible futures. To meet the expressive requirement associated with this focal idea we will now introduce what we call the "focus operator", written $[\mathsf{H} : \phi]$, where ϕ is a formula. Intuitively, this operator restricts attention to future paths of a certain kind—roughly speaking, those of which ϕ is true. Thus formulæ of type $[\mathsf{H} : \phi]\theta$ must be evaluated with respect, not just to a history (h, g), but to a triple (h, g, S), where S is the set of possible futures that are relevant—the ones on which we focus.[11]

Here is the meaning-condition for our focus operator (the "official" definition):

$(h, g, S) \vDash [\mathsf{H} : \phi]\,\theta$ iff $(h, f, S') \vDash \theta$, for all $f \in S'$,

where $S' = \{f \in S\colon\ (h, f, S) \vDash \phi\}$.

If the focus operator does not occur in θ, then this condition can be reduced to the following:

$(h, g, S) \vDash [\mathsf{H} : \phi]\,\theta$ iff $(h, f) \vDash \theta$, for all $f \in S$ such that $(h, f) \vDash \phi$.

It may appear that the focus operator is superfluous. Indeed, in the non-deontic context of the object language of Section 1 it is definable: the schema

$[\mathsf{H} : \phi]\,\theta \leftrightarrow [\mathsf{F}](\phi \to \theta)$

is valid there. However, in a deontic setting this reduction no longer works: how a proposed action is evaluated depends on what the alternatives are supposed to be. There are things that are forbidden under normal circumstances but which in an emergency become permitted or perhaps obligatory; other things that are normally obligatory but which in certain situations become omissible or even forbidden; and so on.

Here is an alternative way of defining the focus operator which avoids introducing a third element of the index:

$(h, g) \vDash^{\mathfrak{M}} [\mathsf{H} : \phi]\theta$ iff $(h, g) \vDash^{\mathfrak{M}^\phi} \theta$,

where $\mathfrak{M}^\phi = (U, E, H^\phi, V)$ and $H^\phi = \{f \in H\colon\ (h, f) \vDash^{\mathfrak{M}} \phi\}$.

Notice that $(\mathfrak{M}^\phi)^\psi = \mathfrak{M}^{\phi\wedge\psi}$.

From the official definition it would appear that a formula $[\mathsf{H} : \phi]\theta$ is really a kind of conditional: "given that ϕ, θ". But looked at in the way suggested by the alternative definition, the operator $[\mathsf{H} : \phi]$ may also be viewed as a model changing operator. While we prefer the first way of *defining* the focus operator, we nevertheless favour the second way of *viewing* the operator. This is why we think of it as a focus changing operator and stick with the admittedly clumsy notation $[\mathsf{H} : \phi]\theta$ rather than, say, $\phi \Rrightarrow \theta$.

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11 In an analogous way one might sometimes want to consider more than the actual set of possible futures—what if such and such were possible? Perhaps there would be some interest in considering, as a kind of dual to the focus operator, a "widening-of-scope" operator.

2.6. Seinsollen and Tunsollen

A question sometimes aired in the philosophical literature concerns the relative primacy of *Seinsollen* (ought-to-be) and *Tunsollen* (ought-to-do). At least three views are possible: (i) that *Seinsollen* is the basic concept and that *Tunsollen* can be defined in terms of it and non-deontic concepts; (ii) that *Tunsollen* is the basic concept and that *Seinsollen* can be defined in terms of it and non-deontic concepts; and (iii) that both concepts are basic and that neither is definable in terms of the other. In this paper we are not taking a stand on this issue.

For our purposes we find the *Tunsollen* approach congenial, but it would certainly be possible to introduce a deontic propositional operator in terms of which our deontic term operators would be definable; it might even be a good idea to do so. Here is one way of doing it. Let [D]—D for *deontic*—be a new unary proposition-forming propositional operator with the truth-condition

$$(h, g, S) \vDash [\mathsf{D}]\phi \text{ iff } (h, g', S) \vDash \phi, \text{ for all } g' \in N(h, S).$$

This operator may perhaps be read as "it is deontically necessary that" or "ideally". But it is too weak to be identified with "it ought to be the case" or "it is obligatory that".

We note the following validities involving the local operators:

$$\mathsf{ob}\,\alpha \to [\mathsf{D}]\langle\mathsf{F}\rangle\mathsf{occurs}\,\alpha,$$

$$\mathsf{pm}\,\alpha \to \langle\mathsf{D}\rangle\langle\mathsf{F}\rangle\mathsf{occurs}\,\alpha,$$

$$\mathsf{fb}\,\alpha \to [\mathsf{D}][\mathsf{F}]\neg\mathsf{occurs}\,\alpha,$$

$$\mathsf{ex}\,\alpha \to [\mathsf{D}]\langle\mathsf{F}\rangle\neg\mathsf{occurs}\,\alpha;$$

for all $\delta \in \{\mathsf{ob}, \mathsf{pm}, \mathsf{fb}, \mathsf{ex}\}$,

$$\delta\alpha \to [\mathsf{UNTIL\ occurs}\,\alpha]\delta\alpha,$$

$$\delta\alpha \to [\mathsf{H}][\mathsf{F}][\mathsf{after}\,\alpha]\neg\delta\alpha.$$

But it might also be possible to go the other way: to define propositional deontic operators in terms of deontic event operators (and additional non-deontic operators). Thus given the deontic term operators as well as the resultative operator ∂ mentioned in Section 1.3, we would be able to define a number of deontic proposition-forming propositional operators. For example, obligation operators O_1 and O_2 can be defined:

$$\mathsf{O}_1\phi \leftrightarrow \mathsf{ob}\,\partial\phi,$$

$$\mathsf{O}_2\phi \leftrightarrow \mathsf{ob}\,\partial[\mathsf{F}]\phi.$$

In the former case, the event $\partial\phi$ is obligatory (that is, the event resulting in ϕ becoming true (at least momentarily)), while in the latter case the event $\partial[\mathsf{F}]\phi$ $(= \partial([\mathsf{F}]\phi))$ is obligatory (that is, the event resulting in ϕ becoming true and remaining true).

Similarly, obligation operators O_3 and O_4 can be defined:

$$\mathsf{O}_3\phi \leftrightarrow \mathsf{ob}^*\partial\phi,$$

$$\mathsf{O}_4\phi \leftrightarrow \mathsf{ob}^*\partial[\mathsf{F}]\phi.$$

3. Deontic logic with both real and deontic actions: an outline

3.1. Pre-formal remarks

Real actions are not the only category of actions of interest to deontic logic. In this section we will briefly consider one other category, that of what we call (pure) deontic actions: actions that change the *legal (normal) position* without changing the *real position*.[12] (The treatment offered here is tentative; a fuller account may be given on another occasion.)

In particular we will discuss four: ordering, permitting, forbidding, and making-omissible a real action or event *a*:

- to *order a*: to make *a* obligatory ("*a* must be done!")
- to *permit a*: to make *a* permissible ("*a* may be done!")

[12] In this paper we only consider pure real actions and pure deontic actions. There is also a category of actions that may be called "real actions with deontic import". For example, writing one's name on a piece of paper has legal significance under certain circumstances, and so constitutes a legal action as well as a real action. It would be interesting to develop a deontic logic in which such actions can be represented. Here, however, we do not.

- to *prohibit* a: to make a forbidden ("a must be omitted!")
- to *declare a omissible*: to make a omissible ("a may be omitted!")

Our current way of developing our model theory suggests the following semi-formal understanding of the four action types:

- *ordering a*: as long as a has not been realized, every legal future includes an occurrence of a path in a, and every occurrence of a path in a will discharge the obligation to do a,
- *permitting a*: as long as a has not been realized, every legal future includes an occurrence of a path in a, and every occurrence of a path in a will exercise the permission to do a,
- *forbidding a*: as long as a has not been realized, every legal future excludes every occurrence of a path in a, and every occurrence of a path in a will violate the prohibition to do a,
- *making a omissible*: as long as a has not been realized, some legal future excludes every occurrence of a path in a, and every occurrence of a path in a goes against the omissibility to do a.

Note that these deontic actions modify the normative position within the current norm. That is to say, the norm does not change, but the normative position does.

When we proceed to formalize these ideas, we have to generalize our previous notions. What will a general frame look like? Suppose we start with a universe U of points as before and a set E of events in U (the points of U being "real points" and the events in E being "real events"). What will now count as a possible history? Real histories are maximal sequences of real events. Evidently, general histories should be sequences of real events and deontic actions.

To represent the deontic actions (there are only four of them in this paper!), assume that a is a real event or action and adopt the following encoding:

$(1, a)$ for 'ordering a',

$(2, a)$ for 'permitting a',

$(3, a)$ for 'forbidding a',

$(4, a)$ for 'making a omissible'.

At the same time let us agree to represent a path $u_1 \cdots u_n$ of points by the sequence $(0, u_1) \cdots (0, u_n)$. Thus in this semantics both real and deontic actions can be represented as sequences of ordered pairs (i, x), where either $i = 0$ and x is a real action, or $i \in \{1, 2, 3, 4\}$ and $x = a$ for some real action a.

For example, if a is a particular real event, then 'ordering a' is the singleton set $\{(1, a)\}$. Furthermore, general histories are maximal sequences where each step represents either a real change in the environment or else a deontic action. For example,

$$(0, u_1)(0, u_2)(0, u_3)(2, a)(0, u_4)(0, u_5)(3, a)(0, u_6)(0, u_7)$$

is what we may call a *general path*, describing a real development $u_1u_2u_3$, followed by a deontic development—permission to do a—followed by a real development u_4u_5, followed by a deontic development—prohibition to do a—followed by a real development u_6u_7. Note that for each general path there is an underlying (corresponding) real path composed of the elements whose first coordinate is 0. Thus in the example given, the real path is $u_1u_2u_3u_4u_5u_6u_7$.

For greater ease we will allow the identification of a general path $(0, u_1) \cdots (0, u_n)$ with the ordered pair $(0, u_1 \cdots u_n)$. Hence, for example, if $p(\#) = q(*)$ then $(0, p)(0, q) = (0, pq)$.

3.2. Formal development

Let (U, E, H) be a given frame (of the kind defined in the previous section). By a *general frame* we shall understand a triple (U^*, E^*, H^*), where

$U^* = \{(0, u): u \in U\}$,

E^* is the set of all sets $\{(0, p): p \in a\}$, where $a \in E$,

H^* is the set of all sequences of ordered pairs (i, x),

where either $i = 0$ and $x \in U$, or else $i \in \{1, 2, 3, 4\}$ and $a \in E$.

All the model theoretic concepts introduced in Section 2 can now be generalized: general articulated history, general past, general future, general situation.

A *(simple general) norm* is a function that assigns to each given general past a set of possible general futures. That is to say, a norm N assigns to a general past h a subset $N(h)$ of the set $\mathrm{cont}(h)$ of all possible general futures.

Deontic actions have the effect of modifying the normative position; that is, the current set of normal futures. In other words, (in this paper) norms don't change, only normative positions do. For example, when an authority (competent to do so) orders a real event or action *a*, it is understood—by the substantive condition—that *a* will occur or be performed in every possible normal general future. But there is also the condition of homogeneity: every possible execution of *a* will occur in some possible normal general future. Thus, writing $h^\frown d$ for the general history formed by a deontic action *d* directly following *h*, it is not enough to define $N(h^\frown(1,a))$ as $N(\{f \in \mathrm{cont}(h)\colon\ f \text{ includes } a\})$, we must also take homogeneity into account. This line of reasoning leads us to the following definition (where *a* is real):

$$N\big(h^\frown(1,a)\big) = \bigcup_{q\in a} N\big(\big\{f \in \mathrm{cont}(h)\colon\ q \text{ occurs in } f\big\}\big).$$

Similarly, for the other three cases:

$$N\big(h^\frown(2,a)\big) = N(h) \cup \bigcup_{q\in a} N\big(\big\{f \in \mathrm{cont}(h)\colon\ q \text{ occurs in } f\big\}\big),$$
$$N\big(h^\frown(3,a)\big) = \big\{f \in N(h)\colon\ f \text{ excludes } a\big\},$$
$$N\big(h^\frown(4,a)\big) = N(h) \cup N\big\{f \in \mathrm{cont}(h)\colon\ f \text{ excludes } a\big\}.$$

Turning to syntax, call the terms in the old object language *real terms*. In our new object language we also need four symbols in the category of term-forming operators that operate on real terms: !!, !, §§, and §. That is to say, if α is a real term, then $!!\alpha$, $!\alpha$, $\S\S\alpha$ and $\S\alpha$ are *deontic terms* (corresponding to our informal concepts of ordering, permitting, forbidding, and making-omissible). Other conventions are modified in obvious ways. For example, if δ is one of the new operators and α is a real term, then $[\mathsf{after}\,\delta\phi]\theta$ is a formula, for all real formulæ ϕ and θ. And if δ and ε are deontic term-to-term operators and α and β are real terms, then $\delta\alpha + \varepsilon\beta$ is a deontic term.

Turning to semantics, the old meaning conditions are readily generalized in obvious ways, but there are also the following new conditions:

$$[\![!!\alpha]\!] = \big\{\big(1, [\![\alpha]\!]\big)\big\},$$
$$[\![!\alpha]\!] = \big\{\big(2, [\![\alpha]\!]\big)\big\},$$
$$[\![\S\S\alpha]\!] = \big\{\big(3, [\![\alpha]\!]\big)\big\},$$
$$[\![\S\alpha]\!] = \big\{\big(4, [\![\alpha]\!]\big)\big\}.$$

We list some new validities. First some schemas reflecting the fact that the deontic actions in our modelling do not change anything in the real world: if α is a real term and ϕ is a real formula (meaning that neither α nor ϕ contains a deontic operator), then the following are valid:

$$[\mathsf{after}\,\delta\alpha]\phi \leftrightarrow \phi, \text{ if } \delta \in \{\mathsf{ob}, \mathsf{pm}, \mathsf{fb}, \mathsf{ex}\}.$$

Then four "success" validities:

$$[\mathsf{after}\,!!\alpha]\,\mathsf{ob}\,\alpha,$$
$$[\mathsf{after}\,!\alpha]\,\mathsf{pm}\,\alpha,$$
$$[\mathsf{after}\,\S\S\alpha]\,\mathsf{fb}\,\alpha,$$
$$[\mathsf{after}\,\S\alpha]\,\mathsf{ex}\,\alpha.$$

And some distribution validities:

$$\big[\mathsf{after}\,!!(\alpha+\beta)\big]\phi \leftarrow \big([\mathsf{after}\,!!\alpha]\phi \wedge [\mathsf{after}\,!!\beta]\phi\big),$$ [13]
$$\big[\mathsf{after}!(\alpha+\beta)\big]\phi \leftrightarrow \big([\mathsf{after}\,!\alpha]\phi \wedge [\mathsf{after}\,!\beta]\phi\big),$$
$$\big[\mathsf{after}\,\S\S(\alpha+\beta)\big]\phi \leftrightarrow \big([\mathsf{after}\,\S\S\alpha]\phi \wedge [\mathsf{after}\,\S\S\beta]\phi\big),$$
$$\big[\mathsf{after}\,\S(\alpha+\beta)\big]\phi \rightarrow \big([\mathsf{after}\,\S\alpha]\phi \wedge [\mathsf{after}\,\S\beta]\phi\big).$$

But, in general, valid schemata involving the concatenation operators ; and ;; are not as simple.

[13] For $\leftarrow$, see Footnote 10.

Appendix A. Three so-called paradoxes

There are two attitudes one may take with respect to the conundrums in the literature on deontic logic known as "paradoxes". One is to see them as challenges and to use them as an inspiration to try to develop a theory that is immune to their paradoxicality. In other words, for a given "paradox", to find a formal system and a way of formalizing the "paradox" within that system. The other is to use existing "paradoxes" to test the range of applicability of a certain system. For any formal system will have its limitations, and the "paradoxes" may help one to be aware of the boundaries of application.

While the first attitude has certainly influenced the writing of this paper, it is to the second alternative we now turn. There are other systems in the literature that can account for the paradoxes (or most paradoxes), but the question here is, how does our particular system handle them? Here we will consider the paradoxes bearing the names of Ross, Chisholm and Forrester.

Seen in this light, instead of talking about *paradoxes*, we might as well talk about *tests*. And this is what we will do here. As will be seen, our system passes the Ross Test and the Chisholm Test, but it fails (the interesting half of) the Forrester Test.

The Ross test. The Ross Paradox is Alf Ross's challenge in [10] to imperative logic to provide a plausible formalization of the imperative "Post this letter!" that does not imply the imperative "Post this letter or burn it!" The parallel challenge to deontic logic is to provide a system in which "Posting the letter is obligatory" does not imply "Posting the letter or burning it, is obligatory". Standard Deontic Logic fails to meet Ross's challenge since it validates the schema

$$\mathsf{O}\phi \to \mathsf{O}(\phi \vee \psi),$$

where O is a traditional proposition-to-proposition operator.

Ross's own advice was to distinguish between what he called the logic of validity and the logic of satisfaction. According to him there are two sides to the concept of obligation: it is one thing for an obligation to be in force ("valid", in his terminology), another to be discharged ("satisfied", in his terminology). We can rephrase his insight by saying that an obligation (of this kind) remains in force as long as it has not been discharged. But once discharged, that particular obligation is no longer in force.

In the logic presented in this paper (which follows the analysis first given in [15]) Ross's example is formalized in a different way:

$$\mathsf{ob}\,\alpha \to \mathsf{ob}\,(\alpha + \beta)$$

or perhaps

$$\mathsf{ob}^*\,\alpha \to \mathsf{ob}^*(\alpha + \beta).$$

It is easy to see that neither schema is valid in our system. To prove this contention it would not be difficult to construct countermodels in abstract terms (although it would be tedious to specify the details). In the former case we would define a model with enough possible histories, a certain action $a = [\![\phi]\!]$ and a certain articulated history hg such that $(h, g) \vDash \mathsf{ob}\,\alpha$. Furthermore, we could let b be any forbidden action and take an event letter β not occurring in α and define $b = [\![\beta]\!]$. As b is forbidden after h, no future in cont(h) will contain any path in b. By the homogeneity condition, then, $(h, g) \nvDash \mathsf{ob}(\alpha + \beta)$, proving the contention. (Notice that this outline is just a re-description of Ross's own example.)

We conclude that our system passes the Ross test.

The Chisholm test. Perhaps best known of all the deontic paradoxes in the literature is the one formulated by R.M. Chisholm in [4]. It turns on the difficulty of finding a model for four propositions of the following kind:

(C_1) It ought to be the case that if X will do B then X does A.
(C_2) If X will not do B, then X should not do A.
(C_3) X ought to do B.
(C_4) X will not do B.

If one tries to formalize these propositions in Standard Deontic Logic (neglecting the tense-logical aspect), the first two are naturally rendered on the format $\mathsf{O}(\phi \to \psi)$ and $\phi \to \mathsf{O}\psi$, respectively; and contradiction results.

In one familiar version of this example, B stands for X's going to see his grandmother, A for notifying her in advance. The situation described is well known: X will fail to do his duty. But with this understanding of the situation it is not clear that (C_1) and (C_2) provide the only way in which to try to express the human predicament facing X. The propositions

(C_1^*) It ought to be the case that if X will not do B then X does not do A.
(C_2^*) If X will do B, then X should do A.

would also be true of the hypothetical situation. Of course, Chisholm chose his formulations with an eye to bringing out the limitation of SDL. Our concern would disappear if we could find a new, binary connective $\mathsf{O}(\phi, \psi)$ (different from the ordinary unary O, although we use the same letter for both operators), meaning something like "ϕ commits to ψ" or "ϕ

makes it obligatory that ψ". Then (C_1) and (C_1) could be rendered as $\mathrm{O}(\phi, \psi)$ and $\mathrm{O}(\neg\phi, \neg\psi)$, respectively. Moreover, if $\mathrm{O}(\phi, \psi) \to (\phi \to \mathrm{O}\psi)$ were generally valid, everybody could be happy. (There are such "solutions" in the literature.)

Our formulation of Chisholm's example follows a similar line. If α and β are two distinct event letters, our recommended translation is:

(C'_1) $[\mathrm{H}\colon \langle \mathrm{F}\rangle \,\mathrm{occurs}\,\beta]\,\mathrm{ob}\,\alpha$,
(C'_2) $[\mathrm{H}\colon \neg\langle \mathrm{F}\rangle \,\mathrm{occurs}\,\beta]\,\neg\mathrm{ob}\,\alpha$,
(C'_3) $\mathrm{ob}\,\beta$,[14]
(C'_4) $\neg\langle \mathrm{F}\rangle \,\mathrm{occurs}\,\beta$.

Our problem is solved if we can find a normed model and an articulated history with respect to which these four formulæ are true. Again we prefer to outline the kind of model desired rather than give a rigorous definition in abstract terms. So let hg be a complete history where h has some nontrivial futures other than g. We assume that we can define sets S_1 and S_2 such that the following conditions are satisfied:

1. S_1 and S_2 are nonempty and $S_1 \cup S_2 = \mathrm{cont}(h)$,
2. $S_1 = \{f \in \mathrm{cont}(h)\colon\ f$ includes an occurrence of a path in $[\![\beta]\!]\}$,
3. $S_2 = \{f \in \mathrm{cont}(h)\colon\ f$ excludes all occurrences of paths in $[\![\beta]\!]\}$,
4. for all $f \in S_1$, each occurrence of a path in $[\![\beta]\!]$ is preceded in f by an occurrence of a path in $[\![\alpha]\!]$,[15]
5. $g \in S_2$.

Our construction must also satisfy the condition we call (ARROW): $N(h, S_1) \subseteq N(h, \mathrm{cont}(h))$. For simplicity, let us adopt a slightly stronger condition:

6. $N(h, S_1) = N(h, \mathrm{cont}(h))$.

There is no doubt that we could construct an abstract model along these lines (homogeneity would be no problem). And that would solve our problem. For then, by item 2 above, $(h, f, S_1) \vDash \mathrm{ob}\,\alpha$, for all $f \in S_1$, and hence

- $(h, g, \mathrm{cont}(h)) \vDash (\mathrm{C}'_1)$.

By a similar argument, by item 3,

- $(h, g, \mathrm{cont}(h)) \vDash (\mathrm{C}'_2)$.

By items 2 and 6,

- $(h, g, \mathrm{cont}(h)) \vDash (\mathrm{C}'_3)$.

And finally, by item 5,

- $(h, g, \mathrm{cont}(h)) \vDash (\mathrm{C}'_4)$.

We conclude that our system passes the Chisholm Test.

The Forrester test. Consider the following sentences [5]:

(F_1) "Don't kill her! But if you do, do it by giving her enough sleeping-pills!"
(F_2) "Don't kill her! But if you do, do it gently!"

Is it possible to model these sentences in out system? In the first case, the answer is yes. The challenge is to find a model in which is satisfied the formula

$$\mathrm{fb}\,\alpha \wedge \big[\mathrm{H} : \langle \mathrm{F}\rangle \,\mathrm{oocurs}\,\alpha\big]\,\mathrm{ob}\,\beta$$

where $a = [\![\alpha]\!]$ and $b = [\![\beta]\!]$ and a is a subevent of b. With respect to that model, we have to find events (actions) a and b such that

[14] Or $[\mathrm{H} : \mathrm{T}]\,\mathrm{ob}\,\beta$, which is logically equivalent to $\mathrm{ob}\,\beta$,

[15] This condition is not forced by the Chisholm example as formulated by (C'_1)–(C'_4); it is not an essential ingredient of the Chisholm paradox (at least not it is as understood in this paper). But it is part of the intuitive picture.

1. $b \subseteq a$,
2. a is forbidden,
3. given that a will be done, b is obligatory.

To find such a model is not difficult. (This version of Forrester's Paradox is a variant of Chisholm's. There is also an analogous situation in probability theory where there are events a and b, b is a subevent of a, and a is improbable while the conditional probability of b, given a, is high.)

But there seems to be no way of formalizing (F_2) in the theory presented here. The problem is that *giving-enough-sleeping-pills* is a sub event of *killing*, but that *killing-gently* is not in the same sense a subevent of *killing*. Every element p of a is a particular realization of a. But, in a different sense of realization, p itself can be realized in different ways—gently, carelessly, cruelly, quickly, slowly, etc. This second sense of 'realization' concerns performance, an aspect to which the present formalism does not do justice. In our modelling we can express *what* an agent does, but not *how* the agent does it.

Thus our system fails the interesting version of the Forrester Test: Forrester's "paradox" marks one limitation of the present modelling. What is needed is an "adverbial" extension of our theory. But such an extension is not presented in this paper.

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