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# OPTIMAL SELECTION WITH RANDOMLY FADING MEMORY

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**Abstract**—In this paper, we will consider optimal selection problems in which the recall of a random number of observations and uncertainty of selection are both allowed at each stage of the selection process. General rules as well as closed form solutions for specific examples with Markov memory system and finite memory are obtained.

*Key Words and Phases*: Optimal selection, memory systems, relative ranks, dynamic programming, backward induction, optimal stopping.

## 1. INTRODUCTION

Recently, the secretary problem and its various generalizations and variations have been the subjects of extensive investigations. In particular, Petruccelli [2] considered the best-choice problems in which there are $N$ potential applicants who will present themselves one by one but in a random order to the employer for a job. At stage $n$, $n = 1, 2, \ldots, N - 1$ of the interview process, the employer can either make an offer to the relatively best applicant among applicants $1, 2, \ldots, n$ or make no offer and proceed to interview applicant $n + 1$. If an offer is made to the applicant $n - r + 1$, $r = 1, 2, \ldots, n$, then the probability that this applicant will accept the offer is $q(r + 1)$. The objective in that problem is to develop an optimal procedure so that the probability of choosing the best applicant is maximized. Petruccelli's result is an extension of that of Yang [5] and Smith [3]. In the latter, Smith considered a secretary problem with uncertainty of employment and without any recall of past applicants. Whereas in the former, Yang treated the case that recall of any past applicants is allowed and yet the most current applicant will accept the offer with certainty if one is made to him.

The optimal selection problem treated in this paper allows both uncertainty of selection and randomness in the memory system. The memory system is referred to as the numbers of past observations that can be recalled for selection. The memory systems of the problems considered by Petruccelli and Yang are perfect ones. Whereas that considered by Smith and Deely [4] and Smith [3] are constant ones. Therefore our results can be considered as an extension of those mentioned in the above.

Basic formulas and the mathematical formulation of the problem are contained in Sec. 2. In Sec. 3, motivation for using the memory system is given and some general rules of the selection problem are obtained. These general rules are not in closed form due to the absence of specifications of both the memory system and probability of selection. In Sec. 4, we consider an example of a nonhomogeneous Markov memory system and sure availability for selection. In Sec. 5, an example of finite memory system and constant probability of selection is treated. This result is an extension of that considered in [3] and [4].

## 2. MATHEMATICAL FORMULATION

Suppose there is a job opening for which there are $N$ potential applicants who will present themselves to the employer one by one but in a random order. Let $X_1, X_2, \ldots, X_n$ be $N$ random variables defined on a probability space $(\Omega, F, P)$. $X_i$ represents the absolute rank of the $i$th applicant among those $N$ potential applicants where $X_i = 1$ meaning the best qualified and $X_i = N$ meaning the least qualified for the job. We assume that there is no tie in qualification can occur. Therefore $(X_1, X_2, \ldots, X_N)$ will be a random permutation of integers $1, 2, \ldots$ and $N$. For $n = 1, 2, \ldots, N$, let $Y_n = Y_n(X_1, \ldots, X_n)$ be a random vector of $n$ components in which the $j$th component represents the relative rank of $X_j$ among $X_1, X_2, \ldots, X_n$. Let $Z_n = n - j + 1$ if and only if the $j$th component of $Y_n$ is 1. In this case the $j$th applicant is said to be the candidate at the stage $n$ and $Z_n$ is the relative position, counting from the $n$th applicant backward, of the candidate. For example $Z_n = 1$ means that the most recently interviewed applicant $n$ is the candidate, etc.

Let $\Theta_1, \ldots, \Theta_N$ be $N$ integer-valued random variables also defined on $(\Omega, F, P)$ with joint probability density function $p_N(\alpha_1, \ldots, \alpha_N) = P(\Theta_1 = \alpha_1, \ldots, \Theta_N = \alpha_N)$ for $\alpha_1 = 1$, $1 \leq \alpha_{i+1} \leq \alpha_i + 1$, $i = 1, 2 \ldots N - 1$. $\{\Theta_n, 1 \leq n \leq N\}$ is called the memory system of the problem and will be assumed to be independent of $\{Y_n, 1 \leq n \leq N\}$ (and hence of $\{Z_n, 1 \leq n \leq N\}$). At stage $n$, the employer will remember only $\Theta_n$ of the most recently interviewed applicants. An offer will be made to the candidate only if $Z_n \leq \Theta_n$.

We let $F_n = \sigma(Y_i, \Theta_i, 1 \leq i \leq n)$ for $n = 1, 2 \ldots N$. Following notations similar to that of [2] and [5], we let $(n, r; \alpha_1 \ldots \alpha_n)$ be the state of the interview process after interview $n$ if $Z_n = r$, $\Theta_i = \alpha_i$, $1 \leq i \leq n$, and if no offer has yet been made to the candidate. $(n, \infty; \alpha_1 \ldots \alpha_n)$ will denote the state of the process if the first $n$ applicants have been interviewed and the candidate has already rejected an offer. If in the state $(n, r; \alpha_1 \ldots \alpha_n)$, $1 \leq r \leq \alpha_n$, an offer is made to the candidate he will accept with probability $q(r)$ and will reject the offer with probability $1 - q(r)$ where $q(\infty)$ and $q(r)$, $r = 1, 2, \ldots N$, satisfy

(i) $q(\infty) = 0$
(ii) $0 < q(1) \leq 1$
(iii) $q(r)$ is nonincreasing in $r (r = 1, 2, \ldots N)$.

The combination of (i) and the assumption that $P(\Theta_{n+1} > \Theta_n + 1 \mid \Theta_i = \alpha_i, 1 \leq i \leq n) = 0$ guarantee that once having rejected an offer or out of the memory, an applicant is no longer available.

Suppose the process is in state $(n, r; \alpha_1 \ldots \alpha_n)$, $1 \leq r \leq \alpha_n$. We then have the option of making an offer to the candidate or of making no offer and interviewing applicant $n + 1$. We let $\pi_b(n, r; \alpha_1 \ldots \alpha_n)$ be the probability of choosing the best applicant in the former case, and $\pi_f(n, r; \alpha_1 \ldots \alpha_n)$ the same probability in the latter case, given that $\Theta_i = \alpha_i$, $1 \leq i \leq n$, and $Z_n = r$. Based only on $F_n$, $1 \leq n \leq N$, the observed relative ranks of applicants and memory system, it is desired to find, for each set of $q$'s and each $N \geq 1$, a procedure to maximize the probability of selecting the best among the $N$ applicants. An optimal procedure is then to make the best applicant among the first $n$ an offer whenever the process is in state $(n, r; \alpha_1 \ldots \alpha_n)$, $1 \leq r \leq \alpha_n$, and $\pi_b(n, r; \alpha_1 \ldots \alpha_n) > \pi_f(n, r; \alpha_1 \ldots \alpha_n)$.

To derive the formulas for $\pi_b(n, r; \alpha_1 \ldots \alpha_n)$, $\pi_f(n, r; \alpha_1 \ldots \alpha_n)$ and $\pi(n, r; \alpha_1 \ldots \alpha_n) \equiv \max\{\pi_b(n, r; \alpha_1 \ldots \alpha_n), \pi_f(n, r; \alpha_1 \ldots \alpha_n)\}$. We first observe that

$$P(Z_{n+1} = r, \Theta_{n+1} = k \mid F_n) = \begin{cases} \dfrac{1}{n+1} P(\Theta_{n+1} = k \mid \Theta_i, 1 \le i \le n), r = 1 \\ \dfrac{n}{n+1} P(\Theta_{n+1} = k \mid \Theta_i, 1 \le i \le n), r = Z_n + 1 \\ 0, \quad \text{otherwise.} \end{cases}$$

and

$$\pi_f(n, Z_n; \Theta_1 \ldots \Theta_n) = E[\pi(n+1, Z_{n+1}; \Theta_1 \ldots \Theta_{n+1}) \mid F_n].$$

Consequently,

$$\pi_f(n, r; \alpha_1 \ldots \alpha_n) = \frac{1}{n+1} E[\pi(n+1, 1; \Theta_1 \ldots \Theta_{n+1}) \mid \Theta_i = \alpha_i, 1 \le i \le n]$$
$$+ \frac{n}{n+1} E[\pi(n+1, r+1; \Theta_1 \ldots \Theta_{n+1}) \mid \Theta_i = \alpha_i, 1 \le i \le n] \tag{2.1}$$

$$\pi_b(n, r; \alpha_1 \ldots \alpha_n) = \begin{cases} \dfrac{n}{N} q(r) + (1 - q(r))\pi(n, \infty; \alpha_1 \ldots \alpha_n) & \text{if } 1 \le r \le \alpha_n \\ 0 \text{ if } r > \alpha_n. \end{cases} \tag{2.2}$$

$$\pi(n, \infty; \alpha_1 \ldots \alpha_n) = \frac{1}{n+1} E[\pi(n+1, 1; \Theta_1 \ldots \Theta_{n+1}) \mid \Theta_i = \alpha_i, 1 \le i \le n]$$
$$+ \frac{n}{n+1} E[\pi(n+1, \infty; \Theta_1 \ldots \Theta_{n+1}) \mid \Theta_i = \alpha_i, 1 \le i \le n] \tag{2.3}$$

$$\pi(N, r; \alpha_1 \ldots \alpha_N) = \begin{cases} q(r) \text{ if } 1 \le r \le \alpha_N \\ 0 \text{ if } r > \alpha_N. \end{cases} \tag{2.4}$$

From (2.3) and (2.4), it is clear that

$$\pi(n, \infty; \alpha_1 \ldots \alpha_n) = \sum_{j=n+1}^{N} \frac{n}{j(j-1)} E[\pi(j, 1; \Theta_1 \ldots \Theta_j) \mid \Theta_i = \alpha_i, 1 \le i \le n]. \tag{2.5}$$

Note that $\pi_f(n, r; \alpha_1 \ldots \alpha_n) \ge \pi_b(n, r; \alpha_1 \ldots \alpha_n)$ whenever $r > \alpha_n$.

Throughout the paper we shall assume the usual values for the vacuous sum and product:

$$\sum_{j=N}^{N-1} a_j = 0 \quad \text{and} \quad \prod_{j=N}^{N-1} b_j = 1.$$

For any two random variables $X$ and $Y$, $X \le Y$ means $P(X \le Y) = 1$.

## 3. GENERAL RESULTS

The use of the memory system $\{\Theta_n, n = 1, 2, \ldots, N\}$ with joint probability density function $p_N(\alpha_1 \ldots \alpha_N)$ for $\alpha_1 = 1$ and $1 \le \alpha_{n-1} \le \alpha_n + 1$ for $n = 1, 2, \ldots, N-1$ is motivated by some recruiting situations in which the $n$th applicant arrives at the random time $T_n$ ($n = 1, 2, \ldots, N$) and each applicant will be notified of the result whether an offer will or will never be made to him (or her) in exactly $\beta$ units after the time of his (or her) interview. At stage $n$ of the process, an offer will be made to applicant $n - r + 1$ only if $Z_n = r$ and $T_{n-r+1}$ falls within the random time interval $[T_n - \beta, T_n]$. For the purpose of selecting the best applicant, it is sufficient to know the probability distribution of the number of applicants who will arrive with $[T_n - \beta, T_n]$, i.e.

$$P_N(\alpha_1, \alpha_2 \ldots \alpha_N) = P(T_n - T_{n-\alpha_n+1} \le \beta < T_n - T_{n-\alpha_n} \text{ for } n = 1, 2 \ldots N).$$

Note that $p_N(\alpha_1, \ldots, \alpha_N) = 0$ if $\alpha_{n+1} > \alpha_n + 1$ for some $n$, since $0 < T_1 < T_2 < \ldots < T_N$ and $T_n - T_{n-\alpha_{n+1}} \le \beta < T_n - T_{n-\alpha_n}$ together imply that $\beta < T_{n+1} - T_{n-\alpha_n}$.

The memory system $\{\Theta_n, n = 1, 2, \ldots, N\}$ considered in this paper is an extension of existing results. For instance, if $T_n = n$, $n = 1, 2, \ldots, N$, then $\beta = 1$, $\beta \ge N$, and $1 < \beta\ (= m) < N$ correspond to the modified secretary problem ($\Theta_n = 1, n = 1, 2 \ldots N$) in [3], the perfect memory system ($\Theta_n = n, n = 1, 2, \ldots, N$) in [2], and finite memory system ($\Theta_n = n$ for $n = 1, 2, \ldots, m$ and $\Theta_n = m$ for $m + 1 \le n \le N$) in [4].

A nonhomogeneous Markov memory with sure availability and finite memory with $q(1) = p$ and $q(r) = q$ for $r = 2, \ldots, N$, will be treated in the next two sections. As for the remainder of this section, we shall take up the optimal selection problem with general memory system $\{\Theta_n, n = 1, 2, \ldots, N\}$ and $\{q(r), r = 1, 2, \ldots \infty\}$ satisfying conditions in
the preceding section.

Let $\hat{\pi}_b(n, r)$ and $\hat{\pi}_f(n, r)$ be $\pi_b(n, r; \alpha_1 \ldots \alpha_n)$ and $\pi_f(n, r; \alpha_1 \ldots \alpha_n)$ respectively when $\Theta_n = 1$ for $n = 1, 2 \ldots N$. Set $\hat{\pi}(n, r) = \max\{\hat{\pi}_b(n, r), \hat{\pi}_f(n, r)\}$. Then we have

*Lemma* 3.1. $s^* \le \inf\{n \ge 1: \pi_f(n, 1; \Theta_1 \ldots \Theta_n) < \pi_b(n, 1; \Theta_1 \ldots \Theta_n)\}$ where

$$s^* = \inf\left\{n \ge 1: \prod_{k=n}^{n=N} (1 + (1 - q(1))/k) \le 1/q(1)\right\}.$$

*Proof.* Note that

$$\hat{\pi}_b(n, 1) = nN^{-1}q(1) \prod_{k=n}^{N-1} (1 + (1 - q(1))/k)$$

and

$$\hat{\pi}_f(n, 1) = nN^{-1}q(1)(1 - q(1))^{-1} \left(\prod_{k=n}^{N-1} (1 + (1 - q(1))/k) - 1\right)$$

for $n = 1, 2, \ldots, N-1$,

and

$$s^* = \inf\{n \ge 1: \hat{\pi}_f(n, 1) < \hat{\pi}_b(n, 1)\}.$$

The lemma follows if

$$\pi_b(N-1, 1; \Theta_1 \ldots \Theta_{N-1}) \le \pi_f(N-1, 1; \Theta_1 \ldots \Theta_{N-1})$$

since this implies that

$$\pi_b(n, 1; \Theta_1 \ldots \Theta_n) \le \pi_f(n, 1; \Theta_1 \ldots \Theta_n) \text{ for } n = 1, 2 \ldots N-2.$$

Otherwise

$$\hat{\pi}_b(n, 1) = \pi_b(n, 1; \Theta_1 \ldots \Theta_n)$$

and

$$\pi_f(n, 1; \Theta_1 \ldots \Theta_n)$$

$$\ge \hat{\pi}_f(n, 1) + nN^{-1}\, q(N-n+1) \cdot E\left[\sum_{\Theta_N = N-n+1}^{\Theta_{N-1}+1} \frac{p_N(\Theta_1, \ldots, \Theta_N)}{p_{N-1}(\Theta_1, \ldots, \Theta_{N-1})} \,\middle|\, \Theta_i, 1 \le i \le n\right]$$

$$\ge \hat{\pi}_f(n, 1).$$

In this case the lemma also follows.

The following two theorems are direct extensions of Theorem 4.3 and Theorem 4.2 in [2]. Their proofs consist of slight modifications of the above-mentioned theorems and are therefore omitted.

Theorem 3.2. Let $h(N) = 0$ and

$$h(n) = \left(\prod_{k=n}^{N-1} (1 + (1 - q(1))/k) - 1\right) \Big/ (1 - q(1)), \text{ for } n = 1, 2 \ldots N-1.$$

Let $s^*$ be as defined in Lemma 3.1. If $n \ge s^*$ is such that

$$q(r+1) \sum_{\Theta_{n+1} = r+1}^{\Theta_n + 1} \frac{p_{n+1}(\Theta_1, \ldots, \Theta_{n+1})}{p_n(\Theta_1, \ldots, \Theta_n)} \Big/ q(r) < \frac{1 - q(1)h(n)}{1 - q(1)h(n+1)} \tag{3.1}$$

for $1 \le r \le \Theta_n$ and $n = 1, 2, \ldots, N$, then the optimal procedure will make offers to the candidates among applicants $n$, $n = s^*, \ldots, N$ as they appear.

*Remark.* If $\Theta_n = n$ for $n = 1, 2, \ldots, N$, then the left side of (3.1) reduces to $q(r+1)/q(r)$. This is exactly the case considered in [2].

Theorem 3.3. If

$$q(r+1) \sum_{\Theta_N = r+1}^{\Theta_{N-1}+1} \frac{p_N(\Theta_1, \ldots, \Theta_N)}{p_{N-1}(\Theta_1, \ldots, \Theta_{N-1})} \Big/ q(r) \ge [N - 1 - q(1)]/(N-1) \text{ for } 1 \le r \le \Theta_{N-1}. \tag{3.2}$$

Then under the optimal strategy an offer is never made until all applicants have been interviewed. The converse is also true.

## 4. A MARKOV MEMORY SYSTEM AND SURE AVAILABILITY

In this section, we consider the case that $q(r) = 1$ for $r = 1, 2, \ldots, N$. Without specifying the memory system $(\Theta_n, 1 \leq n \leq N)$, we have the following result.

THEOREM 4.1. If $q(r) = 1$ for $r = 1, 2, \ldots, N$, $P(\Theta_{n-1} = \alpha_n + 1 \mid \Theta_i = \alpha_i, 1 \leq i \leq n) \leq P(\Theta_n = \alpha_{n-1} + 1 \mid \Theta_i = \alpha_i, 1 \leq i \leq n-1)$, and

$$P(\Theta_{n-1} \geq r + 1 \mid \Theta_i = \alpha_i, 1 \leq i \leq n) > 1 - N^{-1}$$

for all $n$ and $1 \leq r < \alpha_n$, then the optimal procedure is to make no offer prior to the stage

$$r^* \equiv \inf\{k \geq 1: \pi_f(k, \alpha_k; \alpha_1 \ldots \alpha_k) < \pi_b(k, \alpha_k; \alpha_1 \ldots \alpha_k)\}$$

and make an offer to a candidate if $Z_n = \alpha_n$, $n = r^*, \ldots, N-1$. Otherwise interview applicant $N$ and make an offer to the candidate who is still available.

*Proof.* For convenience, we set

$$W(n, r; \alpha_1 \ldots \alpha_n) = rN^{-1}[\pi_f(n, r; \alpha_1 \ldots \alpha_n) - \pi_b(n, r; \alpha_1 \ldots \alpha_n)].$$

Then

$$\begin{aligned} W(n, r; \alpha_1 \ldots \alpha_n) = {} & P(\Theta_{n+1} \geq r + 1 \mid \Theta_i = \alpha_i, 1 \leq i \leq n) \\ & + n^{-1} - 1 + E[W^+(n+1, r+1; \Theta_1 \ldots \Theta_{n+1}) \mid \Theta_i = \alpha_i, 1 \leq i \leq n] \\ & + n^{-1} E[W^-(n+1, 1; \Theta_1 \ldots \Theta_{n+1}) \mid \Theta_i = \alpha_i, 1 \leq i \leq n]. \end{aligned} \tag{4.1}$$

It is clear that

$$W(n, \alpha_n; \alpha_1 \ldots \alpha_n) \leq \ldots \leq W(n, 2; \alpha_1 \ldots \alpha_n) \leq W(n, 1; \alpha_1 \ldots \alpha_n).$$

Therefore it remains to show that

$$W(n, \alpha_n; \alpha_1 \ldots \alpha_n) \quad \text{is decreasing in } n \text{ for } \alpha_1 = 1,\ 1 \leq \alpha_{n+1} \leq \alpha_n + 1.$$

The proof is by backward induction.

(i) Let $n = N - 2$. Then (4.1) implies

$$\begin{aligned} & W(N-2, \alpha_{N-2}; \alpha_1 \ldots \alpha_{N-2}) - W(N-1, \alpha_{N-1}; \alpha_1 \ldots \alpha_{N-1}) \\ & \quad \geq P(\Theta_{N-1} = \alpha_{N-2} + 1 \mid \Theta_i = \alpha_i, 1 \leq i \leq N-2) \\ & \qquad - P(\Theta_N = \alpha_{N-1} + 1 \mid \Theta_i = \alpha_i, 1 \leq i \leq N-1) + \frac{1}{N-2} - \frac{1}{N-1} \\ & \quad \geq \frac{1}{N-2} - \frac{1}{N-1} > 0 \end{aligned}$$

by the fact that $W^-(N, r; \alpha_1 \ldots \alpha_N) = 0$ and the assumption that $P(\Theta_{n+1} = \alpha_n + 1 \mid \Theta_i = \alpha_i, 1 \leq i \leq n) \leq P(\Theta_n = \alpha_{n-1} + 1 \mid \Theta_i = \alpha_i, 1 \leq i \leq n - 1)$.

(ii) Assume that

$$W(n, \alpha_n; \alpha_1 \ldots \alpha_n) > W(n + 1, \alpha_{n+1}; \alpha_1 \ldots \alpha_{n+1}).$$

Consequently

$$\begin{aligned} W(n - 1, \alpha_{n-1}; \alpha_1 \ldots \alpha_{n-1}) &- W(n, \alpha_n; \alpha_1 \ldots \alpha_n) \\ = P(\Theta_n &= \alpha_{n-1} + 1 \mid \Theta_i = \alpha_i, 1 \leq i \leq n - 1) \\ &- P(\Theta_{n+1} = \alpha_n + 1 \mid \Theta_i = \alpha_i, 1 \leq i \leq n) + \frac{1}{n-1} - \frac{1}{n} \\ &+ E[W^+(n, \alpha_{n-1} + 1; \Theta_1 \ldots \Theta_n) \mid \Theta_i = \alpha_i, 1 \leq i \leq n - 1] \\ &- E[W^+(n + 1, \alpha_n + 1; \Theta_1 \ldots \Theta_{n+1} \mid \Theta_i = \alpha_i, 1 \leq i \leq n] \\ &+ (n - 1)^{-1} E[W^+(n, 1; \Theta_1 \ldots \Theta_n) \mid \Theta_i = \alpha_i, 1 \leq i \leq n - 1] \\ &- n^{-1} E[W^+(n + 1, 1; \Theta_1 \ldots \Theta_{n+1}) \mid \Theta_i = \alpha_i, 1 \leq i \leq n] \\ \geq \frac{1}{n-1} &- \frac{1}{n} > 0. \end{aligned}$$

This completes the induction proof.

*Remark.* If $q(r) = 1$ for $r = 1, 2, \ldots, N$, then at any stage of the process the candidate will accept the offer with certainty when one is made to him or her. The conditions

$$P(\Theta_{n+1} = \alpha_n + 1 \mid \Theta_i = \alpha_i, 1 \leq i \leq n) \leq P(\Theta_n = \alpha_{n-1} + 1 \mid \Theta_i = \alpha_i, 1 \leq i \leq n - 1)$$

and $P(\Theta_{n+1} \geq r + 1 \mid \Theta_i = \alpha_i, 1 \leq i \leq n) > 1 - N^{-1}$ for all $n$ and $1 \leq r < \alpha_n$ imply that at each stage the probability of erasing at most one applicant from the memory of the preceding stage is greater than $1 - N^{-1}$ and that the probability of retaining all applicants in the memory of the preceding stage is nonincreasing with respect to $n$.

These conditions are often met in some of the real recruiting situations in which the interarrival times $T_n - T_{n-1}$, $n = 1, 2, \ldots, N$, are increasing and $T_N$ is relatively short. Under these assumptions, the memory system of the problem behaves almost like a perfect one and there is no advantage in making offers to candidates immediately after their interviews since $q(r) = 1$ for $r = 1, 2, \ldots, N$. Therefore, the optimal strategy after stage $r^*$ and prior to stage $N$ is to make an offer to a candidate only when he or she is about to become unavailable as indicated in Theorem 4.1.

Next we consider a nonhomogeneous Markov memory system $(\Theta_n, 1 \leq n \leq N)$ where

$$P(\Theta_{n+1} = k \mid \Theta_n = \alpha_n) = 1/(\alpha_n + 1) \text{ for } k = 1, 2, \ldots, \alpha_n + 1.$$

In this case we write

$$\pi_b(n, r; \alpha_1 \ldots \alpha_n) = \pi_b(n, r; \alpha_n),$$

$$\pi_f(n, r; \alpha_1 \ldots \alpha_n) = \pi_f(n, r; \alpha_n)$$

and

$$\pi(n, r; \alpha_1 \ldots \alpha_n) = \pi(n, r; \alpha_n).$$

This memory system provides models for recruiting situations in which the arrival times $(T_1 \ldots T_N)$ distribute according to $P\{T_{n-1} - T_{n-j+1} \le \beta < T_{n-1} - T_{n-j} \mid T_n - T_{n-\alpha_n+1} \le \beta < T_n - T_{n-\alpha_n}\} = \dfrac{1}{\alpha_n + 1}$ for $j = 1, 2, \ldots, \alpha_n + 1$. This assumption implies that the events of $j$ applicants ($j = 1, 2, \ldots, \alpha_n + 1$) arriving within $[T_{n-1} - \beta, T_{n-1}]$ given that $\alpha_n$ applicants arrived within $[T_n - \beta, T_n]$ are equally likely to occur.

THEOREM 4.2. Let $\{\Theta_n, 1 \le n \le N\}$ be such that

$$P(\Theta_{n+1} = k \mid \Theta_n = \alpha_n) = 1/(\alpha_n + 1) \text{ for } k = 1, 2, \ldots, \alpha_n + 1,$$

and

$$q(r) = 1 \text{ for } r = 1, 2, \ldots, N.$$

$$\text{Let } \hat{s} = \inf\left\{n \ge 1 \colon \sum_{j=n+1}^{N} 1/(j-1) \le 1\right\}.$$

Then

(i) the optimal procedure is to reject applicants prior to $r^*$ and select a candidate as he or she appears at the stage $r^*$ and onward, where

$$r^* = \begin{cases} \inf\{n \ge \hat{s} \colon \Theta_n \le [\![f(n)]\!]\} & \text{if } \{n \ge \hat{s} \colon \Theta_n \le [\![f(n)]\!]\} \ne \phi \\ N & \text{if } \{n \ge \hat{s} \colon \Theta_n \le [\![f(n)]\!]\} = \phi, \end{cases}$$

and $f(n) = (Nn^2 + 2n - Nn - N + 1)/(N - 1)(n + 1)$.

*Note*: $[\![a]\!]$ = the greatest integer that is less than or equal to $a$.

(ii) The probability of choosing the best applicant using the optimal rule is

$$v(N) = \sum_{k=\hat{s}}^{N} \frac{k-1}{2N} \left(\sum_{j=k}^{N} \frac{1}{j-1}\right) \sum_{\alpha_k=1}^{k-2} \sum_{\alpha_{k-1}=k-2}^{\alpha_{k-2}+1} \cdots \sum_{\alpha_3=2}^{3} \sum_{\alpha_2=1}^{2} \left(\prod_{i=2}^{k} \frac{1}{\alpha_i + 1}\right).$$

*Proof*. (i) It is clear that

$$\pi_b(N-1, r; \alpha_{N-1}) > \pi_f(N-1, r; \alpha_{N-1})$$

for

$$1 \le r \le \alpha_{N-1} \le [\![f(N-1)]\!],$$

since

$$[\![f(N-1)]\!] = N - 2.$$

Assume that $1 \le r \le \alpha_k \le [\![f(k)]\!]$ implies that

$$\pi_b(k, r; \alpha_k) > \pi_f(k, r; \alpha_k) \text{ for } k = N-1, N-2 \ldots n+1.$$

Then, for $1 \le r \le \alpha_n \le [\![f(n)]\!]$,

$$\begin{aligned}
\pi_b(n, r; \alpha_n) &- \pi_f(n, r; \alpha_n) \\
&= \frac{n}{N} - \left(\frac{1}{N} + \frac{n(\alpha_n - r + 1)}{N(\alpha_n + 1)} + \frac{rn}{N(n + 1)(\alpha_n + 1)} - \frac{nr}{N(N - 1)(\alpha_n + 1)}\right) \\
&\ge \frac{n(Nn + 1)}{N(N - 1)(\alpha_n + 1)(n + 1)} - \frac{1}{N} \\
&> \frac{n(Nn + 1) - (N - 1)(n + 1)([\![f(n)]\!] + 1)}{N(N - 1)(\alpha_n + 1)(n + 1)} \ge 0.
\end{aligned}$$

To complete the proof of (i), it remains to show that

$$\alpha_n \le [\![f(n)]\!] \text{ implies that } \alpha_{n+1} \le [\![f(n + 1)]\!].$$

This is the consequence of the facts that

$$1 \le \alpha_{n+1} \le \alpha_n + 1 \text{ and } [\![f(n)]\!] + 1 \le [\![f(n + 1)]\!] \text{ for } n \ge \hat{s}.$$

(ii) Let $\tau(r^*)$ be the procedure stated in (i). That is $\tau(r^*) = k$ whenever the $k$th applicant is selected by the procedure described in (i). $\{X_{\tau(r^*)} = 1\}$ is then the event that the best applicant is chosen using rule $\tau(r^*)$. Then

$$v(N) = \sum_{k=\hat{s}}^{N} \frac{k - 1}{N} \left(\sum_{j=k}^{N} \frac{1}{j - 1}\right) P(r^* = k).$$

Note that

$$P(\Theta_1 = 1, \Theta_2 = \alpha_2, \ldots, \Theta_k = \alpha_k) = \frac{1}{2} \prod_{i=2}^{k} \frac{1}{\alpha_k + 1} \quad \text{for}$$

$$1 \le \alpha_{i+1} \le \alpha_i + 1, i = 2, 3 \ldots k.$$

and

$$[\![f(n)]\!] = n - 2 \text{ for } 1 \le n \le N - 2.$$

Therefore

$$\begin{aligned}
P(r^* = k) &= P(\Theta_k \le k - 2, i - 1 \le \Theta_i \le \Theta_{i-1} + 1, i = 2 \ldots k - 1) \\
&= \frac{1}{2} \sum_{\alpha_k = 1}^{k-2} \sum_{\alpha_{k-1} = k-2}^{\alpha_{k-2}+1} \cdots \sum_{\alpha_2 = 1}^{2} \prod_{i=2}^{k} \frac{1}{\alpha_i + 1}.
\end{aligned}$$

Consequently $v(N)$ is as stated in the theorem.

*Remark.* Due to the randomness of the memory system, $r^*$ is also a random variable whose values depend upon the realization of the memory sizes. If all of realizations of $\Theta_n$, $n = 1, 2, \ldots, N$ are large, say $\{n \ge \hat{S}: \Theta_n \le [\![f(n)]\!]\} = \phi$, then the optimal rule is to wait until stage $N$ since there is no risk of losing the candidates. However, if $\{n \geq \hat{s}: \Theta_n \leq [\![f(n)]\!]\} \neq \phi$, then, as revealed by Theorem 4.2, we should choose the first candidate in sight the moment this event occurs. The integer $[\![f(u)]\!]$, $f(n) = Nn^2 + 2n - Nn - N + 1)/(N - 1)(n + 1)$, can be regarded as threshold size of $\Theta_n$ for $n = \hat{s}, \hat{s} + 1 \ldots N$.

## 5. FINITE MEMORY AND UNCERTAIN AVAILABILITY

In this section, we consider a finite memory system $\{\Theta_n, 1 \leq n \leq N\}$, i.e. $\Theta_i = i$ for $i = 1, 2, \ldots, m$ and $\Theta_i = m$ for $i = m + 1, \ldots, N$ and $q(1) = q$, $q(r) = p$ for $r = 2, \ldots, N$. The reason for considering this finite memory system was given in Sec. 3.

THEOREM 5.1. Let $\{\Theta_n, 1 \leq n \leq N\}$ be such that

$$\Theta_n = \begin{cases} n \text{ for } n = 1, 2, \ldots, m \\ m \text{ for } n = m + 1, \ldots, N, \end{cases}$$

and

$$q(1) = q, \ q(r) = p \text{ for } r = 2, \ldots, N.$$

If $p \geq (N - 1 - q)q/(N - 1)$, then
(i) there exists a positive integer $r^* = r^*(N, m, p, q)$ such that the optimal procedure is to make no offer prior to the stage $r^*$ and to make successive offers to eligible candidates at stages $n = r^*, r^* + 1, \ldots, N - 1$ until an applicant accepts. If no one accepts, then interview applicant $N$ and make an offer to the candidate if he or she is still available.
(ii) $r^* = \max\{m, \hat{r}\}$ where

$$\hat{r} = \inf\left\{n \geq 1: \frac{q}{(N - 1)} + \sum_{j=n+1}^{N-1} \left[p + \frac{q}{1 - q}\left(\prod_{k=j}^{N-1}\left(1 + \frac{1 - q}{k}\right) - 1\right)\right] \Big/ (j - 1) < 1\right\}.$$

If $m \geq N/2$ then $r^* = m$.

(iii) If $r^* = m$ then the probability of choosing the best applicant using the optimal rule is

$$v(N) = \frac{1}{N}\left\{p + \sum_{k=2}^{N} (1 - p)^{k-1}(p\chi_{\{i(k) \neq N\}} + q\chi_{\{i(k) = N\}}) \times \sum_{2 \leq i(2) < \ldots < i(k) \leq N} \prod_{j=2}^{k} [i(j) - 1]^{-1}\right\}.$$

If $r^* > m$ this probability is

$$v(N) = \frac{r^* - m}{N}\left\{p + \sum_{k=1}^{N-r^*+m} (1 - p)^{k-1}(p\chi_{\{i(k) \neq N\}} + q\chi_{\{i(k) = N\}}) \times \sum_{A(r^* - m + 1, N, k)} \prod_{j=1}^{k} (i(j) - 1)^{-1}\right\}.$$

(iv) If $m$ is fixed, then $e^{-1/q} \le \lim_{N\to\infty} \frac{r^*}{N} \le \frac{p}{1+p}$ and

$$\frac{1}{q^{1-q}} \le \lim_{N\to\infty} v(N) \le \left(\frac{q^{1-q}}{(q - p(1-q))^q}\right)^{\frac{1}{1-q}}$$

*Proof.* We first define the following notations:

For any $1 \le s \le N$, let $C(s, N)$ be the number of candidates among applicants $s, \ldots, N$, and let $M(s, N, j)$ be the position of the $j$th candidate, $1 \le j \le C(s, N)$. At stage $n$, the candidate is said to be eligible if $Z_n = m$. That is if $C(s, N) = k$ and $M(s, N, j) = i(j), j = 1, 2 \ldots k$, then the candidate $i(j)$ is eligible if and only if

$$X_{i(j)} = \min\{X_1, \ldots, X_{i(j)}, \ldots, X_{i(j)+m-1}\}, \quad \text{or}$$

equivalently $i(j+1) - i(j) > m$. For $C(s, N) = k$ and $M(s, N, j) = i(j), j = 1, 2, \ldots, k$, we let $\hat{k}(s) = \hat{k}(s, N, k)$ be the number of $j$ such that $s \le i(1) < i(2) < \ldots < i(k) \le N$, $i(j) \le N - m$ and if $i(j+1) - i(j) > m$, i.e. $\hat{k}(s)$ is the number of eligible candidates between the stage $s + m - 1$ and stage $N - 1$. Also we let $\sum_{A(s,N,k)}$ denote the summation over all indices $s \le i(1) < i(2) < \ldots < i(k) \le N$. To prove (i), we note that

$$\pi_b(n, r) = \frac{n}{N} q(r) + (1 - q(r)) \sum_{j=n+1}^{N} \frac{n}{j(j-1)} \pi(j, 1) \text{ for } r = 1, 2, \ldots, m,$$

where $q(1) = q$ and $q(r) = p$ for $r = 2, \ldots, m$. And

$$\pi_f(n, m) = \sum_{j=n+1}^{N} \frac{n}{j(j-1)} \pi(j, 1) \text{ for } n = 1, 2, \ldots, N-1.$$

Therefore $\pi_f(n, m) < \pi_b(n, m)$ if and only if

$$\sum_{j=n+1}^{N} \frac{\pi(j, 1)}{j(j-1)} < \frac{1}{N}.$$

This in turn implies that $\pi_f(n+1, m) < \pi_b(n+1, m)$. Therefore to show (i), it remains to show that $\pi_f(n, r) \ge \pi_b(n, r)$ for $r = 1, 2, \ldots, m-1$ and $n = 1, 2, \ldots, N-1$. We shall prove this by backward induction. It is clear that $\pi_f(N-1, r) \ge \pi_b(N-1, r)$ for $r = 2, \ldots, m-1$ and $\pi_f(N-1, 1) \ge \pi_b(N-1, 1)$ since $p \ge q(N-1-q)/(N-1)$. Assume that $\pi_f(k, r) \ge \pi_b(k, r)$ for $k = N-1, \ldots, n+1$ and $r = 1, 2, \ldots, m-1$. Then

$$\pi_f(n, r) = n \sum_{j=n+1}^{n+(m-r)} \frac{\pi(j, 1)}{j(j-1)} + \frac{n}{N} p + n(1-p) \sum_{j=n+(m-r)+1}^{N} \frac{\pi(j-1)}{j(j-1)} \tag{5.1}$$
$$\text{for } n = r^*, r^*+1, \ldots, N, \text{ and } r = 1, 2, \ldots, m-1.$$

This immediately implies that $\pi_f(n, r) \ge \pi_b(n, r)$ for $r = 2, \ldots, m-1$ and $n = r^*, r^* + 1, \ldots, N$. For $r = 1$, it can be shown that $\pi_f(n, 1) \ge \pi_b(n, 1)$ if and only if

$$\sum_{j=n+1}^{N} \frac{\pi(j, 1)}{j(j-1)} \ge \left\{\frac{1}{N}(q - p) + p \sum_{j=n+m}^{N} \frac{\pi(j, 1)}{j(j-1)}\right\} \Big/ q.$$

By the induction assumption to show that $\pi_f(n, 1) \geq \pi_b(n, 1)$ it suffices to show that $\frac{q}{p} \frac{\pi(n+1, 1)}{n(n+1)} \geq \frac{\pi(n+m, 1)}{(n+m)(n+m-1)}$. Since $\pi(n+1, 1) = \pi_f(n+1, 1)$ and $\pi(n+m, 1) = \pi_f(n+m, 1)$, the above inequality follows from a repeated application of (5.1) and the fact that $\pi(N, 1) = q$ and $\pi(N-1, 1) = \frac{1}{N}q + \frac{N-1}{N}p$. This proves that $\pi_f(n, r) \geq \pi_b(n, r)$ for $r = 1, 2, \ldots, m-1$ and $n = r^*, \ldots, N$. For $n = 1, 2, \ldots, r^*-1$, and $r = 1, 2, \ldots, m$, $\pi_f(n, r) \geq \pi_b(n, r)$ since

$$\pi_f(n, r) = \sum_{j=n+1}^{N} \frac{n}{j(j-1)} \pi(j, 1) \text{ and } r^* = \max\{m, \hat{r}\}, \quad \text{where}$$

$$\hat{r} = \inf\left\{n \geq 1: \sum_{j=n+1}^{N} \frac{\pi(j, 1)}{j(j-1)} < 1/N\right\}.$$

(ii) $\hat{r}$ is stated in (ii), since $\pi(N, 1) = q$ and

$$\pi(j, 1) = \pi_f(j, 1) = \frac{j}{N}\left(p + q(1-q)^{-1}\left(\prod_{k=j}^{N-1}\left(1 + \frac{1-q}{k}\right) - 1\right)\right).$$

If $m \geq N/2$ then

$$\sum_{j=m+1}^{N} \frac{\pi(j, 1)}{j(j-1)} \leq \sum_{j=m+1}^{2m} \frac{1}{j(j-1)} = \left(\frac{1}{2m}\right) \leq \frac{1}{N}.$$

Therefore, $m \geq \inf\left\{n \geq 1: \sum_{j=n+1}^{N} \frac{\pi(j, 1)}{j(j-1)} < 1/N\right\}$.

Consequently $r^* = m$.

For the proof of (iii), we let $\tau^*(s)$ denote a procedure of the form given in (i) for arbitrary $1 \leq s \leq N$. By this we mean $\tau^*(s) = k$ whenever the $k$th applicant is selected by the procedure. $\{X_{\tau^*(s)} = 1\}$ is then the event that the best applicant is chosen using rule $\tau^*(s)$. One can show that for $s > 2$,

$$P\{C(s, N) = k\} = \frac{s-1}{N} \sum_{A(s,N,k)} \prod_{j=1}^{k} (i(j) - 1)^{-1} \quad \text{for} \quad 1 \leq k \leq N - s + 1, \quad \text{and}$$

$$P\{C(s, N) = 0\} = \frac{s-1}{N}. \quad \text{For} \quad s = 1, C(1, N) \geq 1 \quad \text{and}$$

$$P\{C(1, N) = k\} = N^{-1} \sum_{2 \leq i(2) \ldots < i(k) \leq N} \prod_{j=2}^{k} (i(j) - 1)^{-1}, k \geq 2,$$

$$P\{C(1, N) = 1\} = 1/N.$$

If $r^* = m$ then $s = 1$, $C(1, N) \geq 1$ and

$$P\{X_{\tau^*(1)} = 1\} = \sum_{k=1}^{N} P\{X_{\tau^*(1)} = 1 \mid C(1, N) = k\} P\{C(1, N) = k)\}.$$

Now that $p\{X_{\tau^*(1)} = 1 \mid C(1, N) = k\}$

$$= (P\chi_{\{i(k) \neq N\}} + q\chi_{\{i(k) = N\}})^{(1-p)^{k-1}}$$

Therefore when $r^* = m$

$$v(N) = \frac{1}{N}\Big(p + \sum_{k=2}^{N} \{p\chi_{[i(k)\neq N]}$$

$$+ q\chi_{[i(k)=N]}\} \sum_{2\le i(2)<\ldots<i(k)\le N} \prod_{j=2}^{k} [i(j) - 1]^{-1}(1 - q)^{\hat{k}-1}\Big).$$

When $r^* > m$, we can argue similarly and obtain $v(N)$ as stated in (iii).
(iv) Let $\hat{\pi}(n, 1)$ be as that described before Lemma 3.1.

$$\text{Then } \pi(j, 1) \le \hat{\pi}(j, 1) = \frac{1}{N} q + \frac{N-1}{N} p. \quad \text{Therefore}$$

$$r^* \le \inf\left\{n \ge m: \sum_{j=n+1}^{N} \frac{\hat{\pi}(j, 1)}{j(j-1)} < \frac{1}{N}\right\}$$

$$= \inf\left\{n \ge m: n > \frac{N(q + (N-1)p)}{N + q + (N-1)p}\right\}.$$

On the other hand

$$\sum_{j=n+1}^{N} \frac{\pi(j, 1)}{j(j-1)} \ge \sum_{j=n+1}^{N} \frac{\pi_b(j, 1)}{j(j-1)} \ge \sum_{j=n+1}^{N} \frac{jq/N}{j(j-1)}$$

$$= q \sum_{j=n+1}^{N} \frac{1}{j-1} \Big/ N.$$

Thus $\bar{s} = \inf\left\{n \ge m: \sum_{j=n+1}^{N} \frac{1}{j-1} < \frac{1}{q}\right\} \le r^*$.

Consequently $e^{-1/q} \le \lim_{N\to\infty} \frac{r^*}{N} \le \frac{p}{1+p}$.

If $m$ is fixed, then $\hat{k} \to k$ as $N \to \infty$. Therefore

$$\lim_{N\to\infty} v(N) \le \lim_{N\to\infty} \bar{v}(N)$$

where $\bar{v}(N)$ is the probability of choosing the best applicant if the perfect memory system is used in place of the finite memory system, and that $\lim_{N\to\infty} \bar{v}(N) = \left(\frac{q^{1+q}}{(q - p(1-q))^q}\right)^{\frac{1}{1-q}}$ is proved in [2]. If we set $p = 0$ then we have the case that $m = 1$. Therefore $\lim_{N\to\infty} v(N) \ge q^{\frac{1}{1-q}}$. This completes the estimate of $\lim_{N\to\infty} v(N)$.

*Remark.* In the case of perfect memory system, i.e. $\Theta_n = n$ for $n = 1, 2, \ldots, N$ and $q(1) = q$, $q(r) = p$ for $r = 2, 3, \ldots, N$, the optimal procedure under the condition $p > (N - 1 - q)q/(N - 1)$ is to make no offer until applicant $N$ is interviewed as proved in [2]. However, if the memory system $\{\Theta_n, 1 \le n \le N\}$ is such that there exists a positive integer $m < N$ such that $P(\Theta_n < m) = 1$ for $n = 1, 2, \ldots, N$ then the above mentioned procedure is no longer optimal even under the condition $p > (N - 1 - q)q/(N - 1)$.

For the case that $p < (N - 1 - q)q/(N - 1)$, we state the following theorem. The proof will be very similar to that of Theorem 5.1 and is therefore omitted.

THEOREM 5.2. Let $\{\Theta_n, 1 \le n \le N\}$ be such that

$$\Theta_n = \begin{cases} n & \text{for } n = 1, 2, \ldots, m \\ m & \text{for } m = 1 + 2, \ldots, N, \end{cases}$$

$$\text{and } q(1) = q,\ q(r) = p \text{ for } r = 2, \ldots, N.$$

If $p < (N - 1 - q)q/(N - 1)$, then (i) there exists a positive integer $r^* = r^*(N, m, p, q)$ such that the optimal procedure is to make no offer prior to the stage $r^*$ and make successive offers to eligible candidates at stages $n = r^*, r^* + 1 \ldots N - 2$ and stage $N - 1$ if $Z_{N-1} = m$ or $Z_{N-1} = 1$ until an applicant accepts. If no one accepts, then interview applicant $N$ and make an offer to the candidate if he or she is still available.
(ii) If $r^* = m$ then the probability of choosing the best applicant using the optimal rule is

$$v(N) = \frac{1}{N}\Bigg\{p + \sum_{k=2}^{N} (1 - p)^{k-1}(p\chi_{\{i(k)<N-1\}} + q\chi_{\{i(k)\ge N-1\}}) \times \sum_{2\le i(2)<\ldots<i(k)\le N} \prod_{j=2}^{k} (i(j) - 1)^{-1}\Bigg\}.$$

(iii) If $r^* > m$ this probability is

$$v(N) = \frac{r^* - m}{N}\Bigg\{p + \sum_{k=1}^{N-r^*+m} (1 - p)^{k-1}(p\chi_{\{i(k)<N-1\}} + q\chi_{\{i(k)\ge N-1\}}) \times \sum_{A(r^*-m+1,N,k)} \prod_{j=1}^{k} (i(j) - 1)^{-1}\Bigg\}.$$

(iv) If $m$ is fixed, then $e^{-1/q} \le \lim_{N\to\infty} \dfrac{r^*}{N} \le \dfrac{p}{1 + p}$

and

$$q^{\frac{1}{1-q}} \le \lim_{N\to\infty} v(N) \le \left(\frac{q^{1+q}}{(q - p(1 - q))^q}\right)^{\frac{1}{1-q}}$$

Another example of interest is that $\{\Theta_n, 1 \le n \le N\}$ is a finite memory system and geometric availability $q(r) = qp^{r-1}$ for $r = 1, 2, \ldots, N$. Since the technique of treating the problem is very similar to that of Theorem 5.1 and Theorem 5.2, we shall not treat the geometric availability case here. The problem of perfect memory and geometric availability was first solved by Yang for $q(r) = p^{r-1}$ and was extended to $q(r) = q\,p^{r-1}$ by Petruccelli.

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