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# Bounds for the Solutions of Boundary Value Problems

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## INTRODUCTION

In Section 1 of this paper bounds for a linear boundary value problem are derived by means of a differential inequality. The idea is to compare the solution for an arbitrary domain of given volume with that for the sphere of the same volume. Similar investigations have already be done by Payne [8] for the torsion problem. The main result, namely, Theorem 1.1, was conjectured by McNabb [7]. Generalizations to boundary value problems with mixed boundary conditions are also indicated. They depend very much upon a geometrical isoperimetric inequality [2]. Section 2 deals with nonlinear Dirichlet problems. With the help of the method of upper and lower solutions [1], and the results of the previous section, the existence of a solution is established. The ideas resemble those of [4], where a simpler class of nonlinear problems has been studied. Bounds for the solutions of a nonlinear problem can also be found in [3].

## 1. BOUNDS FOR A LINEAR PROBLEM

**1.1.** Let $D$ be a domain in $R^N$ with a smooth boundary $\partial D$. Consider the boundary value problem

$$\begin{aligned} \Delta u + \alpha u + 1 &= 0 \qquad \text{in} \qquad D, \\ u &= 0 \qquad \text{on} \qquad D. \end{aligned} \tag{1.1}$$

Let $\lambda_1$ be the lowest eigenvalue of the problem

$$\begin{aligned} \Delta\varphi + \lambda\varphi &= 0 \qquad \text{in} \qquad D \\ \varphi &= 0 \qquad \text{on} \qquad \partial D. \end{aligned} \tag{1.2}$$

*Throughout this paper we shall assume that* $\alpha < \lambda_1$. Since $\alpha$ does not coincide with an eigenvalue of problem (1.2), (1.1) has exactly one solution [5, pp. 136, 152]. Moreover, it is analytic in $D$.

LEMMA 1.1. (a) *If* $\alpha < \lambda_1$, *the solution* $u$ *of* (1.1) *is positive in* $D$.

(b) *Furthermore, we have* $\alpha u + 1 \geqslant 0$ *in* $D$.

*Proof.* Let $v = \alpha u + 1$. Then $v$ is a solution of the boundary value problem $\Delta v + \alpha v = 0$ in $D$ and $v = 1$ on $\partial D$. Suppose that $v$ is negative in a subdomain $D^- \subset D$. Consider in $D^-$ the eigenvalue problem (1.2). It is well known that the eigenfunction $\varphi_1^-(x)$ corresponding to the lowest eigenvalue $\lambda_1^-$ does not change sign and can therefore be taken to be positive in $D^-$ [5, p. 452]. Here $x$ stands for a generic point $(x_1, x_2, ..., x_N)$ in $R^N$. By the Green's identity it follows that

$$-\alpha \int_{D^-} v\varphi_1^- \, dx = \int_{D^-} \varphi_1^- \Delta v \, dx = \int_{D^-} v \Delta \varphi_1^- \, dx = -\lambda_1 \int_{D^-} v\varphi_1^- \, dx,$$

$$(dx = dx_1 \, dx_2 \cdots dx_N), \qquad (1.3)$$

and therefore $\alpha = \lambda_1^-$. On the other hand, we have [5, p. 398] $\lambda_1^- > \lambda_1$ and therfore $\alpha > \lambda$, which contradicts our assumption. Thus, $v = \alpha u + 1 \geqslant 0$ in $D$. Since $\Delta u + \alpha u + 1 = 0$ in $D$, we have $\Delta u \leqslant 0$ in $D$. From the maximum principle and the boundary condition $u = 0$ on $\partial D$, it follows that $u(x) > 0$ in $D$.

We shall use the notations

$$M = \max_{x \in \bar{D}} u(x), \qquad D(t) = \{x \in D; u(x) \geqslant t\},$$

and

$$\Gamma(t) = \{x \in D; \; u(x) = t\}.$$

Because of the analyticity of $u(x)$, $\Gamma(t)$ is an analytic $(N - 1)$-dimensional surface. The function $a(t)$ denotes the volume $\int_{D(t)} dx$, and $t(a)$ is its inverse. $t(a)$ is a continuous, strictly decreasing function with $t(0) = M$ and $t(A) = 0$, where $A = \int_D dx$. We define

$$H(a) = \int_{D(t(a))} [\alpha u + 1] \, dx. \qquad (1.4)$$

Introducing $a$ as a new variable, we obtain

$$H(a) = \int_0^a [\alpha t(\tilde{a}) + 1] \, d\tilde{a}, \qquad (1.5)$$

and, observing Lemma 1.1(b),

$$dH/da \equiv H'(a) = \alpha t(a) + 1 \geqslant 0, \tag{1.6}$$

$$H''(a) = \alpha(dt/da). \tag{1.7}$$

If $dt$ is sufficiently small, the volume between $\Gamma(t)$ and $\Gamma(t + dt)$ is

$$da = \oint_{\Gamma(t)} dn\, dS + o(dt). \tag{1.8}$$

(This formula holds because $\Gamma(t)$ and $\Gamma(t + dt)$ are analytic $(N - 1)$-dimensional surfaces.) Here, $dn$ denotes the piece of the normal between $\Gamma(t)$ and $\Gamma(t + dt)$, and $dS$ is the $(N - 1)$-dimensional surface element. Thus,

$$\frac{da}{dt} = -\oint_{\Gamma(t)} \frac{dS}{|\operatorname{grad} u|}. \tag{1.9}$$

By the Schwarz inequality we have

$$\oint_{\Gamma(t)} \frac{dS}{|\operatorname{grad} u|} \oint_{\Gamma(t)} |\operatorname{grad} u|\, dS \geqslant \left\{\oint_{\Gamma(t)} dS\right\}^2. \tag{1.10}$$

The geometrical isoperimetric inequality [6, p. 195] states that

$$\oint_{\Gamma(t)} dS \geqslant N\omega_N^{1/N} a^{(N-1)/N}, \tag{1.11}$$

where $\omega_N$ is the volume of the unit sphere $\{|x| < 1\}$. Equality holds if and only if $D(t)$ is a sphere. In view of (1.11), (1.1), and (1.10) it follows that

$$\int_{\Gamma(t)} \frac{dS}{|\operatorname{grad} u|} \cdot H(a) \geqslant [N\omega_N^{1/N} a^{(N-1)/N}]^2 . \tag{1.12}$$

Define $p(a) = [N\omega_N^{1/N} a^{(N-1)/N}]^{-2}$. By (1.9) and (1.7) we have

$$-\alpha H(a)/H''(a) \geqslant [p(a)]^{-1}. \tag{1.13}$$

Since $dt/da \leqslant 0$ a.e. and therefore sign $H''(a) = -\text{sign}\,\alpha$, we have

$$\begin{aligned} H''(a) + \alpha p(a)\, H(a) &\geqslant 0 \qquad \text{if} \qquad \alpha > 0, \\ H''(a) + \alpha p(a)\, H(a) &\leqslant 0 \qquad \text{if} \qquad \alpha < 0. \end{aligned} \tag{1.14}$$

The inequality is valid for almost all $a \in (0, A)$. Equality holds if and only if $D(t)$ is a sphere. Since $u = 0$ on $\partial D$, and $u > 0$ in $D$, $H'(A) = 1$, and because of the definition of $H$, $H(0) = 0$.

Consider now problem (1.1) for the case where $D$ is the sphere $D^* = \{|x| < R\}$ with the same volume as $D$, i.e., $\omega_N R^N = A$. Its solution will be denoted by $u^*$. Analogously, we set $H^*(a) = \int_{D^*(t(a))} [\alpha u^* + 1]\, dx$ with $D^*(t) = \{x \in D^*; u^*(x) \geq t\}$, and $\lambda^*$ for the eigenvalues of (1.2) with $D$ replaced by $D^*$.

Assuming that $\alpha < \lambda_1^*$, and observing that $u^*$ is radially symmetric, we derive exactly in the same way as before the differential equation

$$H^{*\prime\prime}(a) + \alpha p(a)\, H^*(a) = 0 \tag{1.15}$$

with the boundary conditions

$$H^{*\prime}(A) = 1 \quad \text{and} \quad H^*(0) = 0.$$

Let us now compare the solutions of (1.14) with that of (1.15). Because of Lemma 1.1(b), $H(a)$ and $H^*(a)$ are positive in $(0, A)$. If we eliminate $p(a)$ in (1.14), we obtain

$$\alpha[H'' H^* - H^{*\prime\prime} H] = \alpha[H' H^* - H^{*\prime} H]' \geq 0.$$

Hence, $N(a) = \alpha[H'(a)\, H^*(a) - H^{*\prime}(a)\, H(a)]$ is a nondecreasing function of $a$. Because of $N(0) = 0$, we have

$$\alpha[H' H^* - H H^{*\prime}] \geq 0 \quad \text{and} \quad 0 \leq \alpha \frac{H' H^* - H H^{*\prime}}{H^{*2}} = \alpha \left(\frac{H}{H^*}\right)'.$$

These results are summarized in

LEMMA 1.2. (a) $N(a) = \alpha[H'(a)\, H^*(a) - H^{*\prime}(a)\, H(a)]$ *is a nondecreasing, positive function in* $(0, A)$.

(b) $Q(a) = \alpha(H(a)/H^*(a))$ *is nondecreasing in* $(0, A)$.

An evaluation of $N(a)$ at $a = A$ leads to

$$\alpha[H^*(A) - H(A)] \geq 0. \tag{1.16}$$

Since $\lim_{a \to 0}(H(a)/H^*(a)) = (\alpha M + 1)/(\alpha M^* + 1)$, $M^* = \max_{x \in \bar{D}^*} u^*(x)$, it follows from Lemma 1.2(b)and (1.16) that

$$\alpha \frac{\alpha M + 1}{\alpha M^* + 1} \leq \alpha \frac{H(A)}{H^*(A)} = \alpha \frac{H(A) - H^*(A)}{H^*(A)} + \alpha \leq \alpha. \tag{1.17}$$

Since $\alpha M^* + 1 \geq 0$ (cf. Lemma 1.1(b)),

$$M \leq M^*. \tag{1.18}$$

(1.18) holds as long as $\alpha \leq \min\{\lambda_1, \lambda_1^*\}$. In view of the Rayleigh–Faber–

Krahn inequality (cf. [9]), $\min\{\lambda_1, \lambda_1^*\} = \lambda_1^*$. If $r = |x|$, a straightforward calculation yields

$$u^*(r) = \frac{1}{\alpha}\left\{\frac{r^{(2-N)/2}J_{(N-2)/2}(\alpha^{1/2}r)}{R^{(2-N)/2}J_{(N-2)/2}(\alpha^{1/2}R)} - 1\right\} \tag{1.19}$$

where $J_k$ is the Bessel function of order $k$, and $R$ is the radius of $D^*$. We have thus proved

THEOREM 1.1. *Let $\alpha < \lambda_1^* = [\omega_N/A]^{2/N} j^2_{(N-2)/2}$, $j_k$ being the first zero of the Bessel function $J_k$. Then*

$$u(x) \leqslant \frac{1}{\alpha}\left\{\frac{(\alpha^{1/2}R)^{(N-2)/2}}{2^{(N-2)/2}\Gamma(N/2)\, J_{(N-2)/2}(\alpha^{1/2}R)} - 1\right\} \tag{1.20}$$

*where $\omega_N R^N = A$. Equality holds if $D$ is a sphere and $x$ is taken at the center.*

*Remark.* For $\alpha = 0$, (1.20) was proved by Payne [8].

EXAMPLE. If $N = 3$, (1.20) yields

$$u(x) \leqslant \frac{1}{|\alpha|}\left\{1 - \frac{|\alpha|^{1/2} R}{\sinh(|\alpha|^{1/2} R)}\right\} \qquad \text{for } \alpha < 0$$

and

$$u(x) \leqslant \frac{1}{\alpha}\left\{\frac{\alpha^{1/2}R}{\sin(\alpha^{1/2}R)} - 1\right\} \qquad \text{for } 0 < \alpha < \left(\frac{\pi}{R}\right)^2.$$

COROLLARY 1.1. *Let $\alpha < [\omega_N/A]^{2/N} j^2_{(N-2)/2}$. Then the equality sign in (1.20) holds if and only if $D$ is a sphere and $x$ is taken at the center.*

*Proof.* If $M = M^*$, it follows from Lemma 1.2(*b*) and (1.17) that

$$\alpha \leqslant \alpha\frac{H(a)}{H^*(a)} \leqslant \alpha\frac{H(A)}{H^*(A)} \leqslant \alpha. \tag{1.21}$$

Hence, $H(a) = H^*(a)$ in $[0, A]$, and therefore $H''(a) + \alpha p(a) H(a) = 0$ for all $a \in (0, A)$. As it was observed under (1.14), this relation implies that $D(t)$ is a sphere for all $t$, which proves the assertion.

*Remark.* From the proof of Corollary 1.1 it follows that, if $H(\hat{a}) = H^*(\hat{a})$ for some $\hat{a} \in (0, A)$, then $H(a) = H^*(a)$ for all $\hat{a} \leqslant a \leqslant A$.

**1.2.** This section deals with some norm estimates for the solution of problem (1.1).

LEMMA 1.3. *If $\alpha < \lambda_1^*$, then*

$$\alpha H(a) \leqslant \alpha H^*(a). \tag{1.22}$$

*Proof.* This statement is an immediate consequence of Lemma 1.2(b) and (1.17). Equation (1.22) leads to the following.

THEOREM 1.2. *If* $\alpha < \lambda_1{}^*$, *then*

$$\int_D u(x)\,dx \leqslant \int_{D^*} u^*(x)\,dx. \tag{1.23}$$

The next considerations are based on

LEMMA 1.4. *If* $0 < \alpha < \lambda_1{}^*$, *then*

$$H'(a) \leqslant H^{*\prime}(a) \qquad \text{in} \qquad (0, A]. \tag{1.24}$$

*Proof.* Let $\delta(a) = H(a) - H^*(a)$. Then $\delta(a)$ satisfies the differential inequality

$$\delta''(a) + \alpha p(a)\,\delta(a) \geqslant 0 \quad \text{in} \quad (0, A), \qquad \delta(0) = \delta'(A) = 0.$$

Hence, $\delta'(a) = \delta'(A) - \int_a^A \delta''(\tilde{a})\,d\tilde{a} \leqslant \alpha \int_a^A p(\tilde{a})\,\delta(\tilde{a})\,d\tilde{a}$. By (1.22), $\delta(a) \leqslant 0$ and therefore $\delta'(a) \leqslant 0$.

Consider the function $t(a)$ defined in Section 1.1 by means of $u(x)$. Let $\tau(r) = t(\omega_N r^N)$. For $\alpha < \lambda_1$, $\tau(r)$ is a mapping from $D^*$ onto $[0, M]$. Lemma 1.4 yields

COROLLARY 1.2. *If* $0 < \alpha < \lambda_1{}^*$, *then*

$$\tau(r) \leqslant u^*(r) \qquad \text{in} \qquad \left[0, \left(\frac{A}{\omega_N}\right)^{1/N}\right]. \tag{1.25}$$

$u^*(r)$ *is defined in* (1.19).

*Geometric Interpretation for* $N = 2$

Consider the body $B \subset R^3$ bounded by $D$ and the surface $(x_1, x_2, u(x_1, x_2))$, where $(x_1, x_2) \in D$. If $B$ is subject to a Schwarz symmetrization [9, p. 191], then it is transformed into a body $B^{**}$ bounded by $D^*$ and by the surface $(x_1, x_2, \tau(r))$ where $(x_1{}^*, x_2{}^*) \in D^*$. Let $B^*$ be the body bounded by $D^*$ and $(x_1, x_2, u^*(r))$, $(x_1{}^*, x_2{}^*) \in D^*$. Corollary 1.2 states that $B^{**} \subseteq B^*$, provided $0 < \alpha < \lambda_1{}^*$.

Let $G(t)$ be a nondecreasing and continuous function in $\bar{R}^+$. From the definition of the Lebesgue integral we conclude that

$$\int_D G(u(x))\,dx = \int_{D^*} G(\tau(r))\,dx.$$

Observing (1.25), we obtain

COROLLARY 1.3. *If* $0 < \alpha < \lambda_1^*$, *then*

$$\int_D G(u(x))\,dx \leqslant \int_{D^*} G(u^*(x))\,dx.$$

*Remark.* If we choose in particular $G(t) = t^p$ $(p \geqslant 0)$, then Corollary 1.3 yields the norm estimates

$$\| u \|_{L^p(D)} \leqslant \| u^* \|_{L^p(D^*)} .$$

For $\alpha < 0$, such an estimate has only been established for $p = 1$ (cf. Theorem 1.2).

**1.3.** Let $D \subset R^2$ be a simply connected domain. Suppose that its boundary is positively oriented and consists of two connected, piecewise analytic arcs $\Gamma_0$ and $\Gamma_1$ $(\Gamma_1 \cap \Gamma_0 = \phi)$. With $s$ we denote the arc-length on $\partial D$ and with $\kappa(s)$ $[0 \leqslant s \leqslant l]$ the curvature on $\Gamma_1$; $\kappa(s)$ is analytic except at the corners where it has to be interpreted as a Dirac measure. Consider the problem

$$\begin{aligned} \Delta u + \alpha u + 1 &= 0 \quad \text{in} \quad D, \\ u &= 0 \quad \text{on} \quad \Gamma_0, \\ (\partial u/\partial n) + \sigma u &= 0 \quad \text{on} \quad \Gamma_1, \end{aligned} \tag{1.26}$$

where $\sigma(s) \geqslant 0$ is a continuous function for $0 \leqslant s \leqslant l$, and $n$ is the outer normal.

Let $\lambda_1$ be the lowest eigenvalue of $\Delta\varphi + \lambda\varphi = 0$ in $D$, $\varphi = 0$ on $\Gamma_0$ and $(\partial\varphi/\partial n) + \sigma\varphi = 0$ on $\Gamma_1$. In the same way as Lemma 1.1 we derive

LEMMA 1.5. *If* $\alpha < \lambda_1$, *then*

(a) $u(x) > 0$ *in* $D$

(b) $\alpha u(x) + 1 \geqslant 0$ *in* $D$.

By the same method as in Section 1.1 we prove

THEOREM 1.3. *If* $\pi \geqslant \int_0^l \max\{0, \kappa\}\,ds \equiv \pi - \nu$, *and if* $\alpha < \nu j_0^2/2A$, *then*

$$u(x) \leqslant \frac{1}{\alpha}\left[\frac{1}{J_0[(2\alpha A/\nu)^{1/2}]} - 1\right]. \tag{1.27}$$

*Equality holds if and only if* $D$ *is the circular sector* $\{r < (2A/\nu)^{1/2},\ 0 \leqslant \theta \leqslant \nu\}$

*($r$, $\theta$ polar coordinates),* $\sigma \equiv 0$, $\Gamma_0 = \{r = (2A/\nu)^{1/2}, 0 \leqslant \theta \leqslant \nu\}$ *and* $\Gamma_1 = \{\theta = 0, \theta = \nu, 0 < r < (2A/\nu)^{1/2}\}$, *and $x$ is taken at the origin.*

*Proof.* (i) For $\alpha = 0$ we refer to [4].

(ii) Suppose that $\alpha \neq 0$ and that $\alpha < \nu j_0^2/2A$. By a result of [2], it follows that $\alpha < \lambda_1$. Let $D(t)$, $\Gamma(t)$, $a(t)$, and $H(a)$ have the same meaning as in Section 1.1. The relations (1.5)–(1.10) remain valid also for this case. Under our assumptions we have [2] instead of (1.11) the inequality

$$\int_{\Gamma(t)} ds \geqslant (2\nu a)^{1/2}. \tag{1.28}$$

Equality holds if and only if $\Gamma(t)$ is a concentric circular arc of the sector described in Theorem 1.3. If we insert (1.28) into (1.10) and observe that $\int_{\Gamma(t)} |\operatorname{grad} u| \, ds \leqslant H(a)$, we get instead of (1.13), $-\alpha H(a)/H''(a) \geqslant 2\nu a$, which yields for $a \in (0, A)$,

$$\alpha H''(a) + (\alpha^2/2\nu a) H(a) \geqslant 0. \tag{1.29}$$

A discussion similar to that for (1.14) implies (1.27). The solution of (1.26) for the extremal domain with $\sigma \equiv 0$ is

$$u^*(r) = \frac{1}{\alpha}\left[\frac{J_0(\alpha^{1/2} r)}{J_0[(2\alpha A/\nu)^{1/2}]} - 1\right]. \tag{1.30}$$

It is easily verified that in this case (1.29) holds with the equality sign.

The remarks of Section 1.2 apply also to this situation. In order to avoid a repetition we shall only mention the final results.

Let $G(t)$ be a positive, nondecreasing and continuous function in $R^+$.

COROLLARY 1.4. *If* $0 < \alpha < \nu j_0^2/2A$ *and if the conditions of Theorem* 1.3 *hold,*

$$\int_D G(u(x))\, dx \leqslant \int_{D^*} G(u^*(r))\, dx,$$

*where $u^*(r)$ is defined in* (1.30).

As a special case we obtain the norm estimates

$$\| u \|_{L^p(D)} \leqslant \| u^* \|_{L^p(D^*)}, \qquad p > 0.$$

*Remark.* Theorems 1.1 and 1.3 remain valid if $u(x)$ satisfies the differential inequality $\Delta u + \alpha u + 1 \geqslant 0$ in $D$ with $u(x) \leqslant 0$ on $\partial D$ or on $\Gamma_0$, and $\alpha < \lambda_1^*$ or $\alpha < \nu j_0^2/2A$, respectively. This is an immediate consequence of the generalized maximum principle.

## 2. Applications to Nonlinear Problems

**2.1.** Consider the nonlinear Dirichlet problem

$$\begin{aligned} \Delta v + f(v) &= 0 \quad \text{in} \quad D \subset R^N, \\ v &= 0 \quad \text{on} \quad \partial D. \end{aligned} \tag{2.1}$$

Suppose that $f(t)$ satisfies the condition

(A) $f(t) \leqslant f(0) + \alpha t \qquad \text{for all } t \geqslant 0, \quad f(0) \geqslant 0,$

where $\alpha < \lambda_1{}^* = [\omega_N/A]^{2/N} j^2_{(N-2)/2}$. The next statement is an immediate consequence of Theorem 1.1 and the remark at the end of Section 1.3.

Corollary 2.1. *Under the assumption* (A), *every positive solution of* (2.1) *satisfies the inequality*

$$v(x) \leqslant \frac{f(0)}{\alpha} \left[ \frac{(\alpha^{1/2}R)^{(N-2)/2}}{2^{(N-2)/2}\Gamma(N/2)\, J_{(N-2)/2}(\alpha^{1/2}R)} - 1 \right], \tag{2.2}$$

*where* $\omega_N R^N = A$.

**2.2.** In this part we use the bounds for the solution of the linear problem (1.1) in order to establish the existence of a positive solution of the problem

$$\begin{aligned} \Delta v + \alpha v + f(v) &= 0 \quad \text{in} \quad D \subset R^N \\ v &= 0 \quad \text{on} \quad \partial D. \end{aligned} \tag{2.3}$$

Let $\alpha < [\omega_N/A]^{2/N} j^2_{(N-2)/2}$ and let $f(t)$ satisfy

($A_1$) $f(t) \geqslant 0$ is a nondecreasing function for $t \geqslant 0$

($A_2$) $f(t)$ is Holder continuous in $\bar{R}^+$

($A_3$) $0 < \min_{t\geqslant 0}(f(t)/t) = 1/m$.

In view of a result by Amann [1], (2.3) has a positive solution, if we can find an upper solution $\psi(x) \geqslant 0$ such that

$$\begin{aligned} \Delta\psi + \alpha\psi + f(\psi) &\leqslant 0 \quad \text{in} \quad D, \\ \psi &= 0 \quad \text{on} \quad \partial D. \end{aligned} \tag{2.4}$$

Lemma 2.1. *Let $u(x)$ be the solution of* (1.1) *in $D$, and let $u^*(x)$ be the solution of the corresponding problem in $D^*$. Furthermore, $t_0 \geqslant 0$ is such that $t_0/f(t_0) = m$. If $M^* = \max_{x\in \bar{D}^*} u^*(x) \leqslant m$, then $\psi(x) = f(t_0)\, u(x)$ is an upper solution.*

*Proof.* Because of Theorem 1.1, we have $\psi(x) \leqslant M^* f(t_0)$ and according follows from $(A_1)$ that

$$0 = \Delta\psi + \alpha\psi + f(t_0) \geqslant \Delta\psi + \alpha\psi + f(\psi).$$

By Lemma 1.1, $\psi(x)$ is positive in $D$, and the proof is thus completed.
Lemma 2.1 and Amann's result [1] imply

THEOREM 2.1. *If the assumptions of Section* 2.2 *hold, and if* $M^* \leqslant m$, *then problem* (2.3) *has a positive solution* $v(x)$. *Moreover*, $v(x) \leqslant f(t_0)\, u(x)$.

**2.3.** In this section we derive an inequality related to problem (2.3). We suppose that $f(t)$ is a positive, real analytic function on $R^+$, and that $\alpha < \lambda_1$, where $\lambda_1$ is the lowest eigenvalue of (1.2). Moreover, we assume that a positive solution $v(x)$ exists. In analogy to Section 1.1, we write $D(t)$ for the region $\{x \in D;\, v(x) \geqslant t\}$, $a(t)$ for $\int_{D(t)} dx$ and $t(a)$ for its inverse. Let

$$H(a) = \int_{D(t(a))} [\alpha v + f(v)]\, dx = \int_0^a [\alpha t(\tilde{a}) + f[t(\tilde{a})]]\, d\tilde{a}.$$

Then, as in 1.1. we get

$$H'(a) = \alpha t(a) + f(t(a)) \qquad \text{and} \qquad H''(a) = \alpha(dt/da) + f'(dt/da).$$

Using the transformations and inequalities of Section 1.1, we obtain

$$\begin{aligned} H''(a) + p(a)\,[\alpha + f'(t(a))]\, H(a) &\geqslant 0 \qquad \text{if} \qquad \alpha + f'(t) > 0, \\ H''(a) + p(a)\,[\alpha + f'(t(a))]\, H(a) &\leqslant 0 \qquad \text{if} \qquad \alpha + f'(t) < 0. \end{aligned} \tag{2.5}$$

For $f(t) \equiv 1$, inequalities (2.5) reduce to (1.14). In certain cases $t$ can be calculated from $H'(a) = \alpha t + f(t)$. If we insert the expression $t = g[H']$ into (2.5), we get a relation between $H$, $H'$, $H''$, and $a$. In general, the inequality is very difficult to integrate. For some special cases we refer to [3].

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