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Plane overpartitions and cylindric partitions

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ABSTRACT

Generating functions for plane overpartitions are obtained using various methods such as nonintersecting paths, RSK type algorithms and symmetric functions. We extend some of the generating functions to cylindric partitions. Also, we show that plane overpartitions correspond to certain domino tilings and we give some basic properties of this correspondence.

1. Introduction

The goal of the first part of this paper is to introduce a new object called plane overpartitions, and to give several enumeration formulas for these plane overpartitions. A *plane overpartition* is a plane partition where (1) in each row the last occurrence of an integer can be overlined or not and all the other occurrences of this integer are not overlined and (2) in each column the first occurrence of an integer can be overlined or not and all the other occurrences of this integer are overlined. An example of a plane overpartition is

$$\begin{array}{llll} 4 & 4 & \bar{4} & \bar{3} \\ \bar{4} & 3 & 3 & \bar{3} \\ \bar{4} & \bar{3} & & \\ 3 & & & \end{array}.$$

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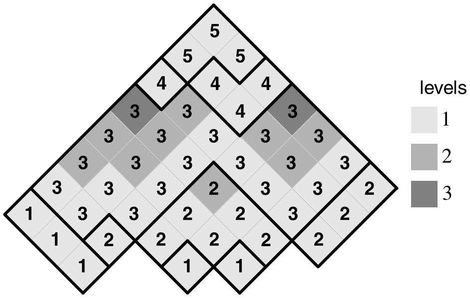

Fig. 1. A plane partition: border components.

This paper takes its place in the series of papers on overpartitions started by Corteel and Lovejoy [6]. The motivation is to show that the generating function for plane overpartitions is:

$$\prod_{n\geqslant 1}\left(\frac{1+q^n}{1-q^n}\right)^n. \tag{1.1}$$

In this paper, we give several proofs of this result and several refinements and generalizations. Namely, we prove the following results.

Result 1. The hook-content formula for the generating function for plane overpartitions of a given shape, see Theorem 3.

Result 2. The hook formula for the generating function for reverse plane overpartitions, see Theorem 5.

Result 3. The generating function formula for plane overpartitions with bounded parts, see Theorem 6.

The goal of the second part of this paper is to extend the generating function formula for cylindric partitions due to Borodin [4] and the following 1-parameter generalized MacMahon formula due to the third author of this paper [31]:

$$\sum_{\substack{\Pi \text{ is a} \\ \text{plane partition}}} A_{\Pi}(t)q^{|\Pi|} = \prod_{n=1}^{\infty}\left(\frac{1-tq^n}{1-q^n}\right)^n, \tag{1.2}$$

where the weight $A_{\Pi}(t)$ is a polynomial in t that we describe below.

A plane partition Π is a Ferrers diagram filled with positive integers that form nonincreasing rows and columns. A set of boxes is connected if one can walk from any box in the set to any other by taking only steps between boxes that are neighbors of each other. A connected component of Π is a connected set of boxes that are filled with the same number. If a box (i, j) belongs to a connected component C then we define its level $h(i, j)$ as the smallest positive integer h such that $(i+h, j+h)$ does not belong to C. A border component of level i is a connected subset of a connected component all of whose boxes have level i. See Fig. 1. We associate to each border component of level i, the weight $(1-t^i)$. The polynomial $A_{\Pi}(t)$ is the product of the weights of its border components. For the plane partition from Fig. 1 it is $(1-t)^{10}(1-t^2)^3(1-t^3)^2$.

We give a new proof of the 1-parameter generalized MacMahon formula. We also extend this formula to two more general objects: skew plane partitions and cylindric partitions. Namely, we prove the following results.

Result 4. 1-parameter generalized formula for the generating function for skew plane partitions, see Theorem 7.

Result 5. 1-parameter generalized formula for the generating function for cylindric partitions, see Theorem 8.

In the rest of this section we give definitions and explain our results in more detail.

A partition λ is a nonincreasing sequence of positive integers $(\lambda_1, \ldots, \lambda_k)$. Each λ_i is a part of the partition and the number of parts is denoted by $\ell(\lambda)$. The weight $|\lambda|$ of λ is the sum of its parts. A partition λ can be graphically represented by the Ferrers diagram that is a diagram formed of $\ell(\lambda)$ left justified rows, where the ith row consists of λ_i cells (or boxes). The conjugate of a partition λ, denoted by λ', is a partition that has the Ferrers diagram equal to the transpose of the Ferrers diagram of λ. For a cell (i, j) of the Ferrers diagram of λ the *hook length* of this cell is $h_{i,j} = \lambda_i + \lambda'_j - i - j + 1$ and the *content* is $c_{i,j} = j - i$. It is well known that the generating function for partitions that have at most n parts is $1/(q)_n$, where $(a)_n := (a; q)_n = \prod_{i=0}^{n-1}(1 - aq^i)$. More definitions on partitions can be found, for example, in [1] or [23].

An *overpartition* is a partition where the last occurrence of an integer can be overlined [6]. Last occurrences in an overpartition are in one-to-one correspondence with corners of the Ferrers diagram and overlined parts can be represented by marking the corresponding corners. The generating function for overpartitions that have at most n parts is $(-q)_n/(q)_n$.

Let λ be a partition. A *plane partition* of shape λ is a filling of the cells of the Ferrers diagram of λ with positive integers that form a nonincreasing sequence along each row and each column. We denote the shape of a plane partition Π by $\mathrm{sh}(\Pi)$ and the sum of all entries by $|\Pi|$, called the weight of Π. It is well known, under the name of MacMahon formula, that the generating function for plane partitions is

$$\sum_{\substack{\Pi \text{ is a} \\ \text{plane partition}}} q^{|\Pi|} = \prod_{i=1}^{\infty}\left(\frac{1}{1-q^i}\right)^i. \tag{1.3}$$

One way to prove this is to construct a bijection between plane partitions and pairs of semi-standard Young tableaux of the same shape and to use the RSK correspondence between these Young tableaux and certain matrices [3].

Recall that a plane overpartition is a plane partition where in each row the last occurrence of an integer can be overlined or not and in each column the first occurrence of an integer can be overlined or not and all the others are overlined. This definition implies that the entries strictly decrease along diagonals, i.e. all connected components are also border components. Therefore, a plane overpartition is a diagonally strict plane partition where some entries are overlined. More precisely, it is easy to check that inside a border component only one entry can be chosen to be overlined or not and this entry is the upper right entry.

Plane overpartitions are therefore in bijection with diagonally strict plane partitions where each border component can be overlined or not (or weighted by 2). Recently, those weighted diagonally strict plane partitions were studied in [7,8,30,31]. The first result obtained was the shifted MacMahon formula that says that the generating function for plane overpartitions is indeed Eq. (1.1). This was obtained as a limiting case of the generating function formula for plane overpartitions which fit into an $r \times c$ box, i.e. whose shapes are contained in the rectangular shape with r rows and c columns.

Theorem 1. *(See [7,30].) The generating function for plane overpartitions which fit in an $r \times c$ box is*

$$\prod_{i=1}^{r}\prod_{j=1}^{c}\frac{1+q^{i+j-1}}{1-q^{i+j-1}}.$$

This theorem was proved in [7,30] using Schur P and Q symmetric functions and a suitable Fock space. In [31] the theorem was proved in a bijective way where an RSK-type algorithm (due to Sagan [26], see also Chapter XIII of [13]) was used to construct a bijection between plane overpartitions and matrices of nonnegative integers where positive entries can be overlined.

In Section 2, we give a mostly combinatorial proof of the generalized MacMahon formula [31]. Namely, we prove the following result.

Theorem 2. *(See [31].)*

$$\sum_{\Pi\in\mathcal{P}(r,c)} A_\Pi(t)q^{|\Pi|} = \prod_{i=1}^{r}\prod_{j=1}^{c}\frac{1-tq^{i+j-1}}{1-q^{i+j-1}}. \tag{1.4}$$

In the above formula, $\mathcal{P}(r,c)$ is the set of plane partitions with at most r rows and c columns. When we set $t=-1$, only the border components of level 1 have a nonzero weight and we get back Theorem 1.

The main result of Section 3 is a hook-content formula for the generating function for plane overpartitions of a given shape. More generally, we give a weighted generating function where overlined parts are weighted by some parameter a.

Let $\mathcal{S}(\lambda)$ be the set of all plane overpartitions of shape λ. The number of overlined parts of a plane overpartition Π is denoted by $o(\Pi)$. For example, $\Pi = \begin{smallmatrix} 4 & 4 & \bar{4} & \bar{3} \\ \bar{4} & 3 & 3 & \bar{3} \\ \bar{4} & \bar{3} \end{smallmatrix}$ is a plane overpartition of shape $(4,4,2)$, with $|\Pi|=35$ and $o(\Pi)=6$.

Theorem 3. *Let λ be a partition. The weighted generating function for plane overpartitions of shape λ is*

$$\sum_{\Pi\in\mathcal{S}(\lambda)} a^{o(\Pi)}q^{|\Pi|} = q^{\sum_i i\lambda_i}\prod_{(i,j)\in\lambda}\frac{1+aq^{c_{i,j}}}{1-q^{h_{i,j}}}. \tag{1.5}$$

We prove this theorem using a correspondence between plane overpartitions and sets of nonintersecting paths that use three kinds of steps. (The work of Brenti used similar paths to compute super Schur functions [5].) Another way to prove this result is to show that plane overpartitions of shape λ are in bijection with super semistandard tableaux of shape λ. This is presented in a remark in Section 3.

We also give the weighted generating function formula for plane overpartitions "bounded" by λ, where by that we mean plane overpartitions such that the ith row of the plane overpartition is an overpartition that has at most λ_i parts and at least λ_{i+1} parts. Let $\mathcal{B}(\lambda)$ be the set of all such plane overpartitions.

Theorem 4. *Let λ be a partition. The weighted generating function for plane overpartitions such that the ith row of the plane overpartition is an overpartition that has at most λ_i parts and at least λ_{i+1} parts is*

$$\sum_{\Pi\in\mathcal{B}(\lambda)} a^{o(\Pi)}q^{|\Pi|} = q^{\sum_i (i-1)\lambda_i}\prod_{(i,j)\in\lambda}\frac{1+aq^{c_{i,j}+1}}{1-q^{h_{i,j}}}. \tag{1.6}$$

Note that it is enough to assign weights to overlined (or nonoverlined) parts only because generating functions where overlined and nonoverlined parts are weighted by a and b, respectively, follow trivially from the above formulas.

The number of nonintersecting paths is given by a determinantal formula (Lemma 1 of [22]). This result was anticipated by Lindström [22] and Karlin and McGregor [15,16], but Gessel and Viennot were first to use it for enumerative purpose of various classes of plane partitions [11,12]. Applying the result and evaluating the determinants we obtain the hook-content formulas. We use a simple involution to show that Stanley's hook-content formula (Theorem 7.21.2 of [28]) follows from our formula (1.5).

From the symmetric function point of view, these formulas are given by Schur functions in a difference of two alphabets, as explained in Section 3.

The end of Section 3 is devoted to *reverse* plane overpartitions. A reverse plane partition of shape λ is a filling of cells of the Ferrers diagram of λ with nonnegative integers that form a nondecreasing sequence along each row and each column. A reverse plane overpartition is a reverse plane partition where (1) only positive entries can be overlined, (2) in each row the last occurrence of a positive integer can be overlined or not and (3) in each column the first occurrence of a positive integer can be overlined or not and all others (if positive) are overlined. An example of a reverse plane overpartition is

$$\begin{array}{cccccc} 0 & 0 & 3 & 4 & 4 & \bar{4} \\ 0 & 0 & 4 & \bar{4} & & \\ 1 & \bar{3} & & & & \\ 3 & \bar{3} & & & & \end{array}.$$

It was proved by Gansner [9] that the generating function for reverse plane partitions of a given shape λ is

$$\prod_{(i,j)\in\lambda} \frac{1}{1-q^{h_{i,j}}}. \tag{1.7}$$

Let $\mathcal{S}^R(\lambda)$ be the set of all reverse plane overpartitions of shape λ. The generating function for reverse plane overpartitions is given by the following hook formula.

Theorem 5. *Let λ be a partition. The generating function for reverse plane overpartitions of shape λ is*

$$\sum_{\Pi\in\mathcal{S}^R(\lambda)} q^{|\Pi|} = \prod_{(i,j)\in\lambda} \frac{1+q^{h_{i,j}}}{1-q^{h_{i,j}}}.$$

We construct a bijection between reverse plane overpartitions of a given shape and sets of non-intersecting paths whose endpoints are not fixed. Using results of [29] we obtain a Pfaffian formula for the generating function for reverse plane partitions of a given shape. Subsequently, we evaluate the Pfaffian and obtain a proof of the hook formula. When λ is the partition with r parts equal to c, this result is the generating function formula for plane overpartitions fitting in an $r\times c$ box (given in Theorem 1).

In Section 4 we make a connection between plane overpartitions and domino tilings. We give some basic properties of this correspondence, such as how a removal of a box or an overline changes the corresponding tiling. This correspondence connects measures on strict plane partitions studied in [31] to measures on domino tilings (see the uniform measure on domino tilings of the Aztec diamond [14] as an example). This connection was expected by similarities in correlation kernels, limit shapes and some other features of these measures, but the connection was not established before.

In Section 5 we propose a bijection between matrices and pairs of plane overpartitions based on ideas of Berele and Remmel [2]. We give another stronger version of the shifted MacMahon formula, as we give a weighted generating function for plane overpartitions with bounded entries. Let $\mathcal{L}(n)$ be the set of all plane overpartitions with the largest entry at most n.

Theorem 6. *The weighted generating functions for plane overpartitions where the largest entry is at most n is*

$$\sum_{\Pi\in\mathcal{L}(n)} a^{o(\Pi)} q^{|\Pi|} = \prod_{j=1}^{n} \frac{\prod_{i=0}^{n}(1+aq^{i+j})}{\prod_{i=1}^{j}(1-q^{i+j-1})(1-a^2q^{i+j})}.$$

In Section 6 we study interlacing sequences and cylindric partitions. We say that a sequence of partitions $\Lambda=(\lambda^1,\ldots,\lambda^T)$ is *interlacing* if λ^i/λ^{i+1} or λ^{i+1}/λ^i is a horizontal strip, i.e. a skew shape having at most one cell in each column. Let $A=(A_1,\ldots,A_{T-1})$ be a sequence of 0's and 1's. We say that an interlacing sequence $\Lambda=(\lambda^1,\ldots,\lambda^T)$ has *profile* A if when $A_i=1$, respectively $A_i=0$, then λ^i/λ^{i+1}, respectively λ^{i+1}/λ^i is a horizontal strip. Interlacing sequences are generalizations of plane partitions. Indeed, plane partitions are interlacing sequences with profile $A=(0,\ldots,0,1,\ldots,1)$.

We now define the *diagram* of an interlacing sequence. See Fig. 2. We start with a square grid; we denote the two directions defined by the grid lines with 0 and 1. A profile $A=(A_1,\ldots,A_{T-1})$

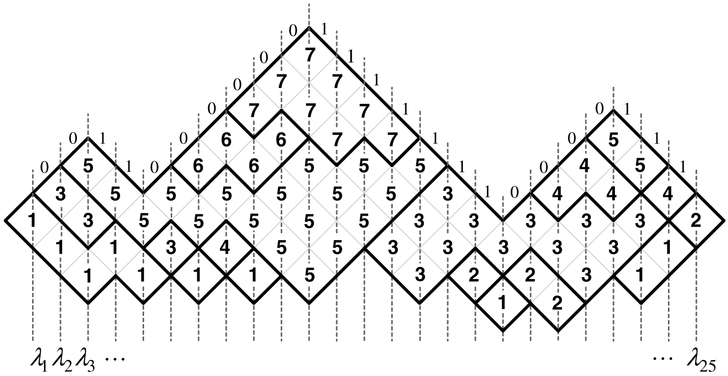

Fig. 2. The diagram of an interlacing sequence.

is represented by a path of length $T+1$ on this grid where the path consists of grid edges whose directions are given by $0, A_1, A_2, \ldots, A_{T-1}, 1$. This path forms the (upper) border of the diagram. Excluding the endpoints of the path we draw the diagonal rays (which form 45° angles with grid lines) starting at the vertices of the path and we index them (from left to right) with integers from 1 to T. A diagram is a connected subset of boxes of a square grid whose (upper) border is given by the profile path and along the ith diagonal ray there are $\ell(\lambda^i)$ boxes. The filling numbers on the ith diagonal ray are parts of λ^i with the largest part at the top. Observe that by the definition of interlacing sequences we obtain monotone sequences of numbers in the direction of grid lines.

A (skew) plane partition and cylindric partition are examples of interlacing sequences. A plane partition can be written as $\Lambda = (\emptyset, \lambda^1, \ldots, \lambda^T, \emptyset)$ with profile $A = (0, 0, \ldots, 0, 1, \ldots, 1, 1)$ and the λ^is are the diagonals of the plane partition. A skew plane partition is an interlacing sequence $\Lambda = (\emptyset, \lambda^1, \ldots, \lambda^T, \emptyset)$ with a profile $A = (0, A_1, \ldots, A_{T-1}, 1)$. A cylindric partition is an interlacing sequence $\Lambda = (\lambda^0, \lambda^1, \ldots, \lambda^T)$ where $\lambda^0 = \lambda^T$, and T is called the period of Λ. A cylindric partition can be represented by the *cylindric diagram* that is obtained from the ordinary diagram by identification of the first and last diagonal.

A *connected component* is defined as before, i.e. a set of connected boxes filled with the same number. The example in Fig. 2 has 18 connected components.

If a box (i, j) belongs to a connected component C then we define its *level* $\ell(i, j)$ as the smallest positive integer ℓ such that $(i+\ell, j+\ell)$ does not belong to C. In other words, a level represents the distance from the "rim", distance being measured diagonally. A *border component* is a connected subset of a connected component where all boxes have the same level. We also say that this border component is of this level. For the example from Fig. 2, border components and their levels are shown in Fig. 3 (different levels are represented by different colors).

Let $(n_1, n_2, \ldots)$ be a sequence of nonnegative integers where n_i is the number of i-level border components of Λ. We set

$$A_\Lambda(t) = \prod_{i \geqslant 1} \bigl(1 - t^i\bigr)^{n_i}. \tag{1.8}$$

For the example above $A_\Lambda(t) = (1-t)^{18}(1-t^2)^4(1-t^3)$.

For a cylindric partition Π, we define *cylindric connected components* and *cylindric border components* in the same way but connectedness is understood on the cylinder, i.e. boxes are connected if they are connected in the cylindric diagram. We define

$$A^{\mathrm{cyl}}_\Pi(t) = \prod_{i \geqslant 1} \bigl(1 - t^i\bigr)^{n_i^{\mathrm{cyl}}},$$

where n_i^{cyl} is the number of cylindric border components of level i.

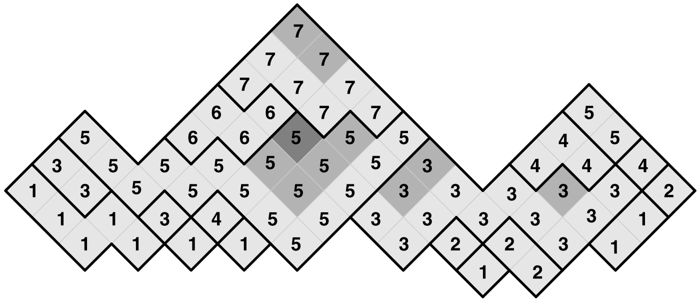

Fig. 3. Border components with levels.

In Section 6 we give a generating function formula for skew plane partitions. Let $\mathrm{Skew}(T, A)$ be the set of all skew plane partitions $\Lambda = (\emptyset, \lambda^1, \ldots, \lambda^T, \emptyset)$ with profile $A = (A_0, A_1, \ldots, A_{T-1}, A_T)$, where $A_0 = 0$ and $A_T = 1$.

Theorem 7. (*Generalized MacMahon formula for skew plane partitions*; *Hall–Littlewood case*)

$$\sum_{\Pi \in \mathrm{Skew}(T,A)} A_\Pi(t) q^{|\Pi|} = \prod_{\substack{0 \leqslant i < j \leqslant T \\ A_i = 0,\, A_j = 1}} \frac{1 - tq^{j-i}}{1 - q^{j-i}}.$$

Note that as profiles are words in $\{0, 1\}$, a profile $A = (A_0, \ldots, A_T)$ encodes the border of a Ferrers diagram λ. Skew plane partitions of profile A are in one-to-one correspondence with reverse plane partitions of shape λ. Moreover, one can check that

$$\prod_{\substack{0 \leqslant i < j \leqslant T \\ A_i = 0,\, A_j = 1}} \frac{1 - tq^{j-i}}{1 - q^{j-i}} = \prod_{(i,j) \in \lambda} \frac{1 - tq^{h_{i,j}}}{1 - q^{h_{i,j}}}.$$

Therefore the theorem of Gansner (Eq. (1.7)) is Theorem 7 with $t = 0$ and our Theorem 5 on reverse plane overpartitions is Theorem 7 with $t = -1$.

This theorem is also a generalization of results of Vuletić [31]. In [31] a 2-parameter generalization of MacMahon formula related to Macdonald symmetric functions was given and the formula is especially simple in the Hall–Littlewood case. In the Hall–Littlewood case, this is a generating function formula for plane partitions weighted by $A_\Pi(t)$. Theorem 7 can be naturally generalized to the Macdonald case, but we do not pursue this here.

Let $\mathrm{Cyl}(T, A)$ be the set of all cylindric partitions with period T and profile $A = (A_1, \ldots, A_T)$. The main result of Section 6 is:

Theorem 8. (*Generalized MacMahon formula for cylindric partitions*; *Hall–Littlewood case*)

$$\sum_{\Pi \in \mathrm{Cyl}(T,A)} A_\Pi^{\mathrm{cyl}}(t) q^{|\Pi|} = \prod_{n=1}^{\infty} \frac{1}{1 - q^{nT}} \prod_{\substack{1 \leqslant i, j \leqslant T \\ A_i = 0,\, A_j = 1}} \frac{1 - tq^{(j-i)_{(T)} + (n-1)T}}{1 - q^{(j-i)_{(T)} + (n-1)T}},$$

where $i_{(T)}$ is the smallest positive integer such that $i \equiv i_{(T)}$ mod T.

The case $t = 0$ is due to Borodin and represents a generating function formula for cylindric partitions. Cylindric partitions were introduced and enumerated by Gessel and Krattenthaler [10]. The result of Borodin could also be proven using Theorem 5 of [10] and the $\mathrm{SU}(r)$-extension of Bailey's

${}_6\psi_6$ summation due to Gustafson (Eq. (7.9) in [10]) [20]. Again Theorem 8 can be naturally generalized to the Macdonald case. The trace generating function for cylindric partitions could also be easily derived from our proof, as done by Okada [24] for the reverse plane partitions case.

The paper is organized as follows. In Section 2 we give a mostly combinatorial proof of the generalized MacMahon formula. In Section 3 we use nonintersecting paths and obtain the hook-length formulas for plane overpartitions and reverse plane partitions of a given shape. In Section 4 we make the connection between domino tilings and plane overpartitions. In Section 5 we construct a bijection between matrices and pairs of plane overpartitions and obtain a generating function formula for plane overpartitions with bounded part size. In Section 6 we give the hook formula for reverse plane partitions contained in a given shape and the 1-parameter generalization of the generating function formula for cylindric partitions. Section 7 contains some concluding remarks.

2. Plane partitions and Hall–Littlewood functions

In this section, we give an alternative proof of the generalization of MacMahon formula due to the third author [31]. Our proof is mostly combinatorial as it uses a bijection between plane partitions and pairs of strict plane partitions of the same shape and the combinatorial description of Hall–Littlewood functions [23, Chapter III, Eq. (5.11)].

Let $\mathcal{P}(r,c)$ be the set of plane partitions with at most r rows and c columns. Given a plane partition Π, let $A_\Pi(t)$ be the polynomial defined in (1.8), as $A_\Pi(t)=\prod_{r \text{ border component}}(1-t^{\text{level}(r)})$.

Recall that Theorem 2 states that

$$\sum_{\Pi\in\mathcal{P}(r,c)} A_\Pi(t)q^{|\Pi|}=\prod_{i=1}^{r}\prod_{j=1}^{c}\frac{1-tq^{i+j-1}}{1-q^{i+j-1}}.$$

Any plane partition Π is in bijection with a sequence of partitions $(\pi^{(1)},\pi^{(2)},\ldots)$. This sequence is such that $\pi^{(i)}$ is the shape of the entries greater than or equal to i in Π for all i. For example if

$$\Pi=\begin{matrix}4433\\3332\\1\end{matrix},$$

the corresponding sequence is $((4,4,1),(4,4),(4,3),(2))$.

Note that the plane partition Π is column strict if and only if $\pi^{(i)}/\pi^{(i+1)}$ is a horizontal strip for all i.

We use a bijection between pairs of column strict plane partitions (Σ,Λ) and plane partitions Π due to Bender and Knuth [3]. We suppose that (Σ,Λ) are of the same shape λ and that the corresponding sequences are $(\sigma^{(1)},\sigma^{(2)},\ldots)$ and $(\lambda^{(1)},\lambda^{(2)},\ldots)$.

Given a plane partition $\Pi=(\Pi_{i,j})$, we define the entries of diagonal x to be the partition $(\Pi_{i,j})$ with $i,j\geqslant 1$ and $j-i=x$. The bijection is such that the entries of diagonal x of Π are $\sigma^{(x+1)}$ if $x\geqslant 0$ and $\lambda^{(-x-1)}$ otherwise. Note that as Λ and Σ have the same shape, the entries of the main diagonal $(x=0)$ are $\sigma^{(1)}=\lambda^{(1)}$.

For example, start with

$$\Sigma=\begin{matrix}4444\\2221\\111\end{matrix}\quad\text{and}\quad\Lambda=\begin{matrix}4433\\3322\\111\end{matrix},$$

whose sequences are $((4,4,3),(4,3),(4),(4))$ and $((4,4,3),(4,4),(4,2),(2))$, respectively and get

$$\Pi=\begin{matrix}4444\\443\\443\\22\end{matrix}.$$

This construction implies that:

$$|\Pi| = |\Sigma| + |\Lambda| - |\lambda|,$$
$$A_\Pi(t) = \frac{\varphi_\Sigma(t)\varphi_\Lambda(t)}{b_\lambda(t)}.$$

Here we have

$$b_\lambda(t) = \prod_{i\geqslant 1} \varphi_{m_i(\lambda)}(t), \qquad \varphi_r(t) = \prod_{j=1}^{r}\bigl(1-t^j\bigr),$$
$$\varphi_\Lambda(t) = \prod_{i\geqslant 1} \varphi_{\lambda^{(i)}/\lambda^{(i+1)}}(t)$$

and

$$\varphi_\theta(t) = \prod_{i\in I}\bigl(1-t^{m_i(\lambda)}\bigr),$$

where θ is a horizontal strip λ/μ, $m_i(\lambda)$ is the multiplicity of i in λ and I is the set of integers such that $\theta'_i = 1$ and $\theta'_{i+1} = 0$. See [23, Chapter III, Sections 2 and 5].

Indeed, the following statements are true.

- Each factor $(1-t^i)$ in $b_\lambda(t)$ is in one-to-one correspondence with a border component of level i that goes through the main diagonal of Π.
- Each factor $(1-t^i)$ in $\varphi_\Sigma(t)$ is in one-to-one correspondence with a border component of level i that ends in a nonnegative diagonal.
- Each factor $(1-t^i)$ in $\varphi_\Lambda(t)$ is in one-to-one correspondence with a border component of level i that starts in a non-positive diagonal.

Continuing with our example, we have

$$\varphi_\Sigma(t) = (1-t)^2\bigl(1-t^2\bigr), \qquad \varphi_\Lambda(t) = (1-t)^3\bigl(1-t^2\bigr), \qquad b_\lambda(t) = (1-t)^2\bigl(1-t^2\bigr),$$

and

$$A_\Pi(t) = (1-t)^3\bigl(1-t^2\bigr) = \frac{(1-t)^2(1-t^2)(1-t)^3(1-t^2)}{(1-t)^2(1-t^2)}.$$

We recall the combinatorial definition of the Hall–Littlewood functions following Macdonald [23]. The Hall–Littlewood function $Q_\lambda(x;t)$ can be defined as

$$Q_\lambda(x;t) = \sum_{\substack{\Lambda\\ \mathrm{sh}(\Lambda)=\lambda}} \varphi_\Lambda(t)x^\Lambda;$$

where $x^\Lambda = x_1^{\alpha_1}x_2^{\alpha_2}\ldots$ and α_i is the number of entries equal to i in Λ. See [23, Chapter III, Eq. (5.11)].

A direct consequence of the preceding bijection is that the entries of Σ are less than or equal to c, and the entries of Λ are less than or equal to r if and only if Π is in $\mathcal{P}(r,c)$. Therefore:

$$\sum_{\Pi\in\mathcal{P}(r,c)} A_\Pi(t)q^{|\Pi|} = \sum_\lambda \frac{Q_\lambda(q,\ldots,q^r,0,\ldots;t)\,Q_\lambda(q^0,\ldots,q^{c-1},0,\ldots;t)}{b_\lambda(t)}.$$

Finally, we need Eq. (4.4) in Chapter III of [23].

$$\sum_\lambda \frac{Q_\lambda(x;t)Q_\lambda(y;t)}{b_\lambda(t)} = \prod_{i,j}\frac{1-tx_iy_j}{1-x_iy_j}; \tag{2.1}$$

With the substitutions $x_i = q^i$ for $1\leqslant i\leqslant r$ and 0 otherwise and $y_j = q^{j-1}$ for $1\leqslant j\leqslant c$ and 0 otherwise, we get the result.

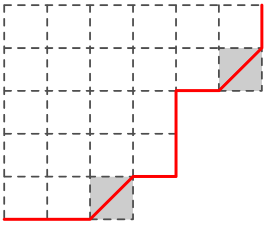

Fig. 4. Paths and overpartitions.

3. Nonintersecting paths

3.1. Plane overpartitions of a given shape

In this section we represent plane overpartitions as nonintersecting paths. We use the determinantal formula for the number of nonintersecting paths (see [12,15,16,22]). Evaluating these determinants we obtain the hook-content formulas from Theorems 3 and 4. A similar approach was used for example in [5] to compute super Schur functions.

We construct a bijection between the set of paths from $(0,0)$ to (x,k) and the set of overpartitions with at most k parts and the largest part at most x. Given an overpartition the corresponding path consists of North and East edges that form the border of the Ferrers diagram of the overpartition except for corners containing an overlined entry where we substitute a pair of North and East edges with an North–East edge. For example, the path corresponding to the overpartition $(6,\bar{6},4,4,\bar{3})$ is shown in Fig. 4. Note that this construction appears also in Proposition 2.2 of [5].

To each overpartition λ we associate a weight equal to $a^{o(\lambda)}q^{|\lambda|}$, where $o(\lambda)$ is the number of overlined parts. To have the same weight on the corresponding path we introduce the following weights on edges. We assign weight 1 to East edges, q^i to North edges on (vertical) level i and weight aq^{i+1} to North–East edges joining vertical levels i and $i+1$. The weight of the path is equal to the product of weights of its edges.

We will need the following lemma.

Lemma 1. *(See [6].) The generating function for overpartitions with at most k parts is given by*

$$\sum_{l(\lambda)\leqslant k} a^{o(\lambda)}q^{|\lambda|} = \frac{(-aq)_k}{(q)_k} \tag{3.1}$$

and the generating function for overpartitions with exactly k parts is given by

$$\sum_{l(\lambda)=k} a^{o(\lambda)}q^{|\lambda|} = q^k\frac{(-a)_k}{(q)_k}. \tag{3.2}$$

For a plane overpartition Π of shape λ we construct a set of nonintersecting paths using paths from row overpartitions where the starting point of the path corresponding to the ith row is shifted upwards by $\lambda_1-\lambda_i+i-1$ so that the starting point is $(0,\lambda_1-\lambda_i+i-1)$. In that way, we obtain a bijection between the set of nonintersecting paths from $(0,\lambda_1-\lambda_i+i-1)$ to (x,λ_1+i-1), where i runs from 1 to $\ell(\lambda)$, and the set of plane overpartitions whose ith row has at most λ_i parts and at least λ_{i+1} parts with x greater or equal to the largest part. The weights of this set of nonintersecting paths (the product of weights of its paths) is equal to $a^{o(\Pi)}q^{|\Pi|}$. Fig. 5 (see also Fig. 8) shows the corresponding set of nonintersecting paths for $x=8$ and the plane overpartition

$$\begin{array}{ccccc} 7 & 4 & \bar{3} & 2 & \bar{2} \\ 3 & 3 & \bar{3} & \bar{2} & \\ \bar{3} & 2 & \bar{1} & & \\ 2 & & & & \end{array}.$$

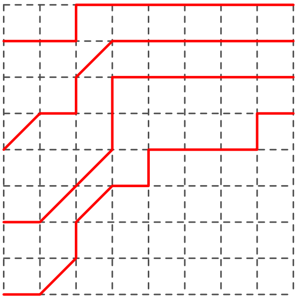

Fig. 5. Nonintersecting paths.

Definition 1. For a partition λ we define $M_\lambda(a;q)$ to be the $\ell(\lambda) \times \ell(\lambda)$ matrix whose (i,j)th entry is given by

$$\frac{(-a)_{\lambda_j+i-j}}{(q)_{\lambda_j+i-j}}.$$

For a partition λ let $\mathcal{B}(\lambda)$ and $\mathcal{S}(\lambda)$ be as in the introduction, i.e. the sets of all plane overpartitions bounded by shape λ and of the shape λ, respectively.

Proposition 1. *Let λ be a partition. The weighted generating function for plane overpartitions whose ith row is an overpartition that has at most λ_i parts and at least λ_{i+1} parts is given by*

$$\sum_{\Pi \in \mathcal{B}(\lambda)} a^{o(\Pi)} q^{|\Pi|} = \det M_\lambda(aq;q).$$

Proof. From (3.1) we have that the limit when x runs to infinity of the number of paths from $(0,0)$ to (x,k) is $(-aq)_k/(q)_k$. Using Lemma 1 of [22] we have that $\det M_\lambda(aq;q)$ is the limit when x goes to infinity of the generating function for $\ell(\lambda)$ nonintersecting paths going from $(0, \lambda_1 + i - 1 - \lambda_i)$ to $(x, \lambda_1 + i - 1)$. Thanks to the bijection between paths and overpartitions this is also the generating function for overpartitions whose ith row overpartition has at most λ_i and at least λ_{i+1} parts. $\square$

Proposition 2. *Let λ be a partition. The weighted generating function for plane overpartitions of shape λ is given by*

$$\sum_{\Pi \in \mathcal{S}(\lambda)} a^{o(\Pi)} q^{|\Pi|} = q^{|\lambda|} \det M_\lambda(a;q).$$

Proof. The proof is the same as the proof of Proposition 1. Here we just use (3.2) instead of (3.1). $\square$

The determinant of $M_\lambda(a;q)$ is given by the following formula:

$$\det M_\lambda(a;q) = q^{\sum_i (i-1)\lambda_i} \prod_{(i,j)\in\lambda} \frac{1 + aq^{c_{i,j}}}{1 - q^{h_{i,j}}}. \tag{3.3}$$

For the proof see for example pages 16–17 of [12] or Theorem 26, (3.11) of [19].

Using (3.3) we obtain the product formulas for the generating functions from Propositions 1 and 2. Those are the hook-content formulas given in Theorems 3 and 4.

Remark 1. We give a bijective proof of formula (3.3). Indeed Krattenthaler [18] showed that the weighted generating function of super semistandard Young tableaux of shape λ is

$$q^{\sum_i (i-1)\lambda_i} \prod_{(i,j)\in\lambda} \frac{1+aq^{c_{i,j}}}{1-q^{h_{i,j}}},$$

where each tableau T is weighted by $a^{o(T)}q^{|T|}$. A super semistandard tableau [18] of shape λ is a filling of cells of the Ferrers diagram of λ with entries from the ordered alphabet $1 < 2 < 3 < \cdots < \bar{1} < \bar{2} < \bar{3} < \cdots$ such that

- the nonoverlined entries form a column-strict reverse plane partition of some shape ν, where ν is a partition contained in λ,
- the overlined entries form a row-strict reverse plane partition of shape λ/ν.

We give here a bijection from super semistandard Young tableaux to plane overpartitions. We start with a super semistandard Young tableau where the nonoverlined (resp. overlined) entries are less than or equal to k (resp. ℓ). First, change all the entries equal to a to $k+1-a$ and all the entries equal to $\bar{a}$ to $\overline{\ell+1-a}$.

For example, with $k=5$ and $\ell=4$, starting with the super semistandard Young tableau

$$\begin{array}{l} 13455\bar{2}\bar{4} \\ 245\bar{1}\bar{2}\bar{3}\bar{4} \\ 35\bar{2}\bar{3}\bar{4} \\ 4\bar{1}\bar{3} \\ \bar{3} \end{array} \quad \text{we obtain} \quad \begin{array}{l} 53211\bar{3}\bar{1} \\ 421\bar{4}\bar{3}\bar{2}\bar{1} \\ 31\bar{3}\bar{2}\bar{1} \\ 2\bar{4}\bar{2} \\ \bar{2} \end{array} \quad .$$

Now we use the order $0 < \bar{1} < 1 < \bar{2} < 2 < \bar{3} < 3 < \cdots$. At first all the nonoverlined entries are active. While there is an active entry, we choose the smallest active entry. If there are more than one, we choose the rightmost one. If this entry is greater than or equal to both of its east and south neighbors (if a neighbor does not exist, we proceed as if it was an entry equal to 0), we declare the entry inactive. Otherwise we swap it with its east or south neighbor, choosing the larger of the two. If they are equal then we choose the east one. We proceed in this way until this entry is greater than or equal to both of its neighbors. When we reach this, we declare the entry inactive. We call this *swapping algorithm*.

Continuing with the preceding example and applying the algorithm to the smallest (and rightmost) active entry until it becomes inactive we obtain:

$$\begin{array}{l} \mathbf{53211}\bar{3}\bar{1} \\ \mathbf{421}\bar{4}\bar{3}\bar{2}\bar{1} \\ \mathbf{31}\bar{3}\bar{2}\bar{1} \\ \mathbf{2}\bar{4}\bar{2} \\ \bar{2} \end{array} \qquad \begin{array}{l} \mathbf{5321}\bar{3}\mathbf{1}\bar{1} \\ \mathbf{421}\bar{4}\bar{3}\bar{2}\bar{1} \\ \mathbf{31}\bar{3}\bar{2}\bar{1} \\ \mathbf{2}\bar{4}\bar{2} \\ \bar{2} \end{array} \qquad \begin{array}{l} \mathbf{5321}\bar{3}\bar{2}\bar{1} \\ \mathbf{421}\bar{4}\bar{3}\mathbf{1}\bar{1} \\ \mathbf{31}\bar{3}\bar{2}\bar{1} \\ \mathbf{2}\bar{4}\bar{2} \\ \bar{2} \end{array} \quad .$$

Then we move the rest of the active entries.

$$\begin{array}{l} \mathbf{532}\bar{4}\bar{3}\bar{2}\bar{1} \\ \mathbf{421}\bar{3}11\bar{1} \\ \mathbf{31}\bar{3}\bar{2}\bar{1} \\ \mathbf{2}\bar{4}\bar{2} \\ \bar{2} \end{array} \qquad \begin{array}{l} \mathbf{532}\bar{4}\bar{3}\bar{2}\bar{1} \\ \mathbf{42}\bar{3}\bar{2}11\bar{1} \\ \mathbf{31}\bar{3}1\bar{1} \\ \mathbf{2}\bar{4}\bar{2} \\ \bar{2} \end{array} \qquad \begin{array}{l} \mathbf{532}\bar{4}\bar{3}\bar{2}\bar{1} \\ \mathbf{42}\bar{3}\bar{2}11\bar{1} \\ \mathbf{3}\bar{4}\bar{3}1\bar{1} \\ \mathbf{2}\bar{2}\mathbf{1} \\ \bar{2} \end{array} \qquad \begin{array}{l} \mathbf{532}\bar{4}\bar{3}\bar{2}\bar{1} \\ \mathbf{42}\bar{3}\bar{2}11\bar{1} \\ \mathbf{3}\bar{4}\bar{3}1\bar{1} \\ \mathbf{2}\bar{2}1 \\ \bar{2} \end{array} \qquad \begin{array}{l} \mathbf{53}\bar{4}\bar{3}2\bar{2}\bar{1} \\ \mathbf{42}\bar{3}\bar{2}11\bar{1} \\ \mathbf{3}\bar{4}\bar{3}1\bar{1} \\ \mathbf{2}\bar{2}1 \\ \bar{2} \end{array} \quad ,$$

$$\begin{array}{lllll}
\mathbf{53}\bar{4}\bar{3}2\bar{2}\bar{1} & \mathbf{53}\bar{4}\bar{3}2\bar{2}\bar{1} & \mathbf{5}\bar{4}3\bar{3}2\bar{2}\bar{1} & \mathbf{5}\bar{4}3\bar{3}2\bar{2}\bar{1} & \mathbf{5}\bar{4}3\bar{3}2\bar{2}\bar{1} \\
\mathbf{4}\bar{4}\bar{3}\bar{2}11\bar{1} & \mathbf{4}\bar{4}\bar{3}\bar{2}11\bar{1} & \mathbf{4}\bar{4}\bar{3}\bar{2}11\bar{1} & \mathbf{4}\bar{4}\bar{3}\bar{2}11\bar{1} & 4\bar{4}\bar{3}\bar{2}11\bar{1} \\
\mathbf{3}\bar{3}21\bar{1} & \mathbf{3}\bar{3}21\bar{1} & \mathbf{3}\bar{3}21\bar{1} & 3\bar{3}21\bar{1} & 3\bar{3}21\bar{1} \\
\mathbf{2}\bar{2}1 & \bar{2}21 & \bar{2}21 & \bar{2}21 & \bar{2}21 \\
\bar{2} & \bar{2} & \bar{2} & \bar{2} & \bar{2}
\end{array}.$$

It can be seen that this is a well-defined mapping from super semistandard Young tableaux to plane overpartitions. Furthermore, it is not too difficult to see that every step in this construction can be reversed and so the mapping is a bijection between super semistandard Young tableaux and plane overpartitions. This bijection is such that the shape and the number of overlined parts is conserved.

Remark 2. Stanley's hook-content formula states that the generating function of semistandard Young tableaux of shape λ where the entries are less than or equal to n is

$$q^{\sum(i-1)\lambda_i}\prod_{(i,j)\in\lambda}\frac{1-q^{n+c_{i,j}}}{1-q^{h_{i,j}}}. \tag{3.4}$$

See Theorem 7.21.2 of [28]. Formula (3.3) is equivalent to Stanley's hook formula by Examples I.2.5 and I.3.3 of [23]. Again we can give a bijective argument.

Formula (3.4) is the weighted generating function of super semistandard tableaux of shape λ with $a=-q^n$. Start with a super semistandard Young tableau T of shape λ and add n to all the overlined entries. Now transform the tableau into a reverse plane overpartition with the swapping algorithm using the order $1>\bar{1}>2>\bar{2}>3>\cdots>0$. This shows that super semistandard Young tableaux T of shape λ with weight $(-q^n)^{o(T)}q^{|T|}$ are in bijection with reverse plane overpartitions Π of the same shape where the overlined entries are greater than n with weight $(-1)^{o(\Pi)}q^{|\Pi|}$.

Now we define a sign reversing involution on these reverse plane overpartitions. Given such a reverse plane overpartition, if there is at least one entry greater than n, we choose the uppermost and rightmost entry greater than n. If this entry is overlined, then we take off the overline, otherwise we overline the entry. Note that in this case the parity of the number of overlined entries is changed and therefore the weight of the reverse plane overpartition is multiplied by -1. If no such entry exists, the given reverse plane partition is a semistandard Young tableau of shape λ where the entries are less than or equal to n.

Now, we give a generating function formula for plane overpartitions with at most r rows and c columns.

Proposition 3. *The weighted generating function for plane overpartitions with at most r rows and c columns is given by*

$$\sum_{c\geqslant\lambda_1\geqslant\cdots\geqslant\lambda_{(r-1)/2}\geqslant 0}\det M_{(c,\lambda_1,\lambda_1,\ldots,\lambda_{(r-1)/2},\lambda_{(r-1)/2})}(aq;q)$$

if r is odd, and by

$$\sum_{c\geqslant\lambda_1\geqslant\cdots\geqslant\lambda_{r/2}\geqslant 0}\det M_{(\lambda_1,\lambda_1,\ldots,\lambda_{r/2},\lambda_{r/2})}(aq;q)$$

if r is even.

In particular, the weighted generating function for all overpartitions is:

$$\sum_{\lambda_1\geqslant\cdots\geqslant\lambda_k}\det M_{(\lambda_1,\lambda_1,\ldots,\lambda_k,\lambda_k)}(aq;q).$$

Proof. This is a direct consequence of Proposition 1. $\square$

We will use this result to get another "symmetric function" proof of the shifted MacMahon formula [7,30,31]:

$$\sum_{\substack{\Pi \text{ is a plane} \\ \text{overpartition}}} q^{|\Pi|} = \prod_{n=1}^{\infty}\left(\frac{1+q^n}{1-q^n}\right)^n.$$

3.2. The weighted shifted MacMahon formula

In this section we give the weighted generalization of the shifted MacMahon formula. We use a symmetric function identity to compute the last sum in Proposition 3 since $\det M_\lambda(a;q)$, as we will soon see, has an interpretation in terms of symmetric functions.

A symmetric function of an alphabet (set of indeterminates, also called letters) $\mathbb{A}$ is a function of letters which is invariant under any permutation of $\mathbb{A} = \{a_1, a_2, \ldots\}$. Recall three standard bases for the algebra of symmetric functions: Schur functions s, complete symmetric functions h and elementary symmetric functions e (see [23]). The rth complete symmetric function h_r and the rth elementary symmetric function e_r are defined by

$$h_r(\mathbb{A}) = \sum_{i_1 \leqslant i_2 \leqslant \cdots \leqslant i_r} a_{i_1} a_{i_2} \ldots a_{i_r}; \qquad e_r(\mathbb{A}) = \sum_{i_1 < i_2 < \cdots < i_r} a_{i_1} a_{i_2} \ldots a_{i_r}.$$

Their generating functions are

$$H_t(\mathbb{A}) = \sum_{r \geqslant 0} h_r(\mathbb{A}) t^r = \prod_{a \in A} \frac{1}{1-ta},$$

$$E_t(\mathbb{A}) = \sum_{r \geqslant 0} e_r(\mathbb{A}) t^r = \prod_{a \in A} (1+ta).$$

Algebraic operations on alphabets, such as addition, subtraction and scalar multiplication, can be defined using the λ-ring framework, see Chapters I and II of [21]. In this framework symmetric functions are seen as ring operators. The addition of two alphabets $\mathbb{A}$ and $\mathbb{B}$ is defined naturally as their disjoint union and is denoted by $\mathbb{A} + \mathbb{B}$. Obviously,

$$H_t(\mathbb{A}+\mathbb{B}) = H_t(\mathbb{A}) H_t(\mathbb{B}).$$

Subtraction and scalar multiplication are defined by

$$H_t(\mathbb{A}-\mathbb{B}) = \frac{H_t(\mathbb{A})}{H_t(\mathbb{B})}, \qquad H_t(c\mathbb{A}) = \big(H_t(\mathbb{A})\big)^c, \quad c \in \mathbb{C}.$$

Analogous relations hold for elementary symmetric functions since $E_t(\mathbb{A}) = H_{-t}(-\mathbb{A})$.

A specialization is an algebra homomorphism between the algebra of symmetric functions and $\mathbb{C}$. If ρ is a specialization and f is a symmetric function we denote its image by $f|_\rho$.

Let $\rho(a)$ be the specialization given by

$$h_n|_{\rho(a)} = \frac{(-a)_n}{(q)_n}.$$

Since, for $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$

$$s_\lambda = \det(h_{\lambda_i - i + j})$$

we have from Definition 1 that

$$s_\lambda|_{\rho(a)} = \det M_\lambda(a;q). \tag{3.5}$$

In the λ-ring framework, the q-binomial theorem (see (2.21) of [1])

$$\sum_{n=0}^{\infty} \frac{(-a)_n}{(q)_n} t^n = \frac{(-at)_\infty}{(t)_\infty}$$

shows that the specialization $\rho(a)$ is equivalent to considering symmetric functions in the difference of the two alphabets $1+q+q^2+\cdots$ and $-a-aq-aq^2-\cdots$. Thus,

$$s_\lambda|_{\rho(a)} = s_\lambda\big((1+q+q^2+\cdots) - (-a-aq-aq^2-aq^3-\cdots)\big).$$

The weighted shifted MacMahon formula can be obtained from (3.5) and Proposition 3. We have

$$\sum_{\substack{\Pi \text{ is a plane}\\ \text{overpartition}}} a^{o(\pi)} q^{|\Pi|} = \sum_{\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_k} s_{(\lambda_1,\lambda_1,\lambda_2,\lambda_2,\ldots,\lambda_k,\lambda_k)}|_{\rho(aq)} = \sum_{\lambda' \text{ even}} s_\lambda|_{\rho(aq)},$$

where λ' is the transpose of λ and a partition is even if it has even parts.

By Ex. 10 b) on p. 79 of [23] we have

$$\sum_{\lambda' \text{ even}} s_\lambda(\mathbb{A}) t^{|\lambda|/2} = H_t\big(e_2(\mathbb{A})\big).$$

If instead of $\mathbb{A}$ we insert a difference of alphabets $\mathbb{A} = x_1 + x_2 + \cdots$ and $\mathbb{B} = y_1 + y_2 + \cdots$, then we obtain the following product formula:

$$\begin{aligned}\sum_{\lambda' \text{ even}} s_\lambda(\mathbb{A}-\mathbb{B}) t^{|\lambda|/2} &= H_t\big(e_2(\mathbb{A}-\mathbb{B})\big)\\ &= H_t\left(\sum_{1\leqslant i<j} x_i x_j + \sum_{1\leqslant i\leqslant j} y_i y_j - \sum_{1\leqslant i,j} x_i y_j\right)\\ &= \prod_{1\leqslant i<j} \frac{1}{1-x_i x_j t} \prod_{1\leqslant i\leqslant j} \frac{1}{1-y_i y_j t} \prod_{1\leqslant i,j} (1-x_i y_j t).\end{aligned}$$

This gives us the weighted shifted MacMahon formula:

Proposition 4. *The weighted generating function formula for plane overpartitions is*

$$\sum_{\substack{\Pi \text{ is a plane}\\ \text{overpartition}}} a^{o(\Pi)} q^{|\Pi|} = \prod_{i=1}^{\infty} \frac{(1+aq^i)^i}{(1-q^i)^{\lceil i/2\rceil}(1-a^2q^i)^{\lfloor i/2\rfloor}}.$$

Proof. Substituting $x_i = q^{i-1}$ and $y_i = -aq^i$ in the preceding product formula, we get

$$\prod_{0\leqslant i<j} \frac{1}{1-q^{i+j}} \prod_{1\leqslant i\leqslant j} \frac{1}{1-a^2q^{i+j}} \prod_{1\leqslant i,j} \big(1+aq^{i+j-1}\big). \qquad \square$$

Remark. This proof was suggested to the authors by one of the anonymous referees. We give a refinement in Section 5.2.

3.3. Reverse plane overpartitions

In this section, we construct a bijection between the set of all reverse plane overpartitions and sets of nonintersecting paths whose endpoints are not fixed. We use this bijection and Stembridge's results [29] to obtain a Pfaffian formula for the generating function for reverse plane overpartitions of a given shape. Evaluating the Pfaffian we obtain the hook formula for reverse plane overpartitions due to Okada [24]. Let $\mathcal{S}^R(\lambda)$ be the set of all reverse plane partitions of shape λ.

Theorem 9. *The generating function for reverse plane overpartitions of shape λ is*

$$\sum_{\Pi \in \mathcal{S}^R(\lambda)} q^{|\Pi|} = \prod_{(i,j)\in\lambda} \frac{1+q^{h_{i,j}}}{1-q^{h_{i,j}}}.$$

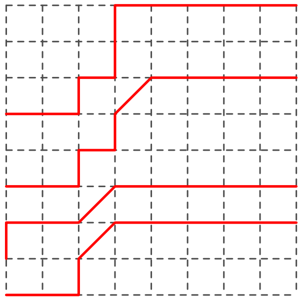

Fig. 6. Nonintersecting paths and reverse plane overpartitions.

We construct a weight preserving bijection between reverse plane overpartitions and sets of nonintersecting paths on a triangular lattice in a similar fashion as in Section 3.1. The lattice consists of East, North and North-East edges. East edges have weight 1, North edges on (vertical) level i have weight q^{i+1} and North-East edges joining vertical levels i and $i+1$ have weight q^{i+1}. The weight of a set of nonintersecting paths p is the product of the weights of their edges and is denoted by $w(p)$. Let Π be a reverse plane overpartition whose positive entries form a skew shape λ/μ and let $\ell = \ell(\lambda)$. Then Π can be represented by a set of ℓ nonintersecting lattice paths such that

- the departure points are $(0, \mu_i + \ell - i)$ and
- the arrivals points are $(x, \lambda_i + \ell - i)$,

for a large enough x and $i = 1, \dots, \ell$. For example let $x = 8$, $\lambda = (5, 4, 2, 2)$ and $\mu = (2, 1)$. Fig. 6 shows the corresponding set of nonintersecting paths for the reverse plane overpartition of shape λ/μ

$$\begin{array}{ccccc} & & 3 & 4 & 4 \\ & 3 & 4 & \bar{4} & \\ 1 & \bar{3} & & & \\ 3 & \bar{3} & & & \end{array}.$$

This implies that all reverse plane overpartitions of shape $\lambda = (\lambda_1, \dots, \lambda_\ell)$ can be represented by nonintersecting lattice paths such that

- the departure points are an ℓ-element subset of $\{(0, i) \mid i \geqslant 0\}$ and
- the arrivals points are $(x, \lambda_i + \ell - i)$,

with $x \to \infty$.

Now, for $r_1 > r_2 > \dots > r_\ell \geqslant 0$ we define

$$W(r_1, r_2, \dots, r_\ell) = \lim_{x\to\infty} \sum_{p \in P(x; r_1, \dots, r_\ell)} w(p),$$

where $P(x; r_1, r_2, \dots, r_\ell)$ is the set of all nonintersecting paths joining an ℓ-element subset of $\{(0, i) | i \geqslant 0\}$ with $\{(x, r_1), \dots, (x, r_\ell)\}$. Note that for $r_1 > r_2 > \dots > r_\ell > 0$ we have

$$W(r_1, \dots, r_\ell, 0) = W(r_1 - 1, \dots, r_\ell - 1). \tag{3.6}$$

By Stembridge's Pfaffian formula for the sum of the weights of nonintersecting paths where departure points are not fixed (Theorem 3.1 of [29]) we obtain

$$W(r_1, r_2, \dots, r_\ell) = \mathrm{Pf}(D),$$

where if ℓ is even D is the $\ell \times \ell$ skew-symmetric matrix defined by $D_{i,j} = W(r_i, r_j)$ for $1 \leqslant i < j \leqslant \ell$, and if ℓ is odd D is the $(\ell + 1) \times (\ell + 1)$ skew-symmetric matrix defined by $D_{i,j} = W(r_i, r_j)$ for $1 \leqslant i < j \leqslant \ell$ and $D_{i,\ell+1} = W(r_i)$ for $1 \leqslant i \leqslant \ell$.

Lemma 2. *Let $r > s \geqslant 0$. Then*

$$W(s) = \frac{(-q)_s}{(q)_s}, \tag{3.7}$$

$$W(r,s) = \frac{(-q)_r}{(q)_r} \cdot \frac{(-q)_s}{(q)_s} \cdot \frac{1-q^{r-s}}{1+q^{r-s}}. \tag{3.8}$$

Proof. From Lemma 1 we have that the generating function for overpartitions with at most n parts is $M(n) = (-q)_n/(q)_n$ and the generating function for overpartitions with exactly n parts is $P(n) = q^n(-1)_n/(q)_n$. This implies that

$$\sum_{i=0}^{n} P(i) = M(n). \tag{3.9}$$

Moreover,

$$W(s) = M(s) = \frac{(-q)_s}{(q)_s} \quad \text{and} \quad W(r,0) = M(r-1) = \frac{(-q)_{r-1}}{(q)_{r-1}}.$$

We prove (3.8) by induction on s. The formula for the base case $s = 0$ holds by the above. So, we assume $s \geqslant 1$.

By Lindström's determinantal formula (Lemma 1 of [22]) we have

$$W(r,s) = \sum_{i=0}^{s} \sum_{j=i+1}^{r} \big(P(r-j)P(s-i) - P(r-i)P(s-j)\big).$$

Summing over j and using (3.9) we obtain

$$W(r,s) = \sum_{i=0}^{s} \big(P(s-i)M(r-i-1) - P(r-i)M(s-1-i)\big).$$

Then

$$W(r,s) = W(r-1,s-1) + P(s)M(r-1) - P(r)M(s-1).$$

It is enough to prove that

$$\frac{P(s)M(r-1) - P(r)M(s-1)}{W(r-1,s-1)} = \frac{2(q^r+q^s)}{(1-q^r)(1-q^s)} \tag{3.10}$$

and (3.8) follows by induction. Now,

$$\begin{aligned} P(s)M(r-1) - P(r)M(s-1) &= q^s \frac{(-1)_s}{(q)_s} \cdot \frac{(-q)_{r-1}}{(q)_{r-1}} - q^r \frac{(-1)_r}{(q)_r} \cdot \frac{(-q)_{s-1}}{(q)_{s-1}} \\ &= \frac{(-q)_{r-1}}{(q)_{r-1}} \cdot \frac{(-q)_{s-1}}{(q)_{s-1}} \cdot \frac{1+q^{r-s}}{1-q^{r-s}} \cdot \frac{2(q^r+q^s)}{(1-q^r)(1-q^s)}. \end{aligned}$$

Using the inductive hypothesis for $W(r-1,s-1)$ we obtain (3.10). □

Let F_λ be the generating function for reverse plane overpartitions of shape λ. Then, using the bijection we have constructed, we obtain

$$F_\lambda = W(\lambda_1 + \ell - 1, \lambda_2 + \ell - 2, \ldots, \lambda_\ell),$$

which, after applying Stembridge's result, gives us a Pfaffian formula. This Pfaffian formula can be expressed as a product after the following observations.

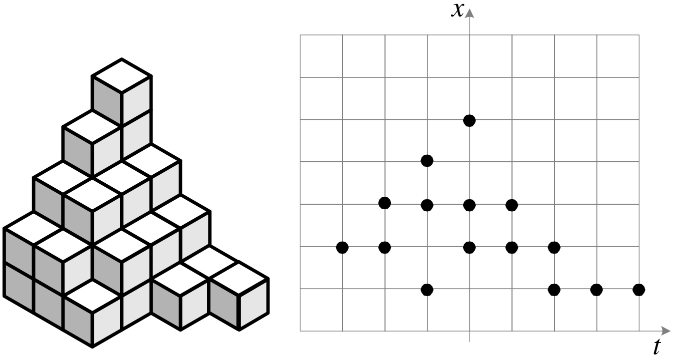

Fig. 7. A strict plane partition and its corresponding 2-dimensional diagram.

Let M be $2n \times 2n$ skew-symmetric matrix. One of definitions of the Pfaffian is the following:

$$\mathrm{Pf}(M) = \sum_{\pi=(i_1,j_1)\ldots(i_n,j_n)} \mathrm{sgn}(\pi) M_{i_1,j_1} M_{i_2,j_2} \cdots M_{i_n,j_n},$$

where the sum is over all perfect matchings (or fixed point free involutions) of $[2n]$. Also, for $r > s > 0$,

$$W(s) = \frac{1+q^s}{1-q^s} W(s-1) \quad \text{and}$$
$$W(r,s) = \frac{1+q^r}{1-q^r} \cdot \frac{1+q^s}{1-q^s} W(r-1,s-1).$$

Then

$$F_\lambda = \prod_{j=1}^{\ell} \frac{1+q^{h_{j,1}}}{1-q^{h_{j,1}}} \cdot F_{\bar{\lambda}},$$

where $\bar{\lambda} = (\lambda_1 - 1, \ldots, \lambda_\ell - 1)$ if $\lambda_\ell > 1$ and $\bar{\lambda} = (\lambda_1 - 1, \ldots, \lambda_{\ell-1} - 1)$ if $\lambda_\ell = 1$ (see (3.6) in this case). Inductively we obtain Theorem 9.

4. Domino tilings

In [30] a measure on (diagonally) strict plane partitions was studied. Strict plane partitions are plane partitions were all diagonals are strict partitions, i.e. strictly decreasing sequences. They can also be seen as plane overpartitions where all overlines are deleted. There are $2^{k(\Pi)}$ different plane overpartitions corresponding to the same strict plane partition Π, where $k(\Pi)$ is the number of connected components of Π.

Alternatively, a strict plane partition can be seen as a subset of $\mathbb{N} \times \mathbb{Z}$ consisting of points (t, x) where x is a part of the diagonal partition indexed by t. See Fig. 7. We call this set the 2-dimensional diagram of that strict plane partition. The connected components are connected sets (no holes) on the same horizontal line. The 2-dimensional diagram of a plane overpartition is the 2-dimensional diagram of its corresponding strict plane partition.

The measure studied in [30] assigns to each strict plane partition a weight equal to $2^{k(\Pi)} q^{|\Pi|}$. The limit shape of this measure is given in terms of the Ronkin function of the polynomial $P(z, w) = -1 + z + w + zw$ and it is parameterized on the domain representing half of the amoeba of this polynomial. This polynomial is also related to plane tilings with dominoes. This, as well as some other features like similarities in correlation kernels [14,30] suggested that a connection between this measure and domino tilings is likely to exist.

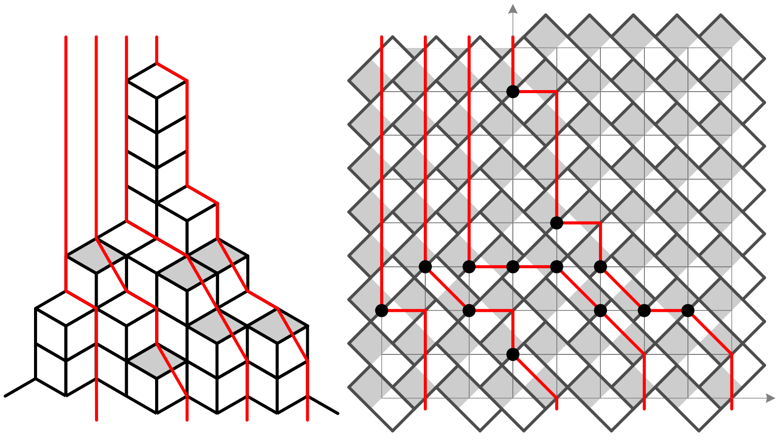

Fig. 8. A plane overpartition and its corresponding domino tiling.

Alternatively, one can see this measure as a uniform measure on plane overpartitions, i.e. each plane overpartition Π has a probability proportional to $q^{|\Pi|}$. In Section 3 we have constructed a bijection between plane overpartitions and sets of nonintersecting paths. See Fig. 8. This figure is obtained from Fig. 5 by a rotation. Note that the paths are incident with all points in the corresponding 2-dimensional diagram. The paths consist of edges of three different kinds: horizontal (joining (t,x) and $(t+1,x)$), vertical (joining $(t,x+1)$ and (t,x)) and diagonal (joining $(t,x+1)$ and $(t+1,x)$). There is a standard way to construct a tiling with dominoes using these paths (see for example [14]). We explain the process below. An example is given in Fig. 8.

We start from $\mathbb{R}^2$ and color it in a chessboard fashion such that $(1/4,1/4)$, $(-1/4,3/4)$, $(1/4,5/4)$ and $(3/4,3/4)$ are vertices of a white square. So, the axes of this infinite chessboard form angles of 45 and 135 degrees with the axes of $\mathbb{R}^2$. A domino placed on this infinite chessboard can be one of the four types: $(1,1)$, $(-1,-1)$, $(-1,1)$ or $(1,-1)$, where we say that a domino is of type (x,y) if (x,y) is the vector parallel to the vector whose starting, respectively end point is the center of the white, respectively black square of that domino.

Now, take a plane overpartition and represent it on this chessboard by its corresponding set of nonintersecting paths. We cover each edge by a domino that satisfies that the endpoints of that edge are midpoints of sides of the black and white square of that domino. In that way, we obtain a tiling of a part of the plane with dominoes of three types: $(1,1)$, $(-1,-1)$ and $(1,-1)$. More precisely, horizontal edges correspond to $(1,1)$ dominoes, vertical to $(-1,-1)$ and diagonal to $(1,-1)$. To tile the whole plane we fill the rest of it by dominoes of the fourth, $(-1,1)$ type. See Fig. 8 for an illustration.

In this way, we have established a correspondence between plane overpartitions and plane tilings with dominoes. We now give some of the properties of this correspondence. First, we describe how a tiling changes when we add or remove an overline or we add or remove a box from a plane overpartition. We require that when we add/remove an overline or a box we obtain a plane overpartition again.

In terms of the 2-dimensional diagram of a plane partition, adding an overline can occur at all places where (t,x) is in the diagram and $(t+1,x)$ is not. Adding an overline at (t,x) means that a pair of horizontal and vertical edges, $((t,x),(t+1,x))$ and $((t+1,x),(t+1,x-1))$, is replaced by one diagonal edge, $((t,x),(t+1,x-1))$. This means that the new tiling differs from the old one by replacing a pair of $(1,1)$ and $(-1,-1)$ dominoes by a pair of $(1,-1)$ and $(-1,1)$ dominoes. See Fig. 9. Removing an overline is the inverse of adding an overline.

We now explain the operation of removing a box. Observe that if a box can be removed from a plane overpartition then the corresponding part is overlined or it can be overlined to obtain a plane overpartition again. So, it is enough to consider how the tiling changes when we remove an overlined

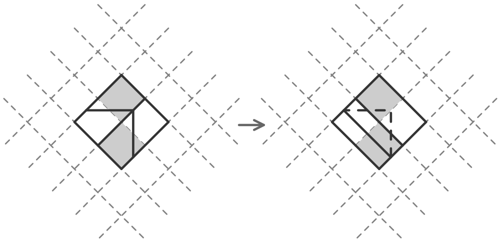

Fig. 9. Adding an overline.

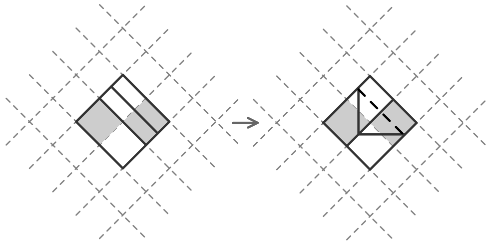

Fig. 10. Removing a box.

box since we have already considered the case of adding or removing an overline. If we remove an overlined box we change a diagonal edge to a pair of vertical and horizontal edges. If the box was represented by (t,x) in the 2-dimensional diagram then the edge $((t,x),(t+1,x-1))$ is replaced by the pair of $((t,x),(t,x-1))$ and $((t,x-1),(t+1,x-1))$. This means that the new tiling differs from the old one by replacing a pair of $(1,-1)$ and $(-1,1)$ dominoes by a pair of $(1,1)$ and $(-1,-1)$ dominoes. See Fig. 10.

All four operations are described by a swap of a pair of adjacent $(1,1)$ and $(-1,-1)$ dominoes and a pair of adjacent of $(1,-1)$ and $(-1,1)$ dominoes.

We conclude this section by the observation that plane overpartitions of a given shape λ and whose parts are bounded by n are in bijection with domino tilings of the rectangle $[-\ell(\lambda)+1,\lambda_1]\times[0,n]$ with certain boundary conditions. These conditions are imposed by the fact that outside of this rectangle nonintersecting paths are just straight lines. We describe the boundary conditions precisely in the proposition below.

Proposition 5. *The set $\mathcal{S}(\lambda)\cap\mathcal{L}(n)$ of all plane overpartitions of shape λ and whose largest part is at most n is in bijection with plane tilings with dominoes where a point $(t,x)\in\mathbb{Z}\times\mathbb{R}$ is covered by a domino of type $(-1,-1)$ if*

- $t\leqslant-\ell(\lambda)$,
- $-\ell(\lambda)<t\leqslant 0$ *and* $x\geqslant n$,
- $t=\lambda_i-i+1$ *for some* i *and* $x\leqslant 0$,

and a domino of type $(-1,1)$ if

- $t>\lambda_1$,
- $0<t\leqslant\lambda_1$ *and* $x\geqslant n+1/2$,
- $t\neq\lambda_i-i+1$ *for all* i *and* $x\leqslant-1/2$.

The boundary conditions for $\lambda=(5,4,3,1)$ and $n=7$ are shown in Fig. 11. The example from Fig. 8 satisfies these boundary conditions.

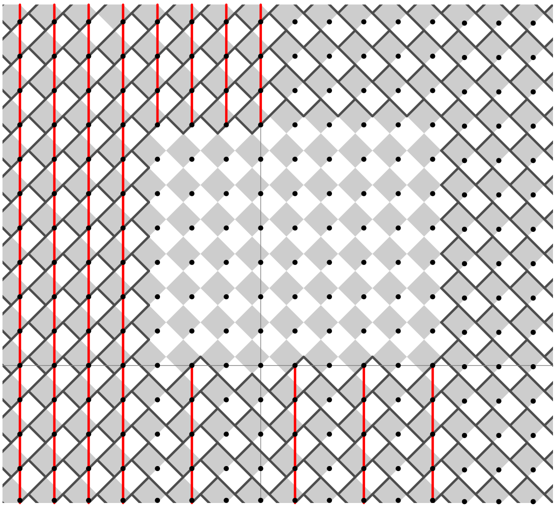

Fig. 11. Boundary conditions.

5. Robinson–Schensted–Knuth (RSK) type algorithm for plane overpartitions

In this section we are going to give a bijection between certain matrices and pairs of plane overpartitions of the same shape. This bijection is obtained by an algorithm similar to the algorithm RS2 of Berele and Remmel [2] which gives a bijection between matrices and pairs of (k,ℓ)-semistandard tableaux. These tableaux are super semistandard tableaux, defined in Section 3, where the nonoverlined (resp. overlined) parts are less than or equal to k (resp. ℓ).

We then apply properties of this algorithm to enumerate plane overpartitions, as done by Bender and Knuth [3] for plane partitions.

5.1. The RSK algorithm

Let $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be a $2n \times 2n$ matrix, made of four $n \times n$ blocks A, B, C and D. The blocks A and D are nonnegative integer matrices, and B and C are $\{0,1\}$ matrices. We denote the set of all such matrices with $\mathcal{M}_n$. We represent a matrix in $\mathcal{M}_n$ by a sequence of pairs of numbers $\binom{i}{j}$, $\binom{i}{\bar{j}}$, $\binom{\bar{i}}{j}$ and $\binom{\bar{i}}{\bar{j}}$.

The encoding of M into pairs is made using the following rules:

- for each nonzero entry a_{ij} of A, we create a_{ij} pairs $\binom{i}{j}$,
- for each nonzero entry b_{ij} of B, we create one pair $\binom{i}{\bar{j}}$,
- for each nonzero entry c_{ij} of C, we create one pair $\binom{\bar{i}}{j}$,
- for each nonzero entry d_{ij} of D, we create d_{ij} pairs $\binom{\bar{i}}{\bar{j}}$.

For example let $M = \begin{pmatrix} 0 & 2 & 1 & 0 \\ 2 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$. After encoding M, we obtain

$$\binom{1}{2}, \binom{1}{2}, \binom{1}{\bar{1}}, \binom{2}{1}, \binom{2}{1}, \binom{2}{\bar{1}}, \binom{\bar{1}}{1}, \binom{\bar{1}}{2}, \binom{\bar{1}}{\bar{2}}, \binom{\bar{2}}{\bar{1}}, \binom{\bar{2}}{\bar{2}}.$$

From now on, we fix the order $\bar{1} < 1 < \bar{2} < 2 < \bar{3} < 3 < \cdots$. We sort the pairs to create a two-line array L such that

- the first line is a nonincreasing sequence,
- if two entries of the first line are equal and overlined (resp. nonoverlined) then the corresponding entries in the second line are in weakly increasing (resp. decreasing) order.

For the example above, after sorting, we obtain the two-line array

$$L = \begin{pmatrix} 2, 2, 2, \bar{2}, \bar{2}, 1, 1, 1, \bar{1}, \bar{1}, \bar{1} \\ 1, 1, \bar{1}, \bar{1}, \bar{2}, 2, 2, \bar{1}, 1, \bar{2}, 2 \end{pmatrix}.$$

We now describe the *insertion algorithm*. It is based on an algorithm proposed by Knuth in [17] and quite similar to the algorithm RS2 of [2].

We first explain how to insert a part j into an overpartition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$. The part j can be inserted at the end of λ if $(\lambda_1, \ldots, \lambda_\ell, j)$ is an overpartition. If this is possible, we insert the part and stop. Otherwise, we find the largest i such that $(\lambda_1, \ldots, \lambda_{i-1}, j, \lambda_{i+1}, \ldots, \lambda_\ell)$ is an overpartition. Then we bump λ_i and replace it with j. At the end we obtain a new overpartition that contains j as a part and whose length is ℓ if we bumped a part or $\ell + 1$ if j was added at the end and no part was bumped. For example if $\lambda = (4, 3, 3, \bar{3}, 2)$ then

- insert 5, obtain $(5, 3, 3, \bar{3}, 2)$ and bump 4,
- insert 3, obtain $(4, 3, 3, 3, 2)$ and bump $\bar{3}$,
- insert $\bar{3}$, obtain $(4, 3, 3, \bar{3}, 2)$ and bump $\bar{3}$,
- insert 1, obtain $(4, 3, 3, \bar{3}, 2, 1)$ and bump nothing.

To insert j into a plane overpartition P insert j in the first row. If nothing is bumped then stop. Otherwise, insert the bumped part in the second row. If something is bumped from the second row, insert it in the third and so on. Stop when nothing is bumped. For example, when $\bar{3}$ is inserted in

$$\begin{matrix} 4 & 3 & 3 & \bar{3} & 2 \\ 3 & \bar{3} & 2 & & \\ 1 & & & & \end{matrix} \quad \text{we obtain} \quad \begin{matrix} 4 & 3 & 3 & \bar{3} & 2 \\ 3 & \bar{3} & 2 & & \\ \bar{3} & & & & \\ 1 & & & & \end{matrix}.$$

We define how to insert a pair $\binom{i}{j}$ into a pair (P, Q) of plane overpartitions of the same shape. We first insert j in P with the insertion algorithm. If the insertion ends in column c and row r of P, then insert i in column c and row r in Q. Finally going from the two-line array L to pairs of plane overpartitions of the same shape works as follows: start with two empty plane overpartitions and insert each pair of L going from left to right. This is identical to the classical RSK algorithm [17].

Continuing with the previous example and applying the insertion algorithm we get

$$P = \begin{matrix} 2 & 2 & 2 \\ \bar{2} & 1 & 1 \\ \bar{2} & \bar{1} & \\ 1 & & \\ \bar{1} & & \\ \bar{1} & & \end{matrix}, \qquad Q = \begin{matrix} 2 & 2 & 2 \\ \bar{2} & 1 & 1 \\ \bar{2} & \bar{1} & \\ 1 & & \\ \bar{1} & & \\ \bar{1} & & \end{matrix}.$$

Let $\mathcal{L}(n)$ be the set of all plane overpartitions with the largest entry at most n.

Theorem 10. *There is a one-to-one correspondence between matrices $M \in \mathcal{M}_n$ and pairs of plane overpartitions of the same shape $(P, Q) \in \mathcal{L}(n) \times \mathcal{L}(n)$. This correspondence is such that*:

- *k appears in P exactly $\sum_i a_{ik} + c_{ik}$ times,*
- *$\bar{k}$ appears in P exactly $\sum_i b_{ik} + d_{ik}$ times,*

- *k appears in Q exactly $\sum_i a_{ik} + b_{ik}$ times,*
- *$\bar{k}$ appears in Q exactly $\sum_i c_{ik} + d_{ik}$ times.*

Proof. The proof is identical to the proof in the case of the RSK algorithm [17] or the RS2 algorithm [2]. Details are given in [27]. □

Theorem 11. *If the insertion algorithm produces (P, Q) with input matrix M, then the insertion algorithm produces (Q, P) with input matrix M^T.*

Proof. The proof is again analogous to the one in [17]. Given a two-line array $\binom{u_1,\dots,u_N}{v_1,\dots,v_N}$, we partition the pairs $\binom{u_\ell}{v_\ell}$ in classes such that $\binom{u_k}{v_k}$ and $\binom{u_m}{v_m}$ are in the same class if and only if:

- $u_k \geqslant u_m$ and if $u_k = u_m$, then u_m is overlined AND
- $v_k \leqslant v_m$ and if $v_k = v_m$, then v_k is overlined.

Then one can sort each class so that the first entries of each pair appear in nonincreasing order and then sort the classes so that the first entries of the first pair of each class are in nonincreasing order. For example if the two-line array is

$$\begin{pmatrix} 2, 2, 2, \bar{2}, \bar{2}, 1, 1, 1, \bar{1}, \bar{1}, \bar{1} \\ 1, 1, \bar{1}, \bar{1}, \bar{2}, 2, 2, \bar{1}, 1, \bar{2}, 2 \end{pmatrix},$$

we get the classes

$$C_1 = \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} \bar{2} \\ \bar{2} \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}, \qquad C_2 = \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\},$$

$$C_3 = \left\{ \begin{pmatrix} 2 \\ \bar{1} \end{pmatrix}, \begin{pmatrix} \bar{2} \\ \bar{1} \end{pmatrix}, \begin{pmatrix} 1 \\ \bar{1} \end{pmatrix}, \begin{pmatrix} \bar{1} \\ 1 \end{pmatrix}, \begin{pmatrix} \bar{1} \\ \bar{2} \end{pmatrix}, \begin{pmatrix} \bar{1} \\ 2 \end{pmatrix} \right\}.$$

If the classes are $C_1, \dots, C_d$ with

$$C_i = \left\{ \begin{pmatrix} u_{i1} \\ v_{i1} \end{pmatrix}, \dots, \begin{pmatrix} u_{in_i} \\ v_{in_i} \end{pmatrix} \right\}$$

then the first row of P is

$$v_{1n_1}, \dots, v_{dn_d}$$

and the first row of Q is

$$u_{11}, \dots, u_{d1}.$$

Moreover one constructs the rest of P and Q using the pairs:

$$\bigcup_{i=1}^{d} \bigcup_{j=1}^{n_i - 1} \begin{pmatrix} u_{i,j+1} \\ v_{ij} \end{pmatrix}.$$

One can adapt the proof of Lemma 1 of [17] for a complete proof. This is done in the master thesis of the second author [27]. As the two-line array corresponding to M^T is obtained by interchanging the two lines of the array and rearranging the columns, the theorem follows. □

This implies that $M = M^T$ if and only if $P = Q$.

Theorem 12. *There is a one-to-one correspondence between symmetric matrices $M \in \mathcal{M}_n$ and plane overpartitions $P \in \mathcal{L}(n)$. In this correspondence:*

- *k appears in P exactly $\sum_i a_{ik} + c_{ik}$ times, and*
- *$\bar{k}$ appears in P exactly $\sum_i b_{ik} + d_{ik}$ times.*

5.2. Enumeration of plane overpartitions

We can get the generating function for plane overpartitions whose largest entry is at most n from Theorem 12. By this bijection, if M is a symmetric matrix of size $2n \times 2n$ with blocks A, B, C and D, each of size $n \times n$ corresponding to a plane partition $\Pi \in \mathcal{L}(n)$, we have

$$|\Pi| = \sum_{i,j} i(a_{ij} + b_{ij} + c_{ij} + d_{ij}) \quad \text{and} \quad o(\Pi) = \sum_{i,j} b_{ij} + d_{ij}.$$

As M is symmetric, i.e. $a_{ij} = a_{ji}$, $d_{ij} = d_{ji}$ and $b_{ij} = c_{ji}$, we can express the above formulas as

$$|\Pi| = \sum_{i,j} (i+j) b_{ij} + \sum_i i(a_{ii} + d_{ii}) + \sum_{i<j} (i+j)(a_{ij} + d_{ij})$$

and

$$o(\Pi) = \sum_{i,j} b_{ij} + 2\sum_{i<j} d_{ij} + \sum_i d_{ii}.$$

Let $\mathcal{O}_n(q,a) = \sum_{\Pi \in \mathcal{L}(n)} a^{o(\Pi)} q^{|\Pi|}$. We are now ready to prove Theorem 6 which states that:

$$\mathcal{O}_n(q,a) = \prod_{j=1}^{n} \frac{\prod_{i=0}^{n}(1 + aq^{i+j})}{\prod_{i=1}^{j}(1 - q^{i+j-1})(1 - a^2 q^{i+j})}.$$

Indeed,

$$\begin{aligned}
\mathcal{O}_n(q,a) &= \sum_{M \in \mathcal{M}_n} \prod_{j=1}^{n} \left(\prod_{i=1}^{n} \left(q^{i+j} a\right)^{b_{ij}} \right) \left(q^{j a_{jj}} \left(a q^{j}\right)^{d_{jj}}\right) \left(\prod_{i=1}^{j-1} q^{(i+j) a_{ij}} \left(a^2 q^{i+j}\right)^{d_{ij}} \right) \\
&= \prod_{j=1}^{n} \frac{\prod_{i=1}^{n}(1 + aq^{i+j})}{(1 - q^j)(1 - aq^j) \prod_{i=1}^{j-1}(1 - q^{i+j})(1 - a^2 q^{i+j})} \\
&= \prod_{j=1}^{n} \frac{\prod_{i=0}^{n}(1 + aq^{i+j})}{\prod_{i=0}^{j-1}(1 - q^{i+j}) \prod_{i=1}^{j}(1 - a^2 q^{i+j})}.
\end{aligned}$$

When n tends to infinity we get back the weighted generating function of Proposition 4. We can also get this result with the method of the proof of Proposition 4 with the substitution $x_i = q^{i-1}$ for $1 \leqslant i \leqslant n+1$ and 0 otherwise and $y_i = -aq^i$ for $1 \leqslant i \leqslant n$ and 0 otherwise.

We can also get some more general results, as in [3].

Theorem 13. *The generating function for plane overpartitions whose parts lie in a set S of positive integers is given by*:

$$\prod_{i \in S} \left(\frac{\prod_{j \in S}(1 + aq^{i+j})}{(1 - q^i)(1 - aq^i)} \prod_{\substack{j \in S \\ j < i}} \frac{1}{(1 - q^{i+j})(1 - a^2 q^{i+j})} \right).$$

For example, as a corollary we get the generating function for plane overpartitions with odd parts. One can get a similar formula for even parts.

Corollary 1. *The generating function for plane overpartitions with odd parts is*

$$\prod_{i=1}^{\infty} \frac{(1 + aq^{2i})^{i-1}}{(1 - q^{2i-1})(1 - aq^{2i-1})(1 - q^{2i})^{\lfloor i/2 \rfloor}(1 - a^2 q^{2i})^{\lfloor i/2 \rfloor}}.$$

6. Interlacing sequences and cylindric partitions

We want to combine results of [4] and [31] to obtain a 1-parameter generalization of the formula for the generating function for cylindric partitions related to Hall–Littlewood symmetric functions.

We use the definitions of interlacing sequences, profiles, cylindric partitions, polynomials $A_\Pi(t)$ and $A_\Pi^{\mathrm{cyl}}(t)$ given in the introduction.

Recall that for an ordinary partition λ the polynomial $b_\lambda(t)$ is defined by

$$b_\lambda(t) = \prod_{i \geqslant 1} \varphi_{m_i(\lambda)}(t),$$

where $m_i(\lambda)$ denotes the number of times i occurs as a part of λ and $\varphi_r(t) = (1-t)(1-t^2)\cdots(1-t^r)$.

For a horizontal strip $\theta = \lambda/\mu$ we define

$$I_{\lambda/\mu} = \{ i \geqslant 1 \mid \theta'_i = 1 \text{ and } \theta'_{i+1} = 0 \},$$
$$J_{\lambda/\mu} = \{ j \geqslant 1 \mid \theta'_j = 0 \text{ and } \theta'_{j+1} = 1 \}.$$

Let

$$\varphi_{\lambda/\mu}(t) = \prod_{i \in I_{\lambda/\mu}} \left(1 - t^{m_i(\lambda)}\right) \quad \text{and} \quad \psi_{\lambda/\mu}(t) = \prod_{j \in J_{\lambda/\mu}} \left(1 - t^{m_j(\mu)}\right). \tag{6.1}$$

Then

$$\varphi_{\lambda/\mu}(t)/\psi_{\lambda/\mu}(t) = b_\lambda(t)/b_\mu(t).$$

For an interlacing sequence $\Lambda = (\lambda^1, \ldots, \lambda^T)$ with profile $A = (A_1, \ldots, A_{T-1})$ we define $\Phi_\Lambda(t)$:

$$\Phi_\Lambda(t) = \prod_{i=1}^{T-1} \phi_i(t), \tag{6.2}$$

where

$$\phi_i(t) = \begin{cases} \varphi_{\lambda^{i+1}/\lambda^i}(t), & A_i = 0, \\ \psi_{\lambda^i/\lambda^{i+1}}(t), & A_i = 1. \end{cases}$$

For $\Lambda = (\lambda^1, \ldots, \lambda^T)$ and $M = (\mu^1, \ldots, \mu^S)$ such that $\lambda^T = \mu^1$ we define

$$\Lambda \cdot M = \left(\lambda^1, \ldots, \lambda^T, \mu^2, \ldots, \mu^S\right)$$

and

$$\Lambda \cap M = \lambda^T.$$

Then

$$A_{\Lambda \cdot M} = \frac{A_\Lambda A_M}{b_{\Lambda \cap M}}, \qquad \Phi_{\Lambda \cdot M} = \Phi_\Lambda \Phi_M. \tag{6.3}$$

For an interlacing sequence $\Lambda = (\lambda^1, \ldots, \lambda^T)$ with profile $A = (A_1, \ldots, A_{T-1})$ we define the reverse $\overline{\Lambda} = (\lambda^T, \ldots, \lambda^1)$ with profile $\overline{A} = (1 - A_{T-1}, \ldots, 1 - A_1)$. Then

$$A_{\overline{\Lambda}} = A_\Lambda, \qquad \Phi_{\overline{\Lambda}} = \frac{b_{\lambda^1} \Phi_\Lambda}{b_{\lambda^T}}. \tag{6.4}$$

For an ordinary partition λ we construct an interlacing sequence $\langle \lambda \rangle = (\emptyset, \lambda^1, \ldots, \lambda^L)$ of length $L + 1 = \ell(\lambda) + 1$, where λ^i is obtained from λ by truncating the last $L - i$ parts. Then

$$A_{\langle \lambda \rangle} = \Phi_{\langle \lambda \rangle} = b_\lambda. \tag{6.5}$$

In [31, Propositions 2.4 and 2.6] it was shown that for a plane partition Π

$$\Phi_\Pi = A_\Pi. \tag{6.6}$$

The following proposition is an analogue of (6.6) for interlacing sequences.

Proposition 6. *If $\Lambda = (\lambda^1, \ldots, \lambda^T)$ is an interlacing sequence then*

$$b_{\lambda^1}\Phi_\Lambda = A_\Lambda.$$

Proof. If we show that the statement is true for sequences with constant profiles then inductively using (6.3) we can show that it is true for sequences with arbitrary profile. It is enough to show that the statement is true for sequences with $(0, \ldots, 0)$ profile because of (6.4). So, let $\Lambda = (\lambda^1, \ldots, \lambda^T)$ be a sequence with $(0, \ldots, 0)$ profile. Then $\Pi = \langle\lambda^1\rangle \cdot \Lambda \cdot \overline{\langle\lambda^T\rangle}$ is a plane partition and from (6.3), (6.4) and (6.5) we obtain that $A_\Pi = A_\Lambda$ and $\Phi_\Pi = b_{\lambda^1}\Phi_\Lambda$. Then from (6.6) it follows that $A_\Lambda = b_{\lambda^1}\Phi_\Lambda$. □

For skew plane partitions and cylindric partitions we obtain the following two corollaries.

Corollary 2. *For a skew plane partition Π we have $\Phi_\Pi = A_\Pi$.*

Corollary 3. *If Π is a cylindric partition given by $\Lambda = (\lambda^0, \ldots, \lambda^T)$ then $\Phi_\Lambda = A_\Pi^{\mathrm{cyl}}$.*

The last corollary comes from the observation that if a cylindric partition Π is given by a sequence $\Lambda = (\lambda^0, \ldots, \lambda^T)$ then

$$A_\Pi^{\mathrm{cyl}}(t) = A_\Lambda(t)/b_{\lambda^0}(t). \tag{6.7}$$

In the rest of this section we prove generalized MacMahon formulas for skew plane partitions and cylindric partitions that are stated in Theorems 7 and 8. The proofs of these theorems were inspired by [4,25,31]. We use a special class of symmetric functions called Hall–Littlewood functions.

6.1. The weight functions

In this subsection we introduce weights on sequences of ordinary partitions. For that we use Hall–Littlewood symmetric functions P and Q. We recall some of the facts about these functions, but for more details see Chapters III and VI of [23]. We follow the notation used there.

Recall that Hall–Littlewood symmetric functions $P_{\lambda/\mu}(x;t)$ and $Q_{\lambda/\mu}(x;t)$ depend on a parameter t and are indexed by pairs of ordinary partitions λ and μ. In the case when $t = 0$ they are equal to ordinary Schur functions and in the case when $t = -1$ to Schur P and Q functions.

The relationship between P and Q functions is given by (see (5.4) of [23, Chapter III])

$$Q_{\lambda/\mu}(x;t) = \frac{b_\lambda}{b_\mu} P_{\lambda/\mu}(x;t). \tag{6.8}$$

Recall that (by (5.3) of [23, Chapter III] and (6.8))

$$P_{\lambda/\mu} = Q_{\lambda/\mu} = 0 \quad \text{unless } \lambda \supset \mu. \tag{6.9}$$

We set $P_\lambda = P_{\lambda/\emptyset}$ and $Q_\lambda = Q_{\lambda/\emptyset}$. Recall that ((4.4) of [23, Chapter III])

$$H(x, y; t) := \sum_\lambda Q_\lambda(x;t) P_\lambda(y;t) = \prod_{i,j} \frac{1 - t x_i y_j}{1 - x_i y_j}.$$

A specialization of an algebra $\mathcal{A}$ is an algebra homomorphism between $\mathcal{A}$ and another algebra over the same ring. If ρ and σ are specializations of the algebra of symmetric functions then we write $P_{\lambda/\mu}(\rho;t)$, $Q_{\lambda/\mu}(\rho;t)$ and $H(\rho, \sigma; t)$ for the images of $P_{\lambda/\mu}(x;t)$, $Q_{\lambda/\mu}(x;t)$ and $H(x, y; t)$ under ρ, respectively $\rho \otimes \sigma$. A multiplication of a specialization ρ by a scalar $a \in \mathbb{C}$ is defined by its images on power sums:

$$p_n(a \cdot \rho) = a^n p_n(\rho).$$

If ρ is the specialization of Λ where $x_1 = a$, $x_2 = x_3 = \cdots = 0$ then by (5.14) and (5.14') of [23, Chapter VI]

$$Q_{\lambda/\mu}(\rho; t) = \begin{cases} \varphi_{\lambda/\mu}(t) a^{|\lambda|-|\mu|} & \lambda \supset \mu,\ \lambda/\mu \text{ is a horizontal strip,} \\ 0 & \text{otherwise.} \end{cases} \tag{6.10}$$

Similarly,

$$P_{\lambda/\mu}(\rho; t) = \begin{cases} \psi_{\lambda/\mu}(t) a^{|\lambda|-|\mu|} & \lambda \supset \mu,\ \lambda/\mu \text{ is a horizontal strip,} \\ 0 & \text{otherwise.} \end{cases} \tag{6.11}$$

Let $T \geqslant 2$ be an integer and $\rho^{\pm} = (\rho_1^{\pm}, \ldots, \rho_{T-1}^{\pm})$ be finite sequences of specializations. For a sequence of partitions $\Lambda = (\lambda^1, \ldots, \lambda^T)$ we set the weight function $W(\Lambda)$ to be

$$W(\Lambda) = q^{T|\lambda^T|} \sum_M \prod_{n=1}^{T-1} P_{\lambda^n/\mu^n}\big(\rho_n^-; t\big) Q_{\lambda^{n+1}/\mu^n}\big(\rho_n^+; t\big),$$

where q and t are parameters and the sum ranges over all sequences of partitions $M = (\mu^1, \ldots, \mu^{T-1})$.

We can also define the weights using another set of specializations $R^{\pm} = (R_1^{\pm}, \ldots, R_{T-1}^{\pm})$ where $R_i^{\pm} = q^{\pm i} \rho_i^{\pm}$. Then

$$W(\Lambda) = \sum_M W\big(\Lambda, M, R^-, R^+\big),$$

where

$$W\big(\Lambda, M, R^-, R^+\big) = q^{|\Lambda|} \prod_{n=1}^{T-1} P_{\lambda^n/\mu^n}\big(R_n^-; t\big) Q_{\lambda^{n+1}/\mu^n}\big(R_n^+; t\big).$$

We will focus on two special sums, namely

$$Z_{\text{skew}} = \sum_{\Lambda = (\emptyset, \lambda^1, \ldots, \lambda^T, \emptyset)} W(\Lambda)$$

and

$$Z_{\text{cyl}} = \sum_{\substack{\Lambda = (\lambda^0, \lambda^1, \ldots, \lambda^T) \\ \lambda^0 = \lambda^T}} W(\Lambda).$$

Let $A^- = (A_1^-, \ldots, A_{T-1}^-)$ and $A^+ = (A_1^+, \ldots, A_{T-1}^+)$ be sequences of 0's and 1's such that $A_k^- + A_k^+ = 1$.

If the specializations $R^{\pm}$ are evaluations given by

$$R_k^{\pm} : x_1 = A_k^{\pm}, x_2 = x_3 = \cdots = 0, \tag{6.12}$$

then by (6.10) and (6.11) the weight $W(\Lambda)$ vanishes unless Λ is an interlacing sequence of profile A^-, and in that case

$$W(\Lambda) = \Phi_\Lambda(t) q^{|\Lambda|}.$$

Then, from Corollaries 2 and 3, we have

$$Z_{\text{skew}} = \sum_{\Pi \in \text{Skew}(T, A)} A_\Pi(t) q^{|\Pi|}$$

and

$$Z_{\text{cyl}} = \sum_{\Pi \in \text{Cyl}(T, A)} A_\Pi^{\text{cyl}}(t) q^{|\Pi|},$$

where A^- in both formulas is given by the fixed profile A of skew plane partitions, respectively cylindric partitions.

6.2. Partition functions

If ρ is $x_1 = r$, $x_2 = x_3 = \cdots = 0$ and σ is $x_1 = s$, $x_2 = x_3 = \cdots = 0$ then

$$H(\rho,\sigma) = \frac{1-trs}{1-rs}. \tag{6.13}$$

Thus, for the specializations $\rho_i^{\pm} = q^{\mp i} R_i^{\pm}$, where $R^{\pm}$ are given by (6.12), we have

$$H\big(\rho_i^+,\rho_j^-\big) = \frac{1-tq^{j-i}A_i^+A_j^-}{1-q^{j-i}A_i^+A_j^-}. \tag{6.14}$$

We now use Proposition 2.2 of [31]:

Proposition 7.

$$Z_{\mathrm{skew}}\big(\rho^-,\rho^+\big) = \prod_{0\leqslant i<j\leqslant T} H\big(\rho_i^+,\rho_j^-\big).$$

This proposition together with (6.14) implies Theorem 7. The generating function formula for skew plane partitions can also be seen as the generating function formula for reverse plane partitions as explained in the introduction.

Each skew plane partition can be represented as an infinite sequence of ordinary partitions by adding infinitely many empty partitions to the left and right side. In that way, the profiles become infinite sequences of 0's and 1's. Theorem 7 also gives the generating function formula for skew plane partitions of infinite profiles $A = (\ldots, A_{-1}, A_0, A_1, \ldots)$:

$$\sum_{\Pi\in \mathrm{Skew}(A)} A_{\Pi}(t)q^{|\Pi|} = \prod_{\substack{i<j\\ A_i=0,\, A_j=1}} \frac{1-tq^{j-i}}{1-q^{j-i}}. \tag{6.15}$$

Similarly for cylindric partitions, using (6.13) together with the following proposition we obtain Theorem 8.

Proposition 8.

$$Z_{\mathrm{cyl}}\big(R^-,R^+\big) = \prod_{n=1}^{\infty}\frac{1}{1-q^{nT}}\prod_{k,l=1}^{T} H\big(q^{(k-l)_{(T)}+(n-1)T}R_k^-, R_l^+\big), \tag{6.16}$$

where $i_{(T)}$ is the smallest positive integer such that $i \equiv i_{(T)} \bmod T$.

Proof. We use

$$\sum_{\lambda} Q_{\lambda/\mu}(x)P_{\lambda/\nu}(y) = H(x,y)\sum_{\tau} Q_{\nu/\tau}(x)P_{\mu/\tau}(y). \tag{6.17}$$

The proof of this is analogous to the proof of Proposition 5.1 ((6.17) for $t=-1$) that appeared in [30]. Also, see Example 26 of Chapter I, Section 5 of [23].

The proof of (6.16) uses the same idea as in the proof of Proposition 1.1 of [4].

We start with the definition of $Z_{\mathrm{cyl}}(R^-,R^+)$:

$$\begin{aligned}
Z_{\mathrm{cyl}}\big(R^-,R^+\big) &= \sum_{\Lambda,M} W\big(\Lambda,M,R^-,R^+\big)\\
&= \sum_{\Lambda,M} q^{|\Lambda|}\prod_{n=1}^{T} P_{\lambda^{n-1}/\mu^n}\big(R_n^-\big)Q_{\lambda^n/\mu^n}\big(R_n^+\big)\\
&= \sum_{\Lambda,M} q^{|M|}\prod_{n=1}^{T} P_{\lambda^{n-1}/\mu^n}\big(qR_n^-\big)Q_{\lambda^n/\mu^n}\big(R_n^+\big).
\end{aligned}$$

If $x = (x_1, x_2, \dots, x_T)$ is a vector we define the shift as $\mathrm{sh}(x) = (x_2, \dots, x_T, x_1)$. We set $R_0^{\pm} = R_T^{\pm}$, $\mu_0 = \mu_T$ and $\nu_0 = \nu_T$. If using the formula (6.17) we substitute the sums over the λ^i's by the sums over the ν^{i-1}'s we obtain

$$\begin{aligned}
Z_{\mathrm{cyl}}(R^-, R^+) &= H(q\,\mathrm{sh}\,R^-; R^+) \sum_{M,N} q^{|M|} Q_{\mu^1/\nu^0}(R_T^+), P_{\mu^0/\nu^0}(qR_1^-) \cdot \\
&\quad \cdot Q_{\mu^2/\nu^1}(R_1^+) P_{\mu^1/\nu^1}(qR_2^-) \cdots Q_{\mu^0/\nu^{T-1}}(qR_{T-1}^+) P_{\mu^{T-1}/\nu^{T-1}}(R_T^-) \\
&= H(q\,\mathrm{sh}\,R^-; R^+) \sum_{M,N} W(\mathrm{sh}\,M, N, q\,\mathrm{sh}\,R^-, R^+) \\
&= H(q\,\mathrm{sh}\,R^-; R^+) Z_{\mathrm{cyl}}(q\,\mathrm{sh}\,R^-, R^+).
\end{aligned}$$

Since $\mathrm{sh}^T = \mathrm{id}$, if we apply the same trick T times, we obtain

$$\begin{aligned}
Z_{\mathrm{cyl}}(R^-, R^+) &= \prod_{i=1}^{T} H(q^i\,\mathrm{sh}^i\,R^-; R^+) \cdot Z_{\mathrm{cyl}}(q^T R^-, R^+) \\
&= \prod_{i=1}^{T} H(q^i\,\mathrm{sh}^i\,R^-; R^+) \cdot Z_{\mathrm{cyl}}(sR^-, R^+),
\end{aligned}$$

where $s = q^T$. Thus,

$$Z_{\mathrm{cyl}}(R^-, R^+) = \prod_{n=1}^{\infty} \prod_{i=1}^{T} H(q^{i+(n-1)T}\,\mathrm{sh}^i\,R^-; R^+) \lim_{n\to\infty} Z_{\mathrm{cyl}}(s^n R^-, R^+).$$

From

$$\lim_{n\to\infty} Z_{\mathrm{cyl}}(s^n R^-, R^+) = \lim_{n\to\infty} Z_{\mathrm{cyl}}(\text{trivial}, R^+) = \prod_{n=1}^{\infty} \frac{1}{1-s^n}$$

and

$$\begin{aligned}
\prod_{i=1}^{T} H(q^i\,\mathrm{sh}^i\,R^-, R^+) &= \prod_{l=1}^{T} \left[\prod_{k=l+1}^{T} H(q^{k-l} R_k^-, R_l^+) \prod_{k=1}^{l} H(q^{T+k-l} R_k^-, R_l^+) \right] \\
&= \prod_{k,l=1}^{T} H(q^{(k-l)_{(T)}} R_k^-, R_l^+),
\end{aligned}$$

we conclude that (6.16) holds. $\square$

Observe that if in Theorem 8 we let $T \to \infty$, i.e. the circumference of the cylinder tends to infinity then we recover (6.15).

7. Concluding remarks

In this paper, we determine generating functions for plane overpartitions with several types of constraints. In particular, we can compute the generating function for plane overpartitions with at most r rows and c columns and the generating function for plane overpartitions with entries at most n. All the proofs can be done with symmetric functions, but we also highlight combinatorial proofs when these are simple. The natural question is therefore to put those constraints together and to compute the generating function of plane overpartitions with at most r rows, c columns and entries at most n. Unfortunately, this generating function cannot be written in terms of a product in the style of Theorems 1–8. For example, when $r = 1$, this generating function can be written as

$$\sum_{k=0}^{n} a^k q^{\binom{k+1}{2}} \frac{(q)_{n-k+c}}{(q)_{n-k}(q)_k(q)_{c-k}}.$$

Computer experiments show that there is a large irreducible factor in the product. Let $\mathbb{A}$ be the alphabet $1 + q + \cdots + q^n$ and $\mathbb{B}$ be the alphabet $-aq - aq^2 - \cdots - aq^n$. For general r we can write this generating function as

$$\sum_{\lambda \subseteq c^r} s_\lambda(\mathbb{A} - \mathbb{B})$$

where the sum is taken over partitions λ such that all the parts of λ' have the same parity as r.

In Section 6 we compute generating functions for cylindric partitions. Their generating functions are elegant products. In the case $t = 0$ (cylindric partitions) and $t = -1$ (cylindric overpartitions), one could certainly adapt the ideas of Gansner [9] to give a constructive proof of the result. We leave this as an open problem.

Finally it would be interesting to generalize the combinatorial techniques used in this paper in Sections 3 and 5 to understand the combinatorics of plane partitions for general t.

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