On the invariant mean and statistical convergence

M. Mursaleen a,∗, Osama H.H. Edely b

a Department of Mathematics, Aligarh Muslim University, Aligarh-202002, India
b Department of Mathematics, Tafila Technical University, Tafila, P.O. Box: 179, Zip code 66110, Jordan

ABSTRACT

Two concepts – one of almost convergence and the other of statistical convergence – play a very active role in recent research on summability theory. The definition of almost convergence introduced by Lorentz [G.G. Lorentz, A contribution to theory of divergent sequences, Acta Math. 80 (1948) 167–190] originated from the concept of the Banach limit, while the statistical convergence introduced by Fast [H. Fast, Sur la convergence statistique, Colloq. Math. 2 (1951) 241–244] was defined through the concept of density. Both involve non-matrix methods of summability and they are incompatible. In this work we define two new kinds of summability methods by using these two mutually incompatible concepts of the Banach limit and of density to deal with those sequences which are statistically convergent but not almost convergent or vice versa.

1. Introduction and preliminaries

Let c and l∞ denote the spaces of all convergent and bounded sequences, respectively, and note that c ⊂ l∞. In the theory of sequence spaces, a beautiful application of the well known Hahn–Banach Extension Theorem gave rise to the concept of the Banach limit. That is, the lim functional defined on c can be extended to the whole of l∞ and this extended functional is known as the Banach limit [1]. In 1948, Lorentz [2] used this notion of a weak limit to define a new type of convergence, known as the almost convergence. Later on, Raimi [3] gave a slight generalization of almost convergence and named it the σ-convergence. There is another notion of convergence known as the statistical convergence which was introduced by Fast [10] who attributed this concept to Hugo Steinhaus. Actually Henry Fast had heard about this concept from Steinhaus, but in fact it was Antoni Zygmund who proved theorems on the statistical convergence of Fourier series in the first edition of his book [4, pp. 181–188] where he used the term “almost convergence” in place of statistical convergence and at that time this idea was not recognized much. Since the term “almost convergence” was already in use (as described above), Fast had to choose a different name for his concept and “statistical convergence” was most suitable. For our convenience, we use a more logical term for the statistical convergence: δ-convergence (because it is related to density δ). Note that these two methods, i.e. σ-convergence and δ-convergence, are non-matrix methods of summability and are incompatible [5].

In this work, we define two new kinds of summability methods by using these two mutually incompatible concepts, i.e. of the invariant mean (or Banach limit) and of density. That is, we define σ-statistical convergence and statistical σ-convergence and establish some interesting results for them.

First we recall the following:

(i) Let σ be a mapping of the set of positive integers N into itself. A continuous linear functional ϕ defined on the space l∞ of all bounded sequences is called an invariant mean (or a σ-mean; cf. [3]) if it is non-negative, normal and ϕ(x) = ϕ((xσ(n))).

∗ Corresponding author. Tel.: +91 571 2720241.
E-mail addresses: mursaleenm@gmail.com (M. Mursaleen), osamaedely@yahoo.com (O.H.H. Edely).

0893-9659/$ – see front matter © 2009 Elsevier Ltd. All rights reserved.
doi:10.1016/j.aml.2009.06.005
(ii) A sequence $x = (x_k)$ is said to be $\sigma$-convergent to the number $L$ if and only if all of its $\sigma$-means coincide with $L$, i.e., $\varphi(x) = L$ for all $\varphi$. A bounded sequence $x = (x_k)$ is $\sigma$-convergent (cf. [6]) to the number $L$ if and only if $\lim_{p \to \infty} t_{pm} = L$ uniformly in $m$, where

$$t_{pm} = \frac{x_m + x_{\sigma^1(m)} + x_{\sigma^2(m)} + \cdots + x_{\sigma^p(m)}}{p + 1}.$$

We denote the set of all $\sigma$-convergent sequences by $V_\sigma$ and in this case we write $x_k \to L(V_\sigma)$ and $L$ is called the $\sigma$-limit of $x$. Note that a $\sigma$-mean extends the limit functional on $c$ in the sense that $\varphi(x) = \lim x$ for all $x \in c$ if and only if $\sigma$ has no finite orbits (cf. [7]) and $c \subset V_\sigma \subset l_\infty$.

If $\sigma$ is a translation then the $\sigma$-mean is called a Banach limit and $\sigma$-convergence is reduced to the concept of almost convergence introduced by Lorentz [2].

(iii) A bounded sequence $x = (x_k)$ is said to be strongly $\sigma$-convergent (cf. [8]) to the limit $L$ if $\lim_{p \to \infty} \frac{1}{p} \sum_{j=1}^{p} |x_{\sigma^j(m)} - L| = 0$, uniformly in $m$.

We denote the set of all strongly $\sigma$-convergent sequences by $[V_\sigma]$ and in this case we write $x_k \to L[V_\sigma]$.

(iv) Let $K \subseteq \mathbb{N}$ and $K_n = \{k \leq n : k \in K\}$. Then the natural density of $K$ (cf. [9]) is defined by $\delta(K) = \lim_n n^{-1}|K_n|$ if the limit exists, where $|K_n|$ denotes the cardinality of $K_n$.

(v) A sequence $x = (x_k)$ of real numbers is said to be statistically convergent to $L$ provided that for every $\epsilon > 0$ the set $K_\epsilon := \{k \in \mathbb{N} : |x_k - L| \geq \epsilon\}$ has natural density zero (cf. [10,11]). In this case $L$ is called the $\epsilon$-limit of $x$.

For $\sigma$-convergence and $\delta$-convergence of double sequences, we refer the reader to [12–14].

**Remark 1.1.** Note that:

(a) a convergent sequence is also $\delta$-convergent as well as $\sigma$-convergent;
(b) a $\sigma$-convergent sequence is also bounded;
(c) a $\delta$-convergent sequence may or may not be bounded.

### 2. Some new definitions and examples

Now, we introduce some new concepts by using the notions of density and the $\sigma$-mean.

**Definition 2.1.** Let $\sigma = (m + 1, m + p)$ denote the cardinality of the set $\{\sigma(m) \leq m \leq \sigma^p(m) : k \in K\}$, and write $N_p = \min_m \sigma_0(m + 1, m + p)$, $N^p = \max_m \sigma_0(m + 1, m + p)$. It is easy to see that the limits $\delta_0(K) = \lim_{p \to \infty} \frac{N_p}{p}$ and $\delta_\sigma(K) = \lim_{p \to \infty} \frac{N^p}{p}$ exist. These are called respectively the lower and upper $\sigma$-density of the set $K$. If $\delta_\sigma(K) = \delta_\sigma(K)$, then the common value $\delta_\sigma(K)$ is called the $\sigma$-density of the set $K$. It is clear that for $K \subseteq \mathbb{N}$, $\delta_\sigma(K) \leq \delta(K) \leq \delta_\sigma(K)$. For $\sigma(n) = n + 1$, $\sigma$-density is reduced to uniform density [15].

**Definition 2.2.** A sequence $x = (x_k)$ is said to be $\sigma$-statistically convergent to $L$ if for every $\epsilon > 0$ the set $K_\epsilon := \{k \in \mathbb{N} : |x_k - L| \geq \epsilon\}$ has $\sigma$-density zero, i.e., $\delta_\sigma(K_\epsilon) = 0$. In this case we write $\sigma(\delta)$- $\lim x = L$. That is,

$$\lim_{p \to \infty} \frac{1}{p} \left| \left\{ \sigma(m) \leq k \leq \sigma^p(m) : |x_k - L| \geq \epsilon \right\} \right| = 0, \text{ uniformly in } m.$$

**Definition 2.3.** A sequence $x = (x_k)$ is said to be statistically $\sigma$-convergent to $L$ if for every $\epsilon > 0$ the set $K_\epsilon := \{k \in \mathbb{N} : \varphi(|x_k - L|) \geq \epsilon\}$ has natural density zero, i.e., $\delta(K_\epsilon) = 0$. In this case we write $\delta(\sigma)$- $\lim x = L$. That is,

$$\lim_{n \to \infty} \frac{1}{n} \left| \left\{ p \leq n : |t_{pm} - L| \geq \epsilon \right\} \right| = 0, \text{ uniformly in } m.$$

For some related methods, we refer the reader to [16,9,17].

Now, we extend the definition of the set $[V_\sigma]$.

**Definition 2.4.** A sequence $x = (x_k)$ is said to be strongly $\sigma_\epsilon$-convergent ($0 < \epsilon < \infty$) to the limit $L$ if $\lim_{p \to \infty} \frac{1}{p} \sum_{j=1}^{p} |x_{\sigma^j(m)} - L|^\epsilon = 0$, uniformly in $m$, and we write this as $x_k \to L[V_\sigma]_\epsilon$. In this case $L$ is called the $[V_\sigma]_\epsilon$-limit of $x$. Note that, for $q = 1$, $[V_\sigma]_1 = \infty = [V_\sigma]$. 

**Remark 2.5.** (i) If $x$ is $\sigma$-statistically convergent then $x$ is statistically convergent and $\sigma(st)$- $\lim x = st$- $\lim x$.

(ii) Statistical convergence implies statistical $\sigma$-convergence, and hence by (i) we get that $\sigma$-statistical convergence $\Rightarrow$ statistical convergence $\Rightarrow$ statistical $\sigma$-convergence.

(iii) $\sigma$-convergence implies statistical $\sigma$-convergence but not $\sigma$-statistical convergence.
Examples 2.6. Let $P$ be the set of all primes. Define $x = (x_k)$ by

$$x_k = \begin{cases} 1; & \text{if } k \in P, \\ 0; & \text{otherwise}. \end{cases}$$

Then it is not convergent but it is $\sigma$-statistically convergent, since $\delta_\sigma(P) = 0$. Therefore from Remark (ii) it is statistically convergent and statistically $\sigma$-convergent.

Examples 2.7. The sequence $x = (x_k)$ defined as

$$x_k = \begin{cases} 1; & \text{if } k \text{ is odd}, \\ 0; & \text{if } k \text{ is even} \end{cases}$$

is $\sigma$-convergent to $1/2$ (for $\sigma(n) = n + 1$) and hence statistically $\sigma$-convergent to $1/2$ but it is neither statistically convergent nor $\sigma$-statistically convergent.

Examples 2.8 ([15]). Let $A = \bigcup_{k=1}^\infty \{10^k + 1, 10^k + 2, \ldots, 10^k + k\}$. Then $\delta(A) = 0, \delta_\sigma(A) = 0, \delta_\sigma(A) = 1$. Let $\sigma(n) = n + 1$. Define $x = (x_k)$ by

$$x_k = \begin{cases} 1; & \text{if } k \in A, \\ 0; & \text{if } k \not\in A. \end{cases}$$

Then $\text{st}\lim x = 0$, but it is not $\sigma$-statistically convergent.

3. Main results

We prove here some relations between $\sigma$-statistical convergence, statistical $\sigma$-convergence, strong $\sigma$-convergence, and $\sigma$-convergence.

In our first theorem we establish the relation between our two newly defined concepts of $\sigma$-statistical convergence and statistical $\sigma$-convergence.

Theorem 3.1. If a sequence $x = (x_k)$ is bounded and $\sigma$-statistically convergent to $L$ then it is statistically $\sigma$-convergent to $L$ but not conversely.

Proof. Let $x = (x_k)$ be bounded and $\sigma$-statistically convergent to $L$. Write $K_\sigma(\epsilon) := \{\sigma(m) \leq k \leq \sigma^p(m) : |x_k - L| \geq \epsilon\}$. Then

$$|t_{pm} - L| = \left| \frac{1}{p} \sum_{j=1}^p x_{\sigma^p(m)} - L \right| = \left| \frac{1}{p} \sum_{k=\sigma(m)} x_k - L \right| = \left| \frac{1}{p} \sum_{k=\sigma(m)} (x_k - L) \right|,$$

$$\leq \frac{1}{p} \sum_{k=\sigma(m)} (x_k - L) \leq \frac{1}{p} (\sup_{k} |x_k - L|)(K_\sigma(m + 1, m + p)),

\leq \frac{1}{p} (\sup_{k} |x_k - L|)(\max_{m} K_\sigma(m + 1, m + p)),

= \frac{1}{p} (\sup_{k} |x_k - L|)N_p \to 0 \quad \text{as } p \to \infty,$$

which implies that $t_{pm} \to L$ as $p \to \infty$, uniformly in $m$. That is, $x$ is $\sigma$-convergent to $L$ and hence statistically $\sigma$-convergent to $L$.

For the converse, consider the case $\sigma(n) = n + 1$ and the sequence $x = (x_k)$ defined as

$$x_k = \begin{cases} 1; & \text{if } k \text{ is odd}, \\ -1; & \text{if } k \text{ is even} \end{cases}$$

Of course this sequence is not $\sigma$-statistically convergent. On the other hand $x$ is $\sigma$-convergent to $0$ and hence statistically $\sigma$-convergent to $0$.

This completes the proof of the theorem. \hfill \Box

The next theorem gives the relation between $\sigma$-statistical convergence and strong $\sigma_q$-convergence.

Theorem 3.2. (a) If $0 < q < \infty$ and a sequence $x = (x_k)$ is strongly $\sigma_q$-convergent to the limit $L$, then it is $\sigma$-statistically convergent to $L$.

(b) If $x = (x_k)$ is bounded and $\sigma$-statistically convergent to $L$ then $x_k \to L[V_\sigma]_q$. 

proof. (a) If $0 < q < \infty$ and $x_k \rightarrow [V_{\sigma}]_q$, then as $p \rightarrow \infty$,

$$0 \leftarrow \frac{1}{p} \sum_{j=1}^{p} |x_{\sigma(j)} - L|^q \geq \frac{1}{p} \sum_{|x_{\sigma(j)} - L|^q}^{p} \geq \frac{e^q}{p} K_\epsilon (m, m + p) \geq \frac{e^q}{N^q}.$$

That is, $\lim_{p \rightarrow \infty} N^q/p = 0$ and so $\delta_\epsilon(K_\epsilon) = 0$, where $K_\epsilon : \{k \leq n : |x_k - L| \geq \epsilon\}$. Hence $x = (x_k)$ is $\sigma$-statistically convergent to $L$.

(b) Suppose that $x = (x_k)$ is bounded and $\sigma$-statistically convergent to $L$. Then for $\epsilon > 0$, we have $\delta_\epsilon(K_\epsilon) = 0$. Since $x \in l_\infty$, there exists $M > 0$ such that $|x_k - L| \leq M$ ($k = 1, 2, \ldots$). For every $m \in \mathbb{N}$, we have

$$\frac{1}{p} \sum_{j=1}^{p} |x_{\sigma(j)} - L|^q = \frac{1}{p} \sum_{k=1}^{\sigma(m)} |x_k - L|^q + \frac{1}{p} \sum_{k=\sigma(m)}^{\sigma(m) + m} |x_k - L|^q = S_1(m, p) + S_2(m, p),$$

where $S_1(m, p) = \frac{1}{p} \sum_{k=1}^{\sigma(m)} |x_k - L|^q$ and $S_2(m, p) = \frac{1}{p} \sum_{k=\sigma(m)}^{\sigma(m) + m} |x_k - L|^q$.

Now if $k \notin K_\epsilon$ then $S_1(m, p) < \epsilon^q$. For $k \in K_\epsilon$, we have $S_2(m, p) \leq (sup |x_k - L|)(sup_{m \geq 0} K_\epsilon(m + 1, m + p)/p) \leq M \frac{2^p}{p} \rightarrow 0$, as $p \rightarrow \infty$, since $\delta_\epsilon(K_\epsilon) = 0$. Hence $x_k \rightarrow [V_{\sigma}]_q$.

This completes the proof of the theorem. \(\square\)

**Corollary 3.3.** From Theorem 3.2, we have

$$l_\infty \cap \sigma(st) \subseteq [V_\sigma] \subset V_\sigma.$$

In the next result we characterize statistically $\sigma$-convergent sequences through the $\sigma$-convergence of subsequences.

**Theorem 3.4.** A sequence $x = (x_k)$ is statistically $\sigma$-convergent to $L$ if and only if there exists a set $K = \{k_1 < k_2 < \cdots < k_n < \cdots\} \subseteq \mathbb{N}$ such that $\delta(K) = 1$ and $\sigma$-lim $x_{k_n} = L$.

**Proof.** Suppose that there exists a set $K = \{k_1 < k_2 < \cdots < k_n < \cdots\} \subseteq \mathbb{N}$ such that $\delta(K) = 1$ and $\sigma$-lim $x_{k_n} = L$. Then there is a positive integer $N$ such that for $n > N$,

$$\varphi(|x_k - L|) < \epsilon.$$  \(\text{(1)}\)

Put $K_\epsilon(\varphi) : \{n \in \mathbb{N} : \varphi(|x_k - L|) \geq \epsilon\}$ and $K' = \{k_{n+1}, k_{n+2}, \ldots\}$. Then $\delta(K') = 1$ and $K_\epsilon(\varphi) \subseteq \mathbb{N} - K'$ which implies that $\delta(K_\epsilon(\varphi)) = 0$. Hence $x = (x_k)$ is statistically $\sigma$-convergent to $L$.

Conversely, let $x = (x_k)$ be statistically $\sigma$-convergent to $L$. For $r = 1, 2, 3, \ldots$, put $K_r(\varphi) : \{j \in \mathbb{N} : \varphi(|x_j - L|) \geq 1/r\}$ and $M_r(\varphi) : \{j \in \mathbb{N} : \varphi(|x_j - L|) < 1/r\}$. Then $\delta(K_r(\varphi)) = 0$ and

$$M_1(\varphi) \supseteq M_2(\varphi) \supseteq \cdots M_r(\varphi) \supseteq M_{r+1}(\varphi) \supseteq \cdots$$  \(\text{(2)}\)

and

$$\delta(M_r(\varphi)) = 1, \quad r = 1, 2, 3, \ldots.$$  \(\text{(3)}\)

Now we have to show that for $j \in M_r(\varphi)$, $(x_k)$ is $\sigma$-convergent to $L$. Suppose that $(x_k)$ is not $\sigma$-convergent to $L$. Therefore there is $\epsilon > 0$ such that $\varphi(|x_k - L|) \geq \epsilon$ for infinitely many terms. Let $M_r(\varphi) : \{j \in \mathbb{N} : \varphi(|x_j - L|) < \epsilon\}$ and $\epsilon > 1/r$ ($r = 1, 2, 3, \ldots$). Then

$$\delta(M_r(\varphi)) = 0,$$  \(\text{(4)}\)

and by (2), $M_r(\varphi) \subseteq M_{r+1}(\varphi)$. Hence $\delta(M_r(\varphi)) = 0$, which contradicts (3) and therefore $(x_k)$ is $\sigma$-convergent to $L$.

This completes the proof of the theorem. \(\square\)

Similarly, along the same lines we can prove the following dual statement:

**Theorem 3.5.** A sequence $x = (x_k)$ is $\sigma$-statistically convergent to $L$ if and only if there exists a set $K = \{k_1 < k_2 < \cdots < k_n < \cdots\} \subseteq \mathbb{N}$ such that $\delta_\epsilon(K) = 1$ and $\lim x_{k_n} = L$.  \(\square\)
Acknowledgement

The research of the first author was supported by the Department of Science and Technology, New Delhi, under grant number SR/S4/M5:505/07.

References