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Appl. Comput. Harmon. Anal. 15 (2003) 163–167
Applied and Computational Harmonic Analysis
www.elsevier.com/locate/acha

Letter to the Editor

The Heisenberg inequality for the discrete Fourier transform ☆

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Received 2 July 2002; accepted 22 July 2002
Communicated by Charles K. Chui

1. Introduction and statement of results

The traditional form of expressing the Heisenberg principle, stating that one cannot simultaneously concentrate in physical and frequency space, is by means of the so called Heisenberg inequality

$$\int x^2 \left|f(x)\right|^2 \mathrm{d}x \int k^2 \left|\hat{f}(k)\right|^2 \mathrm{d}k \geqslant \frac{1}{4}\|f\|^4.$$

This is a very basic property of the Fourier transform on the real line proved by H. Weyl and W. Pauli by using the two most basic tools of analysis: integration by parts and Schwarz's inequality. A nice account is given in [3]. The account in [2] stops short of using Fourier language and this step is taken in [3]. Sometimes a more complicated looking expression is given but it is easy to reduce the more general case to the one above.

Interest in these topics runs in areas ranging from quantum mechanics, where Heisenberg formulated it first, to signal processing where D. Gabor, see [9], may have been the first one to state it. On the other hand, these considerations underlie a lot of the work of C. Shannon and the remarkable series of papers by different combinations of D. Slepian, H. Landau, and H. Pollak. For a nice survey of this work see [11].

There are even applications of this same idea in tomography, [1], where one sees that the conflict between *spatial and contrast resolution* is governed by the same mathematical limitations. The issue of picking good sampling schemes is discussed in [17].

Of a more recent vintage is a beautiful paper by D. Donoho and P. Stark [8]. Here one introduces two subsets P and Q in physical and frequency space, respectively, and then the main characters are the operator of "time limiting", i.e., restricting a function in physical space to the set P and the operator of "band limiting" which cuts off all the frequencies outside the set Q. The crucial ingredient in [8] is the observation that the norm of the operator PQ is properly bounded.

☆ This paper is partially supported by NSF Grant FD9971151.
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doi:10.1016/S1063-5203(03)00033-2

In this version of the Heisenberg principle one obtains a lower bound on the product of the measures of the sets P and Q. The beauty of the proof is that it relies on very simple facts, just as the proof of the classical Heisenberg inequality. This line of work was extended to groups in [10] and to Gelfand pairs in [7].

The operators P and Q mentioned above play a crucial role—when P and Q are intervals—in the work of Slepian, Landau, and Pollak mentioned earlier. They were considered for arbitrary sets in some work of Fuchs mentioned in [8]. In the first instance they give rise to a rich collection of algebraic miracles whose analysis has led in turn to more miracles. For an introduction to this line of work see, for instance, [12,13].

Motivated by the study of the "scattering transform" I once wrote, see [14], a paper showing that the injunction about *double concentration* does not hold for this *nonlinear transform* in spite of the fact that its linearization at zero potential is given by the Fourier transform. Since I was dealing in [14] with the integers as physical space I needed to quote a version of the Heisenberg inequality in the context of the circle (as opposed to that of the real line mentioned above). I could not find one in the literature so I did it myself by adapting Weyl's proof and observing that one little extra condition had to be added to cancel the contribution form boundary terms in the process of integrating by parts. This proof has now appeared in the textbook literature, see [16].

We finally come to the point of this paper: what about the case of the N roots of unity?, i.e., the case of the discrete Fourier transform.

One can expect that some little natural extra condition like the one needed for the circle might be needed, but a bit of experimentation reveals a more complex situation. For example, a constant function violates a naive adaptation of the inequality that holds in the real line. This was the problem that surfaced in the case of the circle too and that lead us to consider only functions that vanish at *the appropriate replacement* of infinity. However, for the case of the roots of unity, we have troubles also with functions supported at the origin.

We present below a form of the Heisenberg inequality inspired in the initial formulation in quantum mechanics and its generalization due to E. Schroedinger and independently to H. Robertson back in 1930, see [4, p. 135]. This generalized form is also given in [3] as exercise 1 on p. 119.

The difficulty in patching up easily the original form of the inequality on the real line can be traced to the trivial fact that you cannot have in finite dimensions two operators whose commutator is a nonzero multiple of the identity. For a reader that may, at this point, feel a bit queasy about this reconnection of Fourier stuff with quantum mechanics let me argue that this kind of interconnection is a very fruitful one. I just recall that the work of G. Wilson, see [5], in connection with earlier work of Kazhdan, Kostant, and Sternberg, see [6], classifies completely all the *approximate realizations* of the *canonical commutation relations* in a finite-dimensional situation. This plays a crucial role in the study of Calogero–Moser systems as well as in the study of the KdV and KP equations. Pursuing this matter here would, however, take us rather far from the task at hand. It is a pleasure to thank Barry McCoy for timely remarks.

2. An appropriate form of the Heisenberg inequality

Our physical space consists of the integers $\bmod N$, or equivalently the set of Nth-roots of unity. Frequency space is a different copy of the same set. On the finite-dimensional space of complex valued functions a defined on physical space we have the DFT defined by the usual rule

$$\hat{a}_j = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} a_k \omega^{-kj}, \quad \omega \equiv \mathrm{e}^{i2\pi/N}.$$

It is convenient to consider a general operator Q meant to represent *position* as well as the operator of centered differences P meant to represent *momentum*. To make these operators selfadjoint we take them as

$$Q = \begin{bmatrix} q_0 & & 0 \\ & \ddots & \\ 0 & & q_{N-1} \end{bmatrix}$$

and

$$P = i \begin{bmatrix} 0 & 1 & 0 & \dots & -1 & \\ -1 & 0 & 1 & \dots & 0 & \\ 0 & -1 & 0 & \dots & 0 & \\ & & & \dots & & \\ 1 & 0 & 0 & \dots & -1 & 0 \end{bmatrix}.$$

For the time being the (real) values of q_i are quite general. One can derive an inequality for any such choice of q_i but we will see later that there is a natural choice.

We then get for their commutator the skew-adjoint operator $[Q, P]$ given by

$$[Q, P] = i \begin{bmatrix} 0 & q_0 - q_1 & \dots & \dots & q_{N-1} - q_0 \\ q_0 - q_1 & 0 & q_1 - q_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ q_{N-1} - q_0 & 0 & \dots & q_{N-2} - q_{N-1} & 0 \end{bmatrix}.$$

It is convenient now to introduce some simple and very general considerations for linear operators that are at the core of the work of Schroedinger and Robertson mentioned earlier.

The expected value of any selfadjoint operator A in state a is defined by the expression

$$\langle A \rangle = \langle Aa, a \rangle.$$

Since the operator A is selfadjoint the expected value of its square is given by

$$\langle A^2 a \rangle = \langle Aa, Aa \rangle = \|Aa\|^2.$$

If A and B are selfadjoint we have

$$AB = \frac{1}{2} T + \frac{i}{2} S$$

with $T = AB + BA$ and $S = 1/i\ [A, B]$ both selfadjoint.

Then the product of the expected values of their squares satisfies the inequality

$$\begin{aligned} \langle A^2 \rangle \langle B^2 \rangle = \|Aa\|^2 \|Ba\|^2 \geqslant \left| \langle Aa, Ba \rangle^2 \right| &= \frac{1}{4} \left| \langle a, (T + iS)a \rangle \right|^2 \\ &= \frac{1}{4} \left| \langle a, Ta \rangle - i \langle a, Sa \rangle \right|^2 = \frac{1}{4} \left(\langle a, Ta \rangle^2 + \langle a, Sa \rangle^2 \right). \end{aligned}$$

If we consider the expected value of the square of the operator P introduced above we get

$$\langle P^2\rangle = \sum_{0}^{N-1} |a_{j+1} - a_{j-1}|^2,$$

which is converted by using the properties of the DFT into the expression

$$\sum_{j=0}^{N-1} \bigl|(\omega^j - \omega^{-j})\hat{a}_j\bigr|^2 = 4\sum_{j=0}^{N-1} \sin^2\frac{2\pi j}{N}|\hat{a}_j|^2.$$

Note. This last step is entirely similar to the step in going from the second factor in the left-hand side of Heisenberg inequality in [2] to the same factor in [3].

At this point we can choose the operator Q denoting position in a fashion that treats physical and frequency space on the same footing. With this in mind we set

$$q_j = \sin\frac{2\pi j}{N}.$$

This gives for the commutator $[Q, P]$ the matrix

$$[Q, P]_{m,n} = i\left(\sin\frac{2\pi m}{N} - \sin\frac{2\pi n}{N}\right)(-1)^{n-m} \quad \text{if } |n-m| = 1$$

and $[Q, P]_{m,n} = 0$ otherwise, whereas for the term $QP + PQ$ we get

$$[Q, P]_{m,n} = i\left(\sin\frac{2\pi m}{N} + \sin\frac{2\pi n}{N}\right)(-1)^{n-m} \quad \text{if } |n-m| = 1$$

and $(QP + PQ)_{m,n} = 0$ otherwise, and our inequality finally reads

$$\begin{aligned} &4\sum \sin^2\left(\frac{2\pi j}{N}\right)|a_j|^2 \sum \sin^2\left(\frac{2\pi k}{N}\right)|\hat{a}_j|^2 \\ &\quad\geqslant \frac{1}{4}\left(\sum_{j=0}^{N-1}\left(\sin\left(\frac{2\pi j}{N}\right) - \sin\left(\frac{2\pi(j+1)}{N}\right)\right)(a_j\bar{a}_{j+1} + \bar{a}_j a_{j+1})\right)^2 \\ &\qquad - \frac{1}{4}\left(\sum_{j=0}^{N-1}\left(\sin\left(\frac{2\pi j}{N}\right) + \sin\left(\frac{2\pi(j+1)}{N}\right)\right)(a_j\bar{a}_{j+1} - \bar{a}_j a_{j+1})\right)^2. \end{aligned}$$

Notice that if a is real (or imaginary) valued the second term vanishes. One should remark that in the classical case of the real line there is also an extra term that is usually left out. This second term corresponds, once again, to the anticommutator of the operators in question. It just happens that for a real valued function, namely an appropriate Gaussian, the first term alone gives equality. Furthermore, whereas the first term has a nice form, just the fourth power of the norm of a, the second term given by

$$i\int \bigl(2xf'(x) + f(x)\bigr)\bar{f}(x)\,\mathrm{d}x$$

is not so nice and compact. In the case of the DFT neither of the two terms has a nice compact form and it may be better to keep both of them.

The issue of equality between the left-hand side and the right-hand side of the inequality above reduces to the question of finding a such that

$$Qa \cong Pa.$$

It turns out that for N odd this has a one-dimensional space of solutions, whereas for N even there is a two-dimensional space of solutions. None of these bear much of a connection with the natural analog of the Gaussian in this context, though. For some work in this regard see [15].

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