Semilinear Elliptic Equations and Supercritical Growth

C. BUDD AND J. NORBURY

Mathematical Institute, University of Oxford,
24/29 St. Giles, Oxford OX1 3LB, England

Received July 22, 1985; revised May 16, 1986

1. Introduction

The recent papers by Brezis and Nirenberg [1] and by Ni [9] examine positive solutions of the problem

\[ \Delta \bar{u} + \lambda \bar{u} + \bar{u}^p = 0 \quad \text{in} \quad \Omega \subset \mathbb{R}^n, \]

and

\[ \bar{u} = 0 \quad \text{on} \quad \partial \Omega, \]

where \( p \leq (n + 2)/(n - 2) \). The upper bound \( p_c = (n + 2)/(n - 2) \) is the limit of values for which the embedding of \( H_0^1(\Omega) \) in \( L^{p_c+1}(\Omega) \) is continuous. The problem (1.1) has also been considered in the survey paper by Lions [8], and numerical calculations for it have been reported by Georg [4].

Brezis and Nirenberg use variational techniques to prove the existence of positive solutions, and these methods break down for \( p > p_c \). The aim of the present paper is to understand, for the case \( p > p_c \), the behaviour of solutions to (1.1) with large supremum-norm. To this end \( \Omega \) is restricted to be the unit ball \( B \subset \mathbb{R}^3 \), as in this case the behaviour of solutions to (1.1) can be described in some detail.

According to the results proved in Gidas, Ni, and Nirenberg [5], all positive solutions of (1.1) must be radial when \( \Omega = B \). With the scaling \( \bar{u}^{p-1} = \lambda \bar{u}^{p-1} \) in the case \( n = 3 \), problem (1.1) reduces to the following ordinary differential equation problem:

\[ \bar{u}_{rr} + \frac{2}{r} \bar{u}_r + \lambda (\bar{u} + \bar{u}^p) = 0 \quad \text{in} \quad 0 < r < 1, \]

and

\[ \bar{u}_r(0) = \bar{u}(1) = 0. \]
Here $u(r)$ is a positive function of the radial distance from the origin of $B$. From the maximum principle we deduce that the maximum value of $u$ is attained at the origin. Hence $u(0) = \sup u \equiv \sup_{0<r<1} u(r)$. The scaling used above is well defined, as it is known from Rabinowitz [12] that any solution of (1.1) with $p > p_c$ must have $\lambda$ bounded away from zero.

Solutions of (1.2) with small $\sup u$ may be examined as regular bifurcations from the trivial solution at $\lambda = \pi^2$ and global existence of $u$ for $\lambda < \pi^2$ is then ensured by the results of Rabinowitz [11]. Because $\lambda \in [\delta, \pi^2]$ for some $\delta > 0$, the branch of solution pairs $(\lambda, u)$ in the product space $\mathbb{R} \times C[0, 1]$ must extend to infinity in such a way that $\sup u \to \infty$ along the branch with $\lambda < \pi^2$. Thus the existence of large $\sup u$ solutions of (1.2) is ensured, and these solutions are considered in detail in the remainder of this paper.

The approach used in this paper is first to rescale the ordinary differential equation in (1.2) by writing

$$r = \lambda^{-\frac{1}{2}}s \quad \text{and} \quad \hat{u}(\lambda^{-\frac{1}{2}}s) = u(s),$$

and then to consider the associated initial value problem

$$u_{ss} + \frac{2}{s} u_s + u + u^p = 0, \quad \text{in } s > 0,$$

(1.3)

and

$$u(0) = \theta, \quad u_s(0) = 0,$$

(1.4)

Existence of $u(s)$ for $s > 0$ for all $p \geq 1$ and all $\theta > 0$ follows from standard theory since $u(0)$ is the global maximum in $0 < s < \lambda^{1/2}$. Comparison arguments show that $u$ cannot remain positive; thus $u(s)$ has a first positive zero $\mu$, so that $u(\mu) = 0$, and we define $\lambda = \mu^2$.

Let $M(s)$ be defined to satisfy (1.3) and the singular initial conditions

$$s^\alpha M \to K \quad \text{and} \quad s^{\alpha+1} M_s \to -\alpha K \quad \text{as } s \to 0;$$

throughout this paper we take

$$\alpha = \frac{2\alpha}{p-1} \quad \text{and} \quad K^{p-1} = \alpha(1-\alpha).$$

The existence of a unique $M$ is proved in Lemma 4.1; again comparison arguments show that $M$ must vanish before $\pi$, and we thus define $\mu_0$ as the first such zero. Then $M(r) = \mu_0^2 M(\mu_0 r)$ is a weak solution in $H^1_0(B)$ of problem (1.1) with $\lambda = \mu_0^2$.

The main results are stated as follows.
THEOREM 1.1. For $p > p_c = 5$ (as throughout $n = 3$) define $\omega^2 = 2p(p - 3)(p - 1)^{-2} - \frac{1}{4}$ and $\theta_0(\tau) = \theta_0 \exp(2\tau/(p - 1) \omega)$ for some constant $\theta_0$ involving $p$ alone. Then, for each sufficiently large $\tau$, there is a solution $u(s)$ of (1.3, 4) satisfying

(i) $u(0) \equiv \theta = \theta_0(\tau)[1 + 0(\theta_0(\tau)^{m(p - 5)/4})]$ and 
(ii) $\mu = \mu_c + E\theta_0(\tau)^{-1(p - 5)/4} \sin \tau[1 + 0(\theta_0(\tau)^{-m(p - 5)/4})]$,

where $m > 0$ and $E$ are constants involving $p$ alone, $\mu_c$ is as defined below (1.5) and $\mu$ is the first zero of $u$.

THEOREM 1.2. For $n = 1, 2, \ldots$ there exists a sequence of numbers $\theta_n \to \infty$ and a corresponding sequence of positive functions $u_n(s) \in C^2(0, \mu_c)$ such that

(i) $u_n(s)$ satisfies (1.3, 4) with $u_n(0) = \theta_n$ for each $n$, 
(ii) $u_n(\mu_c) = 0$ and $u_n(s) > 0$ for $0 \leq s < \mu_c$, and 
(iii) $\int_0^{\mu_c} \{(d/ds)[u_n(s) - M(s)]^2 s^2 ds\} \to 0$ as $n \to \infty$.

COROLLARY 1.3. As $\tau \to \infty$ the functions $\bar{u}(r) \equiv \mu^2 u(\mu r)$ converge to $\bar{M}(r)$ in $H^1_0(B)$.

These results imply the bifurcation diagram shown in Fig. 1, for which numerical calculations have been made for $p = 9$ and comparison made with the asymptotic predictions choosing the values $E = 0.136$ and

![Figure 1](image-url)
$\theta_0 = 2.26$. The asymptotic formulae were checked independently of these choices by evaluating both the ratios $\theta_n/\theta_{n-1}$ when $\lambda = \mu_c^2$ and the ratios of successive values of $(\lambda - \mu_c^2)$ at those $\lambda$ for which $d\lambda/d\theta = 0$. These comparisons indicate that the error bounds implied by the 0 symbols in Theorem 1.1 appear to be excessive for $p > 6$. This information is used further in Fig. 2 to plot the $H^1_0$ norm of $\hat{u}(r)$ against $\mu$.

The methods of this paper can be generalized in several ways, and examples are given in Section 8. In conclusion, we have shown in this paper that, when $\Omega$ is the unit ball in $\mathbb{R}^3$ and $p > p_c$, there is a critical value of $\lambda = \lambda_c(p)$ at which problem (1.1) has an infinity of positive $C^2$ solutions as well as a singular weak solution. This behaviour contrasts markedly with that of solutions of (1.1) when $p \leq p_c$. In particular, it is observed numerically in the range $p \leq p_c$ (and proved in Ni [9] for the restricted range $p \leq 3$) that, for each fixed $\lambda$, problem (1.1) has a unique positive solution.

These differences show that the breakdown of certain variational methods as a means of solving (1.1) is associated with a real change in the structure of the bifurcation diagram for solutions. Formal asymptotic calculations also indicate that the structure of solutions of problem (1.1), as given by the results in Theorem 1.1, is preserved when $\Omega$ is replaced by a domain which is a small smooth perturbation of the unit sphere, although in this case other forms of solution may be present.

![Figure 2](image-url)
2. The Emden–Fowler Equation

We noted in the Introduction that \( u \) attains its maximum at the origin. When this maximum value is large \( u^p \) dominates \( u \), and hence a solution of (1.3, 4) should lie close to that of the Emden–Fowler problem

\[
\begin{align*}
  w_{ss}(\theta, s) + \frac{2}{s} w_s(\theta, s) + w^p(\theta, s) &= 0 \quad \text{in } s > 0, \\
  w(\theta, 0) &= \theta \quad \text{and} \quad w_s(\theta, 0) = 0.
\end{align*}
\]

(2.1)

In this section we consider solutions of (2.1) in the region \( s^\alpha \gg \theta^{-1} \).

Solutions to (2.1) have the group structure

\[
w(\theta, s) = \theta w(1, \theta^{1-p^{-1}v/2}s),
\]

(2.2)

from which the Emden–Fowler coordinate system

\[
a = s^\alpha w(1, s), \quad b = s^{\alpha+1}w_s(1, s), \quad s = e^t
\]

may be derived. On substituting this into (2.1) we obtain the autonomous system

\[
\begin{align*}
  da/dt &= xa + b, \\
  db/dt &= (x - 1)b - a^p.
\end{align*}
\]

(2.3)

Equations (2.1) and (2.3) have been studied in detail; references include the papers of Joseph and Lundgren [7], Crandall and Rabinowitz [3], Wong [13], and Jones and Küpper [6]. For \( p > 3 \) the phase plane for (2.3) has a saddle point at the origin, and a stationary point at \( (K, -xK) \), where \( K^{p-1} = x(1-x) \). As \( p \) passes through the critical value of 5 this point changes from being an unstable repeller for \( p < 5 \), to a centre for \( p = 5 \) (when (2.3) has a Hamiltonian structure with Hamiltonian \( 6H = 3ab + 3b^2 + a^6 \)), to a stable spiral attractor when \( p > 5 \). In the phase plane the trajectory corresponding to a solution of (2.1) is part of the unstable manifold of the origin, and it is well known from the equality of Pohozaev [10] that this solution is strictly positive for \( s \geq 0 \) when \( p \geq 5 \). A result of Chandrasekhar given in Joseph and Lundgren [7] shows the following.

**Lemma 2.1.** For \( p > 5 \) and \( K^{p-1} = x(1-x) \), if \((a(t), b(t))\) is the solution of (2.3) corresponding to a solution \( w(1, s) \) of (2.1), then as \( s \) and \( t \) tend to infinity

\[
a(t) - K = s^\alpha w(1, s) - K \to 0
\]
and

\[ b(t) + \alpha K = s^* + 1 w_s(1, s) + \alpha K \to 0. \]

From this lemma we immediately deduce bounds for \( w(1, s) \) as the lemma implies that the corresponding trajectory \((a(t), b(t))\) lies in a bounded region of the phase plane. Thus there exist constants \( P \) and \( Q \) such that

\[ w(1, s) < P/s^2 \]

and

\[ |w_s(1, s)| < Q/s^{2+1}. \]

The behaviour of \( w(1, s) \) as \( s \) tends to infinity can now be examined in more detail by studying \( \phi(s) \) where \( \phi(s) = w(1, s) - Ks^{-2} \).

**Lemma 2.2.** There is an \( s^* \) sufficiently large such that, in the range \( s \geq s^* \), \( \phi(s) \) satisfies for \( n > 0 \)

\[ \phi^{(n)}(s) = \frac{C s^{-1/2} \sin(\omega \ln s + D)}{s^n + A_n(s) s^{-(n+1)(p-1)}}, \]

where the \( A_n(s) \) are bounded continuous functions of \( s \), \( \omega \) takes the value given in Theorem 1.1, and \( C \) and \( D \) are both constants.

**Proof.** By substituting \( \phi = w(1, s) - Ks^{-2} \) in (2.1), we find that \( \phi(s) \) satisfies

\[ L\phi \equiv \phi_{ss} + \frac{2}{s} \phi_s + \frac{P}{s^2} \alpha(1 - \alpha) \phi = \phi^2 s^{-(p-2)x} f(\phi, s), \tag{2.5} \]

where \(-f(\phi, s) = [(Ks^{-2} + \phi)^p - K s^{-p} - p K s^{-1} s^{-2} \phi] s^{(p-2)x} \phi^{-2}\). The linear differential equation \( L\phi = 0 \) has linearly independent solutions \( s^{-1/2} \cos(\omega \ln s) \) and \( s^{-1/2} \sin(\omega \ln s) \). Using the method of variation of constants we may invert \( L \) to obtain the following integral equation

\[ \phi(s) = C s^{-1/2} \sin(\omega \ln s + D) + s^{-1/2} \]

\[ \times \int_s^\infty \frac{t^{3/2}}{\omega} \sin(\omega \ln(t/s)) \phi^2 t^{-(p-2)x} f(\phi, t) \, dt, \tag{2.6} \]

where \( C \) and \( D \) are both constants. This may be simplified by setting \( \psi = s^{1/2} \phi \) and \( g(\psi, s) = f(s^{-1/2} \psi, s) \). The resulting expression for \( \psi \) is then

\[ \psi(s) = N \psi(s) \equiv \sin(\omega \ln s + D) \]

\[ + \int_s^\infty \omega \sin(\omega \ln(t/s)) \psi^2 t^{-(p-2)x} g(\psi, t) \, dt. \tag{2.7} \]
We take $s$ in the range $s_* \leq s < \infty$ where $s_*$ is suitably large, and consider $N\psi$ as a map from $C[s_*, \infty)$ into itself. We claim that, for suitable $s_*$, the operator $N\psi$ maps the set

$$B_* = \{ \psi \in C[s_*, \infty) \text{ and } \|\psi\| \equiv \sup_{s_* < s < \infty} (|\psi(s)|) \leq 2C \}$$

into itself, and is a contraction mapping on $B_*$. If $|\psi| < 2C$ then $|\psi| < 2Cs^{-1/2}$. Thus, as $\alpha < \frac{1}{2}$ when $p > 5$, we may deduce that if $s \geq 1$, $|\phi| < Hs^{-\alpha}$ for a suitable constant $H$. From this we see, using the Mean Value Theorem, that $|f(\phi, s)|$ and $|g(\psi, s)|$ are bounded above by a constant $G$ in the region $s \geq 1$. We further deduce from the Mean Value Theorem that in $s > 1$

$$|\psi_1 g(\psi_1, s) - \psi_2 g(\psi_2, s)| \leq 2G \cdot 2C|\psi_1 - \psi_2|$$

for all $\psi_1, \psi_2 \in B_*$. Since, for $|g(\psi, s)| \leq G$,

$$\left| \int_s^\infty \frac{1}{\omega} \sin \left[ \omega \ln(t/s) \right] \psi^2 t^{-(p-2)\alpha} g(\psi, t) \, dt \right| \leq \frac{G}{\omega} \|\psi\|^2 \int_s^\infty t^{(1/2)-(p-2)\alpha} \, dt = \frac{4G}{\omega(p-5)} \|\psi\|^2 s^{(5-p)\alpha/4},$$

we see that in $s \geq 1$

$$|N\psi - C \sin(\omega \ln s + D)| \leq A \|\psi\|^2 s^{(5-p)\alpha/4}.$$

A similar calculation shows that

$$|N\psi_1 - N\psi_2| \leq B \|\psi_1 - \psi_2\| s^{(5-p)\alpha/4}.$$

Hence, as $p > 5$, it is possible for each value of $C$ to choose $s_*$ so that $N$ is a contraction mapping of $B_*$ into itself. Thus we define $\psi_0 = C \sin(\omega \ln s + D)$ and the iteration $\psi_{n+1} = N\psi_n$ for $n \geq 0$. The Contraction Mapping Theorem then ensures that this iteration converges to the unique solution $\psi(s)$ of (2.7) in $B_*$. Further, $\psi$ differs from $\psi_0$ by a bounded multiple of $s^{(5-p)\alpha/4}$. By repeated differentiation of (2.6) with respect to $s$, and use of the above bound, we prove the remainder of the lemma.

The values of $C$ and $D$ which correspond to the solution $w(1, s)$ are determined by the initial conditions (to define $D$ uniquely we restrict $D$ to lie in the range (0, $2\pi$]). The behaviour of $w(\theta, s)$ in the range $s \geq s_\theta^{(p-1)/2}$ can now be deduced from that of $w(1, s)$ by means of the group relation given by (2.2). In par-
ticular, for large \( \theta \) this allows the behaviour of \( w(\theta, s) \) in the range \( s < 1 \) to be found, as we show in the following.

**Lemma 2.3.**  (i) As \( s^{(p-1)/2} \) tends to infinity

\[
s^2w(\theta, s) - K \to 0 \quad \text{and} \quad s^{x+1}w_x(\theta, s) + xK \to 0.
\]

(ii) For all \( s > 0 \) there exist constants \( P \) and \( Q \), independent of \( \theta \), such that

\[
0 < w(\theta, s) < Ps^{-a} \quad \text{and} \quad 0 < -w_x(\theta, s) < Qs^{-(1+a)}.
\]  \hspace{1cm} (2.8)

*Proof.* From Lemma 2.1 we see that for all \( \varepsilon > 0 \) there exists \( M(\varepsilon) \) such that \( |s^2w(1, t) - K| < \varepsilon \) if \( t > M(\varepsilon) \). Setting \( t = s^{(p-1)/2} \), and using (2.2), we have for \( s^{(p-1)/2} > M(\varepsilon) \) that

\[
(s^2w(1, s^{(p-1)/2}) - 2^{-a} = |s^2w(\theta, s) - K| < \varepsilon.
\]

A similar calculation holds for \( w_x(\theta, s) < 0 \). This proves (i). To prove (ii) we simply note from (2.4) that

\[
P \geq t^aw(1, t) = s^2\theta w(1, s^{(p-1)/2}) = s^2w(\theta, s)
\]

and

\[
Q \geq t^{x+1}w_x(1, t) = -s^{x+1}\theta^{1+(p-1)/2}t^{(1,s^{(p-1)/2})} = -s^{x+1}w_x(\theta, s).
\]

**Lemma 2.4.** For \( s^{(1/2)(p-1)} \geq s_\ast \) where \( s_\ast \) takes the value implied by Lemma 2.2, and for \( n = 0, 1, 2 \), the solution \( w(\theta, s) \) of 2.1 satisfies

(i) \[
\frac{\partial^n}{\partial \theta^n}(w(\theta, s)) = \left\{ \begin{array}{ll}
Ks^{-a} & \\
0 &
\end{array} \right. + \frac{\partial^n}{\partial \theta^n} \left\{ Cs^{-(1/2)}\theta^{-(p-5)/4} \times \sin(\omega \ln s + \frac{1}{2}(p-1) \omega \ln \theta + D) \right\}
\]

\[+ B_n(s) s^{-(p+3)/(p-1)} \theta^{-n-(p-5)/2}, \]

(ii) \[
\frac{\partial^n}{\partial \theta^n}(w_x(\theta, s)) = \left\{ \begin{array}{ll}
Ks^{-(a+1)} & \\
0 &
\end{array} \right. + \frac{\partial^{n+1}}{\partial \theta^n \partial s}
\]

\[\times \left\{ Cs^{-(1/2)}\theta^{-(p-5)/4} \sin(\omega \ln s + \frac{1}{2}(p-1) \omega \ln \theta + D) \right\}
\]

\[+ C_n(s) s^{-2(p+1)/(p-1)} \theta^{-n-(p-5)/2}. \]

Here \( B_n(s) \) and \( C_n(s) \) are bounded functions of \( s \) in \( s^{(p-1)/2} \geq s_\ast \) with bounds independent of \( \theta \).
Proof. These bounds are obtained following a lengthy calculation which results from substituting (2.2) into the relations for \( w(1, s) \) given in Lemma 2.2. Information on the derivative of \( w(\theta, s) \) with respect to \( \theta \) is obtained by repeated differentiation of (2.2) and use of the bounds given in Lemma 2.2.

3. THE BEHAVIOUR OF \( u \) NEAR THE ORIGIN

In this section we investigate \( u \) in a region close to the origin by studying it as a perturbation of the function \( w(\theta, s) \) introduced in Section 2. The main result is stated as follows.

**Lemma 3.1.** Define \( x(s) = u(s) - w(\theta, s) \) where \( u \) satisfies (1.3, 4). There is a function \( \gamma(M, s) = \alpha(p[K + \varepsilon(M)]^{p-1} + s^2) \), where \( \varepsilon(M) \) has no \( \theta \) dependence and tends to zero as \( M \) tends to infinity. Then, in the range

\[
\begin{align*}
  s^2 < \theta^{-\frac{1}{p+1}} \equiv \theta^m - 1, \\
  |x(s)| \leq 2 \exp(pM^2) \left( \frac{s^{(p-1)/2}}{M} \right)^\gamma R s^{2-\gamma},
\end{align*}
\]

where \( R \) is a constant independent of \( M \) and \( \theta \).

**Proof.** Solutions to (1.3, 4) and to (2.1) are also continuous solutions of the Volterra integral equations

\[
\begin{align*}
  u(s) &= \theta + V(u + u^p) \quad (3.3) \\
  w(s) &= \theta + V(w(\theta, s)^p), \quad (3.4)
\end{align*}
\]

where \( V \) is the Volterra integral operator defined by

\[
(Vf)(s) = \int_0^s t^2(1/s - 1/t) f(t) \, dt.
\]

For \( x(s) = u(s) - w(\theta, s) \) we have

\[
Lx \equiv [I - V(1 + pw(\theta, s)^{p-1})] x = Vw(\theta, s) + VTx, \quad (3.5)
\]

where \( Tx = (w + x)^p - w^p - pw^{p-1}x \). To proceed further we invert the linear operator \( L \) given in (3.5).
LEMMA 3.2. \( L \) has an inverse \( \Gamma : C \to C \). Moreover there is a function \( \gamma(M) \) defined as in Lemma 3.1 such that

(i) if \( s < M^{\theta - (p - 1)/2} \) then \( |\Gamma f(s)| \leq \exp(p M^2) \sup f \);

(ii) if \( s > M^{\theta - (p - 1)/2} \) then \( |\Gamma f(s)| \leq \exp(p M^2)((s\theta/M)^{(p - 1)/2})^\gamma \sup f \).

Proof. To obtain an inverse for \( L \) we make use of the Volterra series

\[ \Gamma = L^{-1} = I + Z + Z^2 + \cdots, \tag{3.6} \]

where \( Z \) is the linear operator \( Zf(s) = V(1 + pw(\theta, s)^{p - 1}) f(s) \). To study \( Z^n \) we first establish some bounds for \( w(\theta, s) \). From the maximum principle it is evident that \( w(\theta, s) < \theta \) for all \( s > 0 \). Also from Corollary 2.3 it is clear that there is a function \( \varepsilon(M) \) independent of \( \theta \) tending to zero as \( M \) tends to infinity such that \( |w(\theta, s)| \leq [K + \varepsilon(M)]/s^\alpha \) when \( s > M^{\theta - (p - 1)/2} \).

From these bounds we may deduce that

\[ |Zf(s)| \leq p\theta^{(p - 1)} \int_0^{M^{\theta - (p - 1)/2}} |f| \, dt^2 \]

\[ + \gamma(M) \int_{M^{\theta - (p - 1)/2}}^s |f| \, d \left[ \log \left( \frac{t^{(p - 1)/2}}{M} \right) \right], \tag{3.7} \]

where \( \gamma(M) \) is obtained from the estimate

\[ s^2 |1 + pw^{p - 1}(\theta, s)| \leq [s^2 + p(K + \varepsilon(M))^{p - 1}] \equiv \gamma(M) \alpha^{-1}. \]

By induction on the bound given in (3.7) a straightforward calculation shows that

(i) in the region \( s < M^{\theta - (p - 1)/2} \),

\[ |Z^n f(s)| < \sup f (s^2 \theta^{p - 1})^{n!/n!}, \tag{3.8} \]

and

(ii) in the region \( s > M^{\theta - (p - 1)/2} \),

\[ |Z^n f(s)| < \sup f \sum_{m=0}^n \beta^m/m! \delta^{n-m}/(n-m)!, \tag{3.9} \]

where \( \beta = p M^2 \) and \( \delta = \gamma(M) \ln(s\theta^{(p - 1)/2}) \). By using the Volterra series (3.6), an estimate for \( \Gamma \) may now be obtained, since we deduce that \( |\Gamma f(s)| \) is bounded by

\[ |\Gamma f(s)| \leq \sum_{n=0}^{\infty} |Z^n f(s)|. \]
Substituting (3.8) into this bound in the range \( s < M\theta^{-(p-1)/2} \) we deduce Lemma 3.2(i); whereas in the range \( s > M\theta^{-(p-1)/2} \)

\[
|\Gamma f(s)| \leq \sup f \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{\beta^m/m! \delta^n-m/(n-m)!},
\]

which on rearranging and summing establishes Lemma 3.2(ii).

Using these properties of \( I' \) we simplify (3.5) to obtain

\[
x = Y(x) = \Gamma vw(\theta, s) + IVT(x).
\]

As \( V \) is a Volterra integral operator with a bounded kernel, and \( \Gamma \) is a bounded linear operator, it follows that \( Y \) is a compact map from \( C(0, s) \) to itself. Using the bounds for \( \omega(\theta, s) \) given in (2.4) a direct calculation immediately shows

\[
|\omega(\theta, s)| \leq Rs^{2-\alpha}, \quad (3.10)
\]

where \( R \) is a constant independent of \( \theta \).

Making the assumption that \( x \) lies in the set

\[
|\omega(s)| < Ps^{-\alpha}, \quad (3.11)
\]

where \( P \) takes the value given in (2.4), a similar calculation using the definition of \( T(x) \) shows that

\[
|VT(x)| \leq Ss^{2-(p-2)\alpha} \sup x^2. \quad (3.12)
\]

Using the results given in Lemma 3.2 and the bounds given in (3.11), (3.12) a simple calculation shows that, if \( s \) lies in the range given in (3.1), then \( Y \) maps the ball \( B \) into itself, where \( B \) is the ball

\[
B = \left\{ x \in C(0, s): \sup_{0 < t < s} x(t) \leq 2 \exp(pM^2) \left( \frac{s^{\theta(p-1)/2}}{\theta} \right) R s^{2-\alpha} \right\}.
\]

Provided that \( s \) lies in this range, assumption (3.11) is satisfied for all such \( x \). From the Schauder principle we deduce that \( Y \) has a fixed point \( x \) lying inside \( B \). This concludes the proof of Lemma 3.1.

Using the bound for \( \omega(\theta, s) - u(s) \) given by Lemma 3.1 we may, in the region \( s^\alpha \ll \theta^{m-1} \), extend our knowledge of \( u(s) \) by studying \( \partial u/\partial \theta \) and \( \partial^2 u/\partial \theta^2 \) as perturbations of \( \partial \omega/\partial \theta \) and \( \partial^2 \omega/\partial \theta^2 \), respectively.
Lemma 3.3. In the region $sa < \theta^{-m-1}$,

(i) $|\partial u/\partial \theta|, \partial w(\theta, s)/\partial \theta| \leq s^{-1/2}\theta^{-(p-1)/4},$

(ii) $|\partial u(s)/\partial \theta - \partial w_s(\theta, s)/\partial \theta| \leq s^{-3/2}\theta^{-(p-1)/4},$

(iii) $|\partial^2 u/\partial \theta^2 - \partial^2 w(\theta, s)/\partial \theta^2| \leq s^{-1/2}\theta^{-(p+3)/4},$

(iv) $|\partial^2 u(\theta, s)/\partial \theta^2 - \partial^2 w_s(\theta, s)/\partial \theta^2| \leq s^{-3/2}\theta^{-(p+3)/4}.$

Proof: Differentiating (3.3, 4) with respect to $\theta$ we obtain

$$\partial u/\partial \theta = 1 + V(1 + pu^{p-1}) \partial u/\partial \theta$$

and

$$\partial w(\theta, s)/\partial \theta = 1 + V(pw(\theta, s)^{p-1}) \partial w(\theta, s)/\partial \theta.$$ 

We define $y(s)$ by

$$y(s) = \partial u/\partial \theta - \partial w(\theta, s)/\partial \theta,$$

so that $y$ satisfies

$$\hat{L}y \equiv (I - V(1 + pu^{p-1})) y = V(1 + p(u^{p-1} - w^{p-1})) \partial w/\partial \theta.$$ (3.13)

In the paper Budd [2] it is shown that if $u(s) > 0$ then $u(s) < w(\theta, s)$. Hence the estimates used in the proof of Lemma 3.2 to obtain $L^{-1}$ hold equally for $(\hat{L})^{-1}$, and hence the conclusions of that lemma apply to $\hat{L} \equiv (\hat{L})^{-1}$. Thus

$$y = \hat{L}V(1 + p(u^{p-1} - w^{p-1})) \partial w/\partial \theta.$$ (3.14)

We determine bounds for $\partial w(\theta, s)/\partial \theta$ by differentiating (2.2), which gives

$$\partial w(\theta, s)/\partial \theta = w(1, s\theta(\theta^{-1})^{1/2}) + \frac{1}{2}(p-1) \theta(\theta^{-1})^{1/2}w_s(1, s\theta(\theta^{-1})^{1/2}).$$

Substituting the bounds (2.4), we deduce that there is a constant $T$ independent of $\theta$ so that

$$|\partial w(\theta, s)/\partial \theta| < Ts^{-\alpha}.$$ (3.15)

Furthermore, from Lemma 2.4, we deduce that, for all $\varepsilon > 0$, there is an $N(\varepsilon)$ independent of $\theta$ such that, if $s > N(\varepsilon) \theta^{-(p-1)/2}$,

$$|\partial w(\theta, s)/\partial \theta| < (C+\varepsilon) s^{-1/2}\theta^{-(p-1)/4} \frac{1}{4}(p-5).$$ (3.16)

Finally, the bound for $u(s) - w(s)$ given in Lemma 3.1 implies

$$|p(u^{p-1} - w^{p-1})| < KS^{-\alpha}s^{-(p-2)\alpha}(s\theta(\theta^{-1})^{1/2})^\gamma,$$ (3.17)

where $K$ is a constant independent of $\theta$. 
When the bounds (3.15), (3.17) are substituted into (3.14) a straightforward calculation yields the bound for \( y \) given in (i). The bound in (ii) then follows by differentiation of (3.14) with respect to \( s \).

The bounds for (iii) and for (iv) follow by a similar calculation using the corresponding estimates for \( \partial^2 w(\theta, s) / \partial \theta^2 \).

4. THE UNSTABLE MANIFOLD \( M \)

To examine the behaviour of solutions to (1.3), (1.4) in regions away from the origin we substitute the Emden–Fowler coordinate system given after (2.2) into the differential equation (1.3). This gives the following dynamical system

\[
\begin{align*}
\frac{da}{dt} &= \alpha a + b, \\
\frac{db}{dt} &= (\alpha - 1) b - a^\alpha - s^2 a, \\
\frac{ds}{dt} &= s. 
\end{align*}
\]  

(4.1)

This system has an invariant plane \( s = 0 \) on which its behaviour is identical to the system in (2.3). The stationary point \((K, -\alpha K, 0)\) in this plane has a stable manifold which includes the solution to (2.1). To extend our knowledge of the complete system (4.1), we construct the unstable manifold of this stationary point in the region \( s > 0 \). In terms of a solution to the differential equation (1.3), this manifold corresponds to a \( C^2 \) function \( M(s) \) for \( s > 0 \) which satisfies the initial conditions given in (1.5). The properties of \( M \) are given by

**Lemma 4.1.** (i) There is a unique \( C^2 \) function \( M \) which satisfies (1.3), (1.5). \( M \) is defined for all \( s > 0 \) and, in particular, there is an \( s_*(p) \) such that for \( s \in [0, s_*] \)

\[
M(s) = Ks^{-\alpha}[1 - \frac{s^2}{(8 - 6\alpha)} + P(s)s^4] 
\]  

(4.2)

and

\[
M_*(s) = -Ks^{-(\alpha + 1)}[\alpha - \frac{s^2(2 - \alpha)}{(8 - 6\alpha)} + Q(s)s^4],
\]

Here the functions \( P(s) \) and \( Q(s) \) are bounded in \([0, s_*]\).

(ii) \( M \) has a zero \( \mu_c \) with \( 0 < \mu_c < \pi \).

**Corollary 4.2.** (i) \( M \) is singular at the origin.
(ii) If $p > 5$ and $B_\mu$ is the ball in $\mathbb{R}^3$ of radius $\mu_c$ then
\[ M \in H^1_0(B_\mu) \cap L^{p+1}(B_\mu). \]

(iii) The function $\tilde{M}(r) \equiv \mu_c^p M(\mu_c r)$ is a weak solution of (1.1) in $H^1_0(B)$ where $B$ is the unit ball in $\mathbb{R}^3$.

Proof of Lemma 4.1. Once existence for $M$ has been established in a neighborhood of the origin, standard theory will ensure global existence as we deduce from the Maximum Principle that $M$ is bounded above in any region not containing the origin. To study $M$ near to the origin we make the substitution
\[ M(s) = K s^{-2}(1 - (N(s) + 1)s^2\beta) \]
with $\beta = (8 - 6\alpha)^{-1}$, and find the structure of $N$. For $s > 0$, $N$ satisfies the following differential equation
\[ LN \equiv \beta s^2 N_{ss} + \beta(6 - 2\alpha) s N_s = -N + s^2(a + bN + cN^2) + s^4G(s, N), \tag{4.3} \]
where $a = (1 - \alpha) p\beta^2 - \beta$, $b = 2(1 - \alpha) p\beta^2 - \beta$ and $c = (1 - \alpha) p\beta^2$. The function $G(s, N)$ is continuous in both variables for $s > 0$. To obtain a regular solution for $N$ we must also impose the initial conditions
\[ N(0) = N_s(0) = 0. \tag{4.4} \]

By using the linearly independent solutions $\psi = 1$ and $\psi = s^{-(5 - 2\alpha)}$ of the differential equation $L\psi = 0$, we can recast (4.3), (4.4) as the following Volterra integral equation
\[ N = V(-N + s^2(a + bN + cN^2) + s^4G(s, N)). \tag{4.5} \]

Here $V$ is the linear Volterra integral operator
\[ (Vf)(s) = \frac{(8 - 6\alpha)}{(5 - 2\alpha)} \int_0^s t^{-1}(1 - (t/s)^{5 - 2\alpha}) f(t) \, dt. \]

Let $A$ be the Banach space
\[ A = \{ N : N = s^2f, f \text{ continuous } \|N\|_A \equiv \sup_{0 < s < s^*} |f| \} \]
and $B_d$ the ball $\{ N \in A : \|N\|_A \leq d \}$. We define the operator $Y$ acting on $N \in A$ by
\[ YN = V(-N + s^2(a + bN + cN^2) + s^4G(S, N)). \tag{4.6} \]
By choosing a suitable \( d \) and a sufficiently small \( s(p) \) so that \( s < s(p) \), a simple calculation shows that \( Y \) maps \( B_d \) into itself. A similar calculation employing the Lipschitz continuity of \( G \) with respect to \( N \) shows further that \( Y \) is a contraction upon \( B \). Hence, applying the Contraction Mapping Theorem, we deduce that \( Y \) has a fixed point lying inside \( B_d \). The result (i) now follows immediately. To prove (ii) we observe that \( sM(s) \) satisfies

\[
\int_0^\pi sM^p \sin s \, ds = 0.
\]

Letting \( \varepsilon \to 0 \) and using the properties of \( sM \) and \( (sM)_s \) for \( s \to 0 \), we deduce that \( M \) has a zero \( \mu_c < \alpha \).

The corollary follows from the description of \( M \) given above.

As an example of the application of Lemma 4.1 we calculated \( M(s) \) for the exponent value \( p = 7 \). In this case \( \alpha = \frac{1}{4} \) and \( K = 0.77827... \). Using a symbolic manipulation program we may calculate the coefficients of \( M \) to obtain

\[
M(s) = Ks^{-2} \left( 1 - \frac{s^2}{6} + \frac{s^4}{504} + \frac{s^6}{1059 \cdot 897} - \frac{s^8}{10027 \cdot 1447} + \cdots \right).
\]

By numerically integrating (1.3), (1.5), for \( p = 7 \) we find \( \mu_c = 2.831... \).

5. The Behaviour of \( u \) Away from the Origin

In Section 3 \( u \) was studied as a small perturbation of \( \omega(\theta, s) \) in the region \( s^6 \ll \theta^{m-1} \). In this section we study \( u \), in a region bounded away from the origin, as a small perturbation of the function \( M(s) \) constructed in Section 4. We thus define \( y(s) = u(s) - M(s) \), and using (1.3) we see that

\[
\mathcal{L}y = y_{ss} + \frac{2}{s} y_s + p \frac{\alpha}{s^2} (1 - \alpha) y + \bar{N}(s) y = y^2 \bar{N}(s), \tag{5.1}
\]

where

\[
\bar{N}(s) = pM^{p-1} - p\alpha(1 - \alpha) s^{-2}.
\]

(From Lemma 4.1, \( \bar{N}(s) < N_0 \) for \( 0 \leq s \leq \mu_c \).) Further

\[
- \hat{\mathcal{N}} = [(M + y)^p - M^p - pM^{p-1}y] y^2;
\]

thus, for \( |y| < 1 \), \( |\hat{\mathcal{N}}| \) is bounded above by \( p(p - 1)(M + 1)^{p-2} \).

First we solve (5.1) for \( y \) given that \( y(\mu_c) \) and \( y_s(\mu_c) \) are both small. We do this in terms of two linearly independent solutions \( \phi_1, \phi_2 \) of the differential
equation $\hat{L}\phi = 0$. However, we wish to find out the behaviour of $y$ when $s$ is small. Thus we characterize the $\phi_i$ by their singular behaviour near $s = 0$. In fact, we wish to relate $\phi_i$ to the solutions of the operator $L$ defined in (2.5).

**Lemma 5.1.** In the limit $s \to 0$, for $i = 1$ and 2,

$$\phi_i(s) = s^{-1/2} \begin{cases} \cos \\ \sin \end{cases} (\omega \ln s)(1 + O(s^2)),
$$

$$[\phi_i(s)]_s = s^{-1/2} \begin{cases} \cos \\ \sin \end{cases} (\omega \ln s) + O(s^{1/2}).$$

**Proof.** From the definitions

$$\hat{L}\phi = L\phi + \tilde{N}(s) \phi. \quad (5.3)$$

Using the linearly independent solutions for $L\phi = 0$ given after (2.5) we may, by the method of variation of constants and making the substitution

$$\psi(s) = s^{1/2} \phi(s),$$

we recast (5.3) as

$$\psi(s) = A \sin(\omega \ln s + B) - \int_0^s \omega^{-1} t \sin(\omega \ln(t/s)) \psi(t) \tilde{N}(t) \, dt.$$

On the assumption that $\int_0^s |\psi(s)| \, ds$ exists, this may be rewritten as

$$\tilde{L}\psi \equiv \psi(s) - \int_0^s \omega^{-1} t \sin(\omega \ln(t/s)) \psi(t) \tilde{N}(t) \, dt = C \sin(\omega \ln s + D). \quad (5.4)$$

Because $\tilde{N}(t)$ is bounded above, $\tilde{L}\psi$ may be inverted by using a Volterra series to deduce in the limit $s \to 0$ that

$$\psi(s) - C \sin(\omega \ln s + D)(1 + O(s^2)).$$

This form for $\psi(s)$ justifies the existence of the integral as assumed above. By choosing suitable constants $C$ and $D$ we obtain $\phi_1$ and $\phi_2$. Estimates for the derivatives of $\phi_i$ follow on differentiating (5.4) and substituting the above bounds.

Our main result is thus

**Lemma 5.2.** Suppose that $a$ and $b$ are given with $a^2 + b^2 = 1$. For $s$ lying in the range $s^a > e^{a/(p-1)}$, Eq. (5.1) has a solution $y(s)$ which satisfies
(i) \( y(s) = e[a\psi_1(s) + b\psi_2(s)] + A(\varepsilon, s) e^{2s^{-(p-3)/(p-1)}} \) and
(ii) \( y_i(s) = e[a\psi_1(s) + b\psi_2(s)] + B(\varepsilon, s) e^{2s^{-(p-2)/(p-1)}} \). \( (5.5) \)

where \( A(\varepsilon, s) \) and \( B(\varepsilon, s) \) are bounded with bounds depending upon \( p \) alone.

Proof. To show that \( y \) satisfies (5.5) we solve (5.1) by recasting it as an integral equation using the two functions \( \phi_i \) given in (5.2). Setting \( x(s) = s^{1/2} y(s) \) and \( \psi_i(s) = s^{1/2} \phi_i(s) \) we have

\[
x(s) = Fx = c[a\psi_1(s) + b\psi_2(s)] + \int_s^{\mu_c} \frac{1/2}{\omega} [\sin \omega \ln(t/s)](1 + O(t^2)) x^2 \tilde{N}(t) dt.
\]

From Lemma 5.1 we deduce that \( \psi_1 \) and \( \psi_2 \) are bounded by \( N \) in \( 0 \leq s \leq \mu_c \), and claim that \( F \) is a contraction map on the set \( B \) where

\[
B = \{ x \in C[s^*, \mu_c] : \sup_{s^* < s < \mu_c} x(s) < 3N \varepsilon \}
\]

and \( s^* > K \varepsilon^{4/(p-5)} \) for a constant \( K \).

To prove this we observe that

\[
\int_s^{\mu_c} t^{1/2} (1 + O(t^2)) \ dt \leq DS(5 - p)^{a/4}
\]

for some constant \( D \). The function \( \tilde{N}(t) \) is bounded independently of \( x \) by \( (1 + M(t))^{p-2} \). From Lemma 4.1, \( M(t) \) is bounded by a multiple of \( t^{-a} \). From Lemma 5.1 we know that the functions \( \psi_i(s) \) are bounded by a constant \( N \) for all \( s \in [0, \mu_c] \). Hence, using (5.7), we find that, if for \( s \geq s^* \)

\[
|x(s)| < 3N \varepsilon \text{ and } |y(s)| < 3N \varepsilon,
\]

then

(i) \( |Fx(s)| < 2N \varepsilon + A \varepsilon^2 s^{(5-p)a/4} \) and
(ii) \( |Fx - Fy| < \sup |x - y| B \varepsilon s^{(5-p)a/4}, \)

where \( A \) and \( B \) are constants independent of \( \varepsilon \). Choosing \( s^* > C \varepsilon^{4/(p-5)} \) for a suitable \( C \) will thus ensure that \( F \) maps \( B \) to itself and is a contraction on \( B \). Hence, if we define

\[
x_0 = c[a\psi_1 + b\psi_2] \quad \text{and} \quad x_{n+1} = F(x_n) \quad \text{for } n \geq 0,
\]

the Contraction Mapping Theorem ensures that this sequence converges to a function \( x \in B \) which uniquely solves the integral equations. We see that

\[
|x - x_0| \leq |x_0 - x_1|(1 - B \varepsilon^{(5-p)a/4} - 1)^{-1},
\]
so that
\[ |x - x_0| \lesssim \frac{2\pi}{B} s^{5/(5 - \rho)} (1 - B\pi/(5 - \rho))^{-1}. \]

From this bound we deduce (i), and (ii) then follows by differentiating equation (5.6).

**Corollary 5.3.** If \( s > s_\star \) then, for \( \tan \phi = b/a \),
\[
u(s) = M(s) + \varepsilon s^{-1/2} \sin (\omega \ln s + \phi)(1 + O(s^2))
+ A(c, s) e^{2s - (\rho-3)/2}. \tag{5.8}\]

**Corollary 5.4.** There is a function \( f(s) \) which is differentiable and bounded independently of \( \varepsilon \) near to \( s = \mu \) which satisfies
\[
u(s) = \varepsilon [a\phi_1 + b\phi_2] + \varepsilon f + M(s)\]
and
\[ f(\mu) = f_\ast(\mu) = 0. \]

**Proof.** This is a direct consequence of the method of proof of Lemma 5.2.

6. **The Solution of (1.3, 4) in \( s > 0 \)**

In Section 3, \( u \) was studied as a perturbation of \( w(\theta, s) \) in an inner region \( s^2 \ll \theta^m \) and in Section 2 a description of \( w(0, s) \) was given for the range \( s^2 \gg \theta^{-1} \). Furthermore in Section 5 we obtained a description of \( u \) as a perturbation of \( M \) in an outer region \( s^2 \gg e^{4(\rho-5)} \). In this section we choose \( \theta \) sufficiently large and \( \varepsilon \) sufficiently small to ensure that the outer and inner regions intersect. Precise values of \( \theta \) and \( \varepsilon \) are then determined so that we have a solution \( u(s) \) to (1.3, 4) in the region \( s > 0 \). The main result is as follows.

**Lemma 6.1.** There is a \( C^2 \) solution of (1.3, 4) which also satisfies
\[
u(\mu) = \varepsilon [a\phi_1(\mu) + b\phi_2(\mu)] \]
and
\[
u(\mu) = \varepsilon [a\phi_1(s) + b\phi_2(s)]|_{\mu}, \tag{6.1}\]
where \( \phi_1, \phi_2 \) are the functions described in Lemma 5.1.
Define \( \theta_* \) and \( \varepsilon_* \) by

\[
\frac{1}{2} \omega(p - 1) \ln \theta_* + D = \phi = \tan^{-1}(b/a)
\]

and

\[
\varepsilon_* = C \theta_*^{-(p - 5)/4},
\]

where \( C \) and \( D \) take the values given in Lemmas 2.2(4). Then, for sufficiently large \( \theta_* \)

\[
\theta = \theta_*(1 + 0(\theta_*^{-(p - 5)/4}))
\]

and

\[
\varepsilon = \varepsilon_*(1 + 0(\theta_*^{-(p - 5)/4})).
\]

**Proof.** A solution \( U(s) \) is constructed as follows. An inner solution \( u_1(s) \) is obtained which satisfies (1.3), (1.4) and for which the behaviour in the inner region may be determined. Similarly an outer solution \( u_0(s) \) is obtained which satisfies (1.3) and (6.1). The variables \( \theta \) and \( \varepsilon \) are then chosen to ensure that, at a fixed \( s = S \) chosen to satisfy

\[
\varepsilon^{4/(p - 5)} \ll S^\alpha \ll \theta^{m - 1}
\]

(and thus lying in the intersection of the inner and outer regions), we may satisfy the two conditions

\[
u_1(S) = u_0(S)
\]

and

\[
[u_1(s) - u_0(s)]|_{s = S} = 0.
\]

From this choice of \( \theta \) and \( \varepsilon \) we deduce the existence of a \( C^2 \) function \( U(s) \) defined by \( U(s) = u_1(s) \) for \( s \leq S \) and by \( U(s) = u_0(s) \) for \( s \geq S \). Thus \( U(s) \) satisfies (1.3, 4) and (6.1). We claim that the values of \( \theta \) and \( \varepsilon \) required to satisfy these conditions may be obtained as small perturbations of the values of \( \theta_* \) and \( \varepsilon_* \) given in (6.2). To show this we define the function \( F(\theta, \varepsilon) \) by

\[
F(\theta, \varepsilon) = (S^{1/2}(u_1(S) - u_0(S)), (s^{3/2}(u_1(s) - u_0(s)))|_{s = S}).
\]

Taking \( \theta = \theta_* \) and \( \varepsilon = \varepsilon_* \) we find a bound for \( F(\theta_*, \varepsilon_*) \) by making use of
the behaviour of $u_i(s)$ determined by Lemmas 2.2(4) and 3.1, and the
behaviour of $u_0(s)$ given by Lemmas 4.1 and 5.3. Accordingly we find
\[ |S^{-1/2}F(\theta, \varepsilon)| \leq |A| (S^{2-\varepsilon}(\theta^2(\rho-1/2)^2) + |B| (\varepsilon^2 S^{-(\rho-3)/4} + |D| (\varepsilon S^{3/2}). \] (6.4)

Where the vectors $A$, $B$, $C$, and $D$ are bounded independently of $S$, $\theta$, and $\varepsilon$.
For large $\theta$ and small $\varepsilon$, the dominant contribution to $F(\theta, \varepsilon)$ is
\[ |S^{-1/2}F(\theta, \varepsilon)| \leq |B| \varepsilon^2 S^{-(\rho-3)/4} + \text{smaller terms}. \] (6.5)

We now seek values of $\theta$ and $\varepsilon$ which are small perturbations of $\theta$ and $\varepsilon$
and for which $F(\theta, \varepsilon) = 0$. By using Lemmas 2.4, 3.3 and Corollary 5.3, we
may evaluate the Jacobian of $F$ at $(\theta, \varepsilon)$, as follows:
\[ \frac{\partial F(\theta, \varepsilon)}{\partial (\theta, \varepsilon)} = \begin{bmatrix}
C(1/4(5-p) \sin \delta + 1/2 \omega (p-1) \cos \delta) \theta^{(p-1)/4}, \\
C(1/4(5-p) \cos \delta - 1/2 \omega (p-1) \sin \delta) \theta^{(p-1)/4},
\end{bmatrix} \] + smaller order terms, (6.6)

where $\delta = \omega \ln S + 1/2 \omega (p-1) \ln \theta + D = \omega \ln S + \phi$. To simplify this
expression we define the function $G(x, y)$ by
\[ G(x, y) = F(\theta + (p-1)/4x, \varepsilon + y). \]

Using the bounds for $F$ given in (6.4), (6.5) and the second Mean Value
theorem together with (6.4) and the results of Lemmas 2.4 and 3.4, we
express $G(x, y)$ in the form

\[ G(x, y) = C + \frac{1}{y^2} S^{-(p-5)/4} + \text{smaller order terms}, \] (6.7)

where $C$ is a constant independent of $(x, y)$ which is bounded above by
$|B| \varepsilon^2 S^{-(p-5)/4}$. Also $|E|$ is bounded independently of $x$, $y$, $\theta$, and $\varepsilon$. Thus
\[ G(x, y) = C + L \left( \begin{array}{c}
x \\
y
\end{array} \right) + T(x, y), \]

where $L$ is a linear operator which, from a direct calculation, is seen to be
invertible. If we define the operator $J$ mapping $\mathbb{R}^2$ into itself by
\[ J(x, y) = - (L^{-1} C + L^{-1} T(x, y)), \]
then, provided that \( \theta_* \) is suitably large, a direct calculation shows that \( J \) maps the set \( B \) into itself, where \( B \) is the ball

\[
B = \{(x, y): (x^2 + y^2)^{1/2} \leq 4\varepsilon_*^2 S^{-\alpha(p-5)/4}|B|/\omega(p-1)C\}.
\]

We may therefore apply the Brouwer Fixed Point Theorem to conclude that \( J \) has a fixed point in \( B \). This point \( (x, y) \) satisfies both \( G(x, y) = 0 \) and

\[
(x^2 + y^2)^{1/2} \leq A\varepsilon_*^2 S^{-\alpha(p-5)/4},
\]

where \( A \) is a constant independent of \( \varepsilon_* \), \( \theta_* \), and \( S \). By substituting for \( \theta \) and \( \varepsilon \), and then taking \( S \) to have the upper limiting value of \( \theta_*^{-\alpha} \), we may deduce the values given in (6.3).

By using this lemma we obtain the Proof of Theorem 1.1. From the behaviour of \( u \) near \( \mu_c \) given by Corollary 5.4 we see that \( u \) has a zero at \( \mu = \mu_c - x \), where, in the limit \( \varepsilon \to 0 \),

\[
x = \varepsilon[a\varphi_1(\mu_c) + b\varphi_2(\mu_c)](1 + O(\varepsilon))/M_s(\mu_c).
\]

As \( a = \cos \phi \) and \( b = \sin \phi \) we see that for, suitable constants \( \phi_* \) and \( A \),

\[
x = \varepsilon A \sin(\phi + \phi_*)(1 + O(\varepsilon)).
\]

We now define the variables \( \tau \), \( E \) and \( \theta_0 \) by

\[
\tau = \phi + \phi_*,
\]

\[
E = CA,
\]

\[
\theta_0 = \exp(-\alpha/\omega(D + \phi_*)),
\]

where \( C \) and \( D \) take the values given in Lemma 6.1. Thus we have the expressions for \( \mu \) and \( \theta \) given by Theorem 1.2, where we assume here that \( \tau \) is taken sufficiently large for the estimates to hold.

To complete the proof of the theorem we must establish that \( \mu \) is the first zero of \( u \). This result follows from the previous estimates. From the Maximum Principle we may deduce that \( M_s \) does not vanish in \((0, \mu_c)\) and, as \( M_s \) is unbounded as \( s \to 0 \), we have that \( M_s \) is bounded away from zero on \([0, \mu_c]\). However, from the results of Section 4 we know that \( u_s - M_s \) is \( O(\varepsilon) \) if \( s^2 \gg \varepsilon^{4/(p-5)} \). Thus if \( \varepsilon \) is sufficiently small we deduce that \( u_s \) does not vanish if \( s^2 \gg \varepsilon^{4/(p-5)} \) and hence \( u \) cannot vanish if \( \varepsilon^{4/(p-5)} \ll s^2 < \mu^2 \). Similarly \( w(\theta, s) \) is bounded away from zero. From the results of Section 2 we concluded that \( |\mu - W| \ll W \) if \( s^2 \ll \theta^{-m-1} \). Thus, as \( \varepsilon^{4/(p-5)} \ll \theta^{-m-1} \), we see that \( u \) is positive for \( 0 \leq s < \mu \).
Proof of Theorem 1.2. By taking the sequence of numbers \( \tau = n\pi \) for \( n = 1, 2, \ldots \), we have the existence of a sequence of numbers \( \theta_n \) tending to infinity and of functions \( u_n \), such that \( u_n \) satisfies (1.3, 4), \( u_n(0) = \theta_n \), \( u_n(\mu_c) = 0 \). To prove the theorem we must now establish that

\[
\int_0^{\infty} \left( \frac{d}{ds} (u_n(s) - M(s)) \right)^2 s^2 ds \to 0 \quad \text{as} \quad n \to \infty. \quad (6.9)
\]

We consider the two ranges \( s < S \) and \( s > S \) separately, where, for each \( \theta_n \), \( S \) takes the value implies by Lemma 6.1. For \( s > S \) the results of Lemmas 2.4-2.6 imply that there is a constant \( P \) independent of \( \theta \) for which

\[
|u_n(s)| < Ps^{-(\alpha + 1)} \quad \text{for} \quad s < S.
\]

Similarly, Lemma 4.1 shows that there is a constant \( R \) such that \( M_s \) is bounded above by \( Rs^{-(\alpha + 1)} \). Hence

\[
\int_0^S \left( \frac{d}{ds} (u_n(s) - M(s)) \right)^2 s^2 ds \leq (P + R)^2 S^{1-2\alpha}/(1-2\alpha).
\]

In the region \( s > S \) we may deduce from the results of Section 4 that

\[
\left| \frac{d}{ds} (u_n(s) - M(s)) \right| < s_n s^{-3/2} + O(c_n^{2} s^{-(1 + (\rho - 3)/(\rho - 1))}),
\]

\[\text{FIGURE 3}\]
where $\varepsilon_n$ is the value for $\varepsilon$ corresponding to the value $\theta_n$ for $\theta$. Hence

$$
\int_0^S \left( \frac{d}{ds} (u_n(s) - M(s)) \right) \frac{s^2}{ds} ds \leq \varepsilon^2 \log S \leq \theta_n^{-(p-5)/2} \log \theta_n,
$$

as $n \to \infty$, $\theta_n \to \infty$, and $S \to 0$. Thus both contributions to the integral in (6.9) tend to zero as $n \to \infty$.

This establishes Theorem 1.2, and the Corollary 1.3 given in the Introduction follows from similar estimates to those given above.

The solution $u(s)$ obtained in the proof of Theorem 1.1 can be well represented by using the Emden–Fowler coordinate system given after (2.2). A graph of its behaviour is given in Fig. 3.

7. NUMERICAL AND FORMAL RESULTS

By use of an ordinary differential equation solver we integrate Eq. (1.3) for various values of $\theta$ to determine $\mu(\theta)$. In addition, we solve (1.3) with the initial conditions

$$
M(t, s) = Kt^{-2} \quad \text{and} \quad M_s(t, s) = -\alpha Kt^{-(1+\alpha)} \quad \text{for some } t > 0.
$$

By determining $r(t)$ such that $M(t, r(t)) = 0$, and then letting $t \to 0$, we obtain a value for $\mu_c$ as the limiting value of $r(t)$ As we change $p$, $\mu_c$ varies in the manner given in Table I.

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>$\mu_c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>20.0</td>
<td>3.084...</td>
</tr>
<tr>
<td>15.0</td>
<td>3.051...</td>
</tr>
<tr>
<td>10.0</td>
<td>2.969...</td>
</tr>
<tr>
<td>9.5</td>
<td>2.954...</td>
</tr>
<tr>
<td>9.0</td>
<td>2.937...</td>
</tr>
<tr>
<td>8.5</td>
<td>2.917...</td>
</tr>
<tr>
<td>8.0</td>
<td>2.893...</td>
</tr>
<tr>
<td>7.5</td>
<td>2.865...</td>
</tr>
<tr>
<td>7.0</td>
<td>2.831...</td>
</tr>
<tr>
<td>6.5</td>
<td>2.790...</td>
</tr>
<tr>
<td>6.0</td>
<td>2.738...</td>
</tr>
<tr>
<td>5.5</td>
<td>2.671...</td>
</tr>
<tr>
<td>5.0</td>
<td>2.584...</td>
</tr>
<tr>
<td>4.5</td>
<td>2.516...</td>
</tr>
</tbody>
</table>
Then we compare the predicted asymptotic values given in Theorem 1.1 with the numerical results. To make these comparisons we determine values for the constants $\theta_0$ and $E$ by looking at only two points. However, having made these choices, the remaining values for $u$ computed numerically are in very close agreement with the asymptotic results.

As $p$ tends to 5 from above the asymptotic limit of $\mu(\theta)$ for large $\theta$ tends to a value of 2.584. However, when $p = 5$ the asymptotic limit is $\pi/2$. This apparent non-uniformity in the limit $p \to 5$ may be resolved by studying the bifurcation curve for $u$ in a neighborhood of the first quadratic turning point $(T_p) (\theta_{T_p}, \mu_{T_p})$. A formal asymptotic calculation, confirmed by numerical results, shows that, if $0 < p - 5 \leq 1$,

\begin{align}
(\text{i}) \quad \theta_{T_p}^2 &\sim 96(p - 5)^{-1} \text{ and } \\
(\text{ii}) \quad \mu_{T_p} &\sim (\pi/2) + (p - 5)^{1/2} \pi 8^{-1/2}. \quad (7.1)
\end{align}

A further formal calculation shows that, in a neighborhood of this turning point, the curve $\mu(\theta)$ is given by

\begin{equation}
\mu(\theta) \sim \pi 2 + 3^{1/2} \pi \left[ \theta^{-2} + \frac{(p - 5)}{96} \theta^2 \right], \quad (7.2)
\end{equation}

where $(p - 5)$ is small and $1 \leq \theta \leq (p - 5)^{1/2}$. These formulae demonstrate...
that the bifurcation curve for the case $p = 5$ is the limit of the curve in a neighbourhood of $(\theta_{T_p}, u_{T_p})$ in the case $p > 5$. This behaviour is indicated in Fig. 4.

8. Generalisations

a. Changing the Dimension of the Space

The differential equation in (1.2) may be reformulated for an arbitrary (not necessarily integral) space dimension $n$ to give the problem

$$\ddot{u}_r + \frac{(n-1)}{r} \dot{u}_r + \lambda (\ddot{u} + \ddot{u}) = 0,$$

$$\ddot{u}_r(0) = \ddot{u}(1) = 0.$$  \hspace{1cm} (8.1)

When $p = p_c$ it is shown in Brezis and Nirenberg [1] that the dependence of the solution of (8.1) upon $\lambda$ is qualitatively different for the two cases $2 < n < 4$ and $n \geq 4$. In particular for $n \geq 4$ Brezis and Nirenberg prove the existence of a positive solution $\bar{u}(\lambda)$ for all $\lambda$ less than the square of the first zero of $J_{(n-2)/2}$, whereas for the case $2 < n < 4$ there is a lower bound $\lambda_*$ below which nontrivial solutions do not exist. This situation is quite different for the case $p > p_c$ however, and methods identical to those given in Sections 2–6 show the following.

**Lemma 8.1.** For $2 < n < 10$ and for $p > p_c$ the bifurcation diagram for (8.1) is qualitatively similar to that described in Theorem 1.1 with $\alpha, K$, and $\omega$ given in the theorem taking the values:

$$\alpha = 2/(p - 1), \quad K^{n-1} = (n - 2 - \alpha), \quad \omega^2 = px(n - 2 - \alpha) - 1/4(n - 2)^2.$$

b. Changing the Form of the Lower Order Term

As a first step in inquiring into the behaviour of the general problem

$$\ddot{u}_r + \frac{2}{r} \dot{u}_r + f(\ddot{u}) = 0, \quad \ddot{u}_r(0) = \ddot{u}(1) = 0,$$

where

$$f(u) \to u^p \quad \text{as} \quad u \to \infty$$

and

$$f(u) \to u^q \quad \text{as} \quad u \to 0,$$
we may study solutions of
\[ u_{rr} + \frac{2}{r} u_r + u^q + u^p = 0 \quad \text{for} \quad 1 < q < 5 < p, \]
(8.2)
\[ u_r(0) = u(\mu) = 0, \quad u(0) = \theta. \]

Unlike solutions of problem (1.2), those of (8.2) do not bifurcate from the trivial solution. For solutions with small supremum norm we can compare solutions of (8.2) with those of the problem

\[ u_{rr} + \frac{2}{r} u_r + u^q = 0, \quad u_r(0) = u(\mu) = 0, \quad u(0) = \theta. \]
(8.3)

This equation satisfies the group relation given in (2.2), and from that relation we deduce that (8.3) has a solution provided that

\[ \mu - \mu_* \theta^{-(q-1)/2} > 0, \]
(8.4)

where \( \mu_* \) is a function of \( q \) alone. Thus (8.4) gives an approximate description of the bifurcation diagram for solutions of the problem (8.2) in the case of small \( \theta \).

The large-norm solutions of (8.2) may be investigated by using techniques similar to those discussed in this paper, and we can establish the following proposition.

**Lemma 8.2.** (i) For \( 0 < q \leq 3 \) the large-norm structure of the bifurcation diagram for (8.2) is qualitatively the same as that given in Theorem 1.1. Moreover the constants \( \alpha, K, \) and \( \omega \) are the same as those given in Theorem 1.1.

(ii) A function \( M \) exists which is equivalent to the function given in Section 4. \( M \) satisfies the differential equation in (8.2) for \( r > 0 \) and it has the singular initial conditions:

\[ r^2 M(r) - K \to 0 \quad \text{and} \quad r^{q+\alpha + 1} M_r(r) + \alpha K \to 0 \quad \text{as} \quad r \to 0. \]

Near to the origin \( M(r) \) has the form

\[ M(r) = \sum_{n=0}^{\infty} a_n r^{n \gamma - \alpha}, \]

where \( \gamma = 2(p - q)/(p - 1) \). The condition \( q \leq 3 \) is sufficient to ensure that \( M(r) \) has a zero, although numerical calculations have indicated that \( M \) has a zero for all \( q < 5 \).
c. Exponential Growth

When studying systems related to combustion problems we wish to allow for exponential growth rates. For a two-dimensional space, Dirichlet problems with an exponential nonlinearity behave in a similar manner to problems in higher dimensions with a polynomial nonlinearity growing at the critical rate \( p = p_c \). When the space dimension is greater than two, the solutions have a structure similar to that given in Theorem 1.1.

As an example we shall look at large-norm solutions of the equation

\[
\frac{2}{s} u_s + \frac{1}{s} u + (u + e^u - 1) = 0,
\]

\[
(u(0) = u(\mu) = 0).
\]

**Lemma 8.3.** Let \( t^* = (1 + \sqrt{7\tau})/2\pi \). Then (8.5) has a solution \( u(s) \) satisfying

\[
u(0) + \theta_* \rightarrow \theta_* \quad \text{as} \quad \tau \rightarrow \infty,
\]

and

\[
\mu \rightarrow \mu_c + B \sin \tau/\exp(1/4\theta_*) \quad \text{as} \quad \tau \rightarrow \infty
\]

where \( \mu_c, B \) and \( C \) are constants with \( \mu_c \approx 1.38 \ldots \).

To establish this result we proceed, as before, to examine the behaviour near the origin of solutions to the differential equations in (8.5). Near \( s = 0 \) (8.5) is well approximated for large values of \( u(0) \) by the problem

\[
\frac{2}{s} w_s + \frac{1}{s} w + e^w = 0,
\]

\[
w_s(0) = 0.
\]

The solutions to this problem have a group relation which is similar to that given in (2.2) for the Emden–Fowler equation. This is given by

\[
w(s) = 2\theta + w(se^\theta).
\]

From this relation we derive the coordinate system

\[
a(t) = w(s) + 2 \ln(s), \quad b(t) = sw(s), \quad s = e^t.
\]
When substituted into the differential equation in (8.5) this leads to the dynamical system

$$\begin{align*}
d a/dt &= b + 2, \\
-d b/dt &= -b - e^a + s^2(1 + 2 \ln s - a), \\
ds/dt &= s.
\end{align*}$$

(8.8)

This system has an invariant plane at $s=0$ as in (4.1). On this plane its behaviour is similar to that of solutions of (2.3). In particular there is a stationary point at $(a, b, s) = (\ln 2, -2, 0)$ which is a spiral attractor. This point also has an unstable manifold $M$ which, on transforming back to the original equation, corresponds to a function $M$ which satisfies the differential equation in (8.5). This function has the singular initial conditions

$$M(s) \to \ln(2/s^2) \quad \text{and} \quad M'(s) \to -2/s \quad \text{as} \quad s \to 0.$$ 

Near the origin $M(s)$ has the form

$$M(s) = \ln(2/s^2)(1 - s^2/4 + \cdots) + s^2(1 + \ln 2)/6 + \cdots$$

and $M(s)$ has a zero at $\mu_c < \pi$.

Combining the structure of $u(s)$ near the origin with that of $M(s)$ given above, we may use the techniques in Sections 2–6 to verify Lemma 8.3.


\textbf{REFERENCES}