Journal of Approximation Theory **104**, 330–344 (2000)
doi:10.1006/jath.2000.3451, available online at http://www.idealibrary.com on IDEAL®

On the Rate of Convergence of Two Bernstein–Bézier Type Operators for Bounded Variation Functions, II[1]

Xiao-Ming Zeng

Department of Mathematics, Xiamen University, Xiamen 361005, People's Republic of China
E-mail: xmzeng@jingxian.xmu.edu.cn

Communicated by József Szabados

Received November 25, 1998; accepted in revised form December 30, 1999

The rates of convergence of two Bernstein–Bézier type operators $B_n^{(\alpha)}$ and $L_n^{(\alpha)}$ for functions of bounded variation have been studied for the case $\alpha \geqslant 1$ by the author and A. Piriou (1998, *J. Approx. Theory* **95**, 369–387). In this paper the other case $0 < \alpha < 1$ is treated and asymptotically optimal estimations of $B_n^{(\alpha)}$ and $L_n^{(\alpha)}$ for functions of bounded variation are obtained. Besides, some interesting behaviors of the operators $B_n^{(\alpha)}$ and $L_n^{(\alpha)}$ $(\alpha > 0)$ for monotone functions and functions of bounded variation are also given.

1. INTRODUCTION

Let $P_{nk}(x) = \binom{n}{k} x^k (1-x)^{n-k}$ be the Bernstein basis functions. Let $J_{nk}(x) = \sum_{j=k}^{n} P_{nj}(x)$ be the Bézier basis functions. For a function f defined on $[0, 1]$, the Bernstein–Bézier operator $B_n^{(\alpha)}$ applied to f is

$$B_n^{(\alpha)}(f, x) = \sum_{k=0}^{n} f(k/n)\, Q_{nk}^{(\alpha)}(x), \tag{1}$$

and for a function $f \in L_1[0, 1]$, the Bernstein–Kantorovich–Bézier operator $L_n^{(\alpha)}$ applied to f is

$$L_n^{(\alpha)}(f, x) = (n+1) \sum_{k=0}^{n} Q_{nk}^{(\alpha)}(x) \int_{I_k} f(t)\, dt, \tag{2}$$

where $\alpha \geqslant 1$, or $0 < \alpha < 1$, $Q_{nk}^{(\alpha)}(x) = J_{nk}^{\alpha}(x) - J_{n,k+1}^{\alpha}(x)$ $(J_{n,n+1}(x) \equiv 0)$, and $I_k = [k/(n+1), (k+1)/(n+1)]$ $(0 \leqslant k \leqslant n)$.

[1] Project 19871068 supported by NSFC and Fujian Provincial Science Foundation of China.

0021-9045/00 $35.00

The approximation behaviors of the operators $B_n^{(\alpha)}$, $L_n^{(\alpha)}$ have many differences between the case $\alpha \geqslant 1$ and the case $0 < \alpha < 1$. For example, on the rate of convergence of the operators $B_n^{(\alpha)}(f, x)$ to $f(x) \in C[0, 1]$, Li and Gong (cf. [10, p. 106]) obtained

$$\|B_n^{(\alpha)}(f, x) - f(x)\|_{C[0,1]} \leqslant \begin{cases} (1+\alpha/4)\,\omega(n^{-1/2}, f), & \alpha \geqslant 1, \\ M\omega(n^{-\alpha/2}, f), & 0 < \alpha < 1, \end{cases}$$

where $\omega(\delta, f)$ is the modulus of continuity of $f(x)$. Obviously, the rate of convergence $\omega(n^{-1/2}, f)$ in the case $\alpha \geqslant 1$ is better than the rate of convergence $\omega(n^{-\alpha/2}, f)$ in the case $0 < \alpha < 1$. Again, for the case $\alpha \geqslant 1$, Liu [10, p. 107] proved an inverse theorem that

$$|B_n^{(\alpha)}(f, x) - f(x)| \leqslant M(\max[(x(1-x)/n)^{1/2}, n^{-1}])^{\beta} \qquad (0 < \beta < 1)$$

on $(0, 1)$ implies that $f(x) \in \mathrm{Lip}\,\beta$, while the inverse theorem for the case $0 < \alpha < 1$ is left unsolved up to now. Approximation of functions of bounded variation with the operators $B_n^{(\alpha)}$ and $L_n^{(\alpha)}$ for the case $\alpha \geqslant 1$ is studied in [1]. In this paper the other case $0 < \alpha < 1$ is treated. This paper proves that the rates of convergence of the operators $B_n^{(\alpha)}$ and $L_n^{(\alpha)}$ for functions of bounded variation in the case $0 < \alpha < 1$ are as good as in the case $\alpha \geqslant 1$ (except the differences in estimate coefficients), which shows that the asymptotical behavior of the operators $B_n^{(\alpha)}$ in space $BV[0, 1]$ is somewhat different from that in space $C[0, 1]$. In addition, in the last part of the paper, some interesting behaviors of the operators $B_n^{(\alpha)}$ and $L_n^{(\alpha)}$ $(\alpha > 0)$ for monotone functions and functions of bounded variation are studied. The main results of this paper are as follows:

THEOREM 1. *Let $0 < \alpha \leqslant 1$ and f be a function of bounded variation on $[0, 1]$ $(f \in BV[0, 1])$. Then for every $x \in (0, 1)$ and $n > \frac{256}{25}(x(1-x))^{-1}$ we have*

$$\left| B_n^{(\alpha)}(f, x) - \frac{1}{2^{\alpha}} f(x+) - \left(1 - \frac{1}{2^{\alpha}}\right) f(x-) \right| \leqslant \frac{A_{\alpha}}{n(x(1-x))^{2-\alpha}} \sum_{k=1}^{n} \bigvee_{x - x/\sqrt{k}}^{x + (1-x)/\sqrt{k}} (g_x) + \frac{1}{\sqrt{nx(1-x)}} (|f(x+) - f(x-)| + \varepsilon_n(x)\,|f(x) - f(x-)|), \qquad (3)$$

where A_α is a positive constant depending only on α,

$$\varepsilon_n(x)=\begin{cases}1, & \text{if } \quad x=k'/n, \quad \text{for some } \ k'\in\mathbf{N}\\ 0, & \text{if } \quad x\neq k/n, \quad \text{for all } \ k\in\mathbf{N},\end{cases}$$

$\bigvee_a^b(g_x)$ is the total variation of g_x on $[a, b]$, and

$$g_x(t)=\begin{cases}f(t)-f(x+), & x<t<1;\\ 0, & t=x;\\ f(t)-f(x-), & 0\leqslant t<x.\end{cases}$$

THEOREM 2. *Let $0<\alpha\leqslant 1$ and f be a function of bounded variation on $[0, 1]$. Then for every $x\in(0, 1)$ and $n>\frac{256}{25}(x(1-x))^{-1}$ we have*

$$\left|L_n^{(\alpha)}(f, x)-\frac{1}{2^\alpha}f(x+)-\left(1-\frac{1}{2^\alpha}\right)f(x-)\right|$$
$$\leqslant\frac{B_\alpha}{n(x(1-x))^{2-\alpha}}\sum_{k=1}^{n}\bigvee_{x-x/\sqrt{k}}^{x+(1-x)/\sqrt{k}}(g_x)+\frac{2\,|f(x+)-f(x-)|}{\sqrt{nx(1-x)}}, \tag{4}$$

where B_α is a positive constant depending only on α.

In Section 3 we will show that the estimations (3) and (4) are asymptotically optimal. From Theorem 1, Theorem 2, and the Korovkin Theorem (cf. [12, p. 27]), Corollary 1 of [1] now follows for all $\alpha>0$. By this we get

If $f(t)$ is bounded on $[0, 1]$, and if $x\in(0, 1)$ is a discontinuity point of the first kind of $f(t)$, then for any number C lying strictly between $f(x+)$ and $f(x-)$, we are able to choose a suitable α such that

$$\lim_{n\to\infty}B_n^{(\alpha)}(f, x)=C \qquad \text{and} \qquad \lim_{n\to\infty}L_n^{(\alpha)}(f, x)=C.$$

2. PRELIMINARIES

We need some preliminary results for proving Theorems 1 and 2. We first recall the Lebesgue–Stieltjes integral representations (cf. [1, (21), (22)]),

$$B_n^{(\alpha)}=\int_0^1 f(t)\,d_tK_{n,\alpha}^{(1)}(x, t),$$
$$K_{n,\alpha}^{(1)}(x, t)=\begin{cases}\sum_{k\leqslant nt}Q_{nk}^{(\alpha)}(x), & 0<t\leqslant 1,\\ 0, & t=0,\end{cases} \tag{5}$$

and

$$L_n^{(\alpha)}(f, x) = \int_0^1 f(t)\, K_{n,\alpha}^{(2)}(x, t)\, dt,$$
$$K_{n,\alpha}^{(2)}(x, t) = \sum_{k=0}^{n} (n+1)\, Q_{nk}^{(\alpha)}(x)\, \chi_k(t), \tag{6}$$

where χ_k is the characteristic function of the interval I_k with respect to $I = [0, 1]$.

Again, similar to [2, p. 272], we define $H_{n,\alpha}(x, t)$ and $R_{n,\alpha}(x, t)$ on $[0, 1]$ as

$$H_{n,\alpha}(x, t) = 1 - K_{n,\alpha}^{(1)}(x, t-), \qquad 0 \leqslant t < 1,$$
$$H_{n,\alpha}(x, 1) = 0, \tag{7}$$

and

$$R_{n,\alpha}(x, t) = 1 - \int_0^t K_{n,\alpha}^{(2)}(x, u)\, du, \qquad 0 \leqslant t < 1,$$
$$R_{n,\alpha}(x, 1) = 0. \tag{8}$$

For proving Theorems 1 and 2 we need to estimate the quantities

$$\left| \left(\sum_{nx < k \leqslant n} P_{nk}(x) \right)^{\alpha} - \frac{1}{2^{\alpha}} \right|, \qquad Q_{nk}^{(\alpha)}(x), \qquad K_{n,\alpha}^{(1)}(x, t),$$

$$\int_0^t K_{n,\alpha}^{(2)}(x, u)\, du, \qquad H_{n,\alpha}(x, t), \qquad \text{and} \qquad R_{n,\alpha}(x, t).$$

Below we give these estimations.

LEMMA 1. *For $0 < \alpha \leqslant 1$ and $x \in (0, 1)$, as $n > \frac{256}{25}(x(1-x))^{-1}$ we have*

$$\left| \left(\sum_{nx < k \leqslant n} P_{nk}(x) \right)^{\alpha} - \frac{1}{2^{\alpha}} \right| < \frac{1}{\sqrt{nx(1-x)}}. \tag{9}$$

Proof. By the mean value theorem it follows that

$$\left| \left(\sum_{nx < k \leqslant n} P_{nk}(x) \right)^{\alpha} - \frac{1}{2^{\alpha}} \right| = \alpha(\gamma_{nk}(x))^{\alpha - 1} \left| \sum_{nx < k \leqslant n} P_{nk}(x) - \frac{1}{2} \right|, \tag{10}$$

where $\gamma_{nk}(x)$ lies between 1/2 and $\sum_{nx<k\leqslant n} P_{nk}(x)$. From the proof of Lemma 2 of [1] we know that

$$\left|\sum_{nx<k\leqslant n} P_{nk}(x)-\frac{1}{2}\right|<\frac{0.8(2x^2-2x+1)}{\sqrt{nx(1-x)}}\leqslant\frac{0.8}{\sqrt{nx(1-x)}}. \tag{11}$$

It follows for $n>\frac{256}{25}(x(1-x))^{-1}$ that

$$\sum_{nx<k\leqslant n} P_{nk}(x)>\tfrac{1}{4}, \tag{12}$$

which implies

$$\gamma_{nk}(x)>\tfrac{1}{4}.$$

Hence from (10), (11), and the fact that $3.2\alpha<4^\alpha$ $(0<\alpha\leqslant 1)$, we get

$$\left|\left(\sum_{nx<k\leqslant n} P_{nk}(x)\right)^\alpha-\frac{1}{2^\alpha}\right|<\alpha 4^{1-\alpha}\frac{0.8}{\sqrt{nx(1-x)}}<\frac{1}{\sqrt{nx(1-x)}}.$$

The proof is complete.

LEMMA 2. *Let $0<\alpha\leqslant 1$ and $x\in(0,1)$. Then for $k=0, 1, 2, ..., n$, there holds*

$$\alpha P_{nk}(x)\leqslant Q_{nk}^{(\alpha)}(x)\leqslant P_{nk}^\alpha(x). \tag{13}$$

Proof. Since $Q_{nk}^{(\alpha)}(x)=J_{nk}^\alpha(x)-J_{n,k+1}^\alpha(x)$, $P_{nk}(x)=J_{nk}(x)-J_{n,k+1}(x)$, by the mean value theorem we get the left hand inequality of (13). Again, note that for $0<\alpha\leqslant 1$, there holds

$$(P_{nk}(x)/J_{nk}(x))^\alpha\geqslant P_{nk}(x)/J_{nk}(x)$$

and

$$(J_{n,k+1}(x)/J_{nk}(x))^\alpha\geqslant J_{n,k+1}(x)/J_{nk}(x).$$

Hence

$$(P_{nk}(x)/J_{nk}(x))^\alpha+(J_{n,k+1}(x)/J_{nk}(x))^\alpha\geqslant 1,$$

which derives the right hand inequality of (13).

From Theorem 1 of [3, p. 365] we can observe that the right hand inequality of (13) will derive an estimate order $n^{-\alpha/2}$ $(n\to\infty)$ for $Q_{nk}^{(\alpha)}(x)$. A better estimate order $n^{-1/2}$ $(n\to\infty)$ for some specific $Q_{nk'}^{(\alpha)}(x)$ is as follows:

LEMMA 3. *Let* $0<\alpha\leqslant 1$ *and* $x\in(0,1)$. *Then for* $n>\frac{256}{25x(1-x)}$ *and* $k'=nx$, *there holds*

$$Q_{nk'}^{(\alpha)}(x)<\frac{4\alpha}{4^{\alpha}}P_{nk'}(x)<\frac{1}{\sqrt{nx(1-x)}}. \tag{14}$$

Proof. Using the mean value theorem we have

$$\begin{aligned} Q_{nk'}^{(\alpha)}(x) &= \alpha\gamma_{nk'}^{\alpha-1}(x)[J_{nk'}(x)-J_{n,k'+1}(x)] \\ &= \alpha(1/\gamma_{nk'}(x))^{1-\alpha}P_{nk'}(x), \end{aligned} \tag{15}$$

where $J_{n,k'+1}(x)<\gamma_{nk'}<J_{n,k'}(x)$. Noticing that $k'=nx$, from (12) we have for $n>\frac{256}{25x(1-x)}$

$$\gamma_{nk'}(x)>J_{n,k'+1}(x)\geqslant\sum_{nx<k\leqslant n}P_{nk}(x)>1/4.$$

By Theorem 1 of [3, p. 365] and from (15) we deduce that

$$Q_{nk'}^{(\alpha)}(x)<\alpha 4^{1-\alpha}P_{nk'}(x)<\alpha 4^{1-\alpha}\frac{1}{\sqrt{2e}\sqrt{nx(1-x)}}<\frac{1}{\sqrt{nx(1-x)}}.$$

LEMMA 4. *For* $0<\alpha\leqslant 1$ *and* $0\leqslant t<x<1$ *there holds*

$$K_{n,\alpha}^{(1)}(x,t)\leqslant K_{n,1}^{(1)}(x,t)\leqslant\frac{x(1-x)}{n(x-t)^2}. \tag{16}$$

Proof. The right hand inequality of (16) is well known (see, e.g., [4, p. 6]). Hence we only need to prove the left hand inequality of (16). Noting the expression (5) we have

$$\begin{aligned} \sum_{k\leqslant nt}Q_{nk}^{(\alpha)}(x) &= J_{n0}^{\alpha}(x)-J_{n1}^{\alpha}(x)+J_{n1}^{\alpha}(x)-J_{n2}^{\alpha}(x)+\cdots \\ &\quad +J_{n,[nt]-1}^{\alpha}(x)-J_{n,[nt]}^{\alpha}(x)+J_{n,[nt]}^{\alpha}(x)-J_{n,[nt]+1}^{\alpha}(x) \\ &= J_{n0}^{\alpha}(x)-J_{n,[nt]+1}^{\alpha}(x)=1-\left(\sum_{k=[nt]+1}^{n}P_{nk}(x)\right)^{\alpha}. \end{aligned}$$

Note that $0<\alpha\leqslant 1$ and $(\sum_{k=[nt]+1}^{n}P_{nk}(x))\leqslant 1$. Hence

$$\begin{aligned} 1-\left(\sum_{k=[nt]+1}^{n}P_{nk}(x)\right)^{\alpha} &\leqslant 1-\left(\sum_{k=[nt]+1}^{n}P_{nk}(x)\right) \\ &= \sum_{k\leqslant nt}P_{nk}(x)=\sum_{k\leqslant nt}Q_{nk}^{(1)}(x). \end{aligned}$$

Lemma 4 is proved.

LEMMA 5. *For $0<\alpha\leqslant 1$ and $0\leqslant t<x<1$, as $n>(3x(1-x))^{-1}$ we have*

$$\int_0^t K_{n,\alpha}^{(2)}(x,u)\,du\leqslant\int_0^t K_{n,1}^{(2)}(x,u)\,du\leqslant\frac{2x(1-x)}{n(x-t)^2}. \tag{17}$$

Proof. Let $t\in[k^*/(n+1),(k^*+1)/(n+1))$. Then we can write $t=(k^*+\varepsilon)/(n+1)$ $(0\leqslant\varepsilon<1)$. So

$$\begin{aligned}
\int_0^t K_{n,\alpha}^{(2)}(x,u)\,du &= \int_0^t \sum_{k=0}^{n}(n+1)\,Q_{nk}^{(\alpha)}(x)\,\chi_k(u)\,du\\
&= \sum_{k=0}^{n}(n+1)\,Q_{nk}^{(\alpha)}(x)\int_0^t \chi_k(u)\,du\\
&= \sum_{k=0}^{k^*-1} Q_{nk}^{(\alpha)}(x)+(n+1)\,Q_{nk^*}^{(\alpha)}(x)\int_{k^*/(n+1)}^{(k^*+\varepsilon)/(n+1)} 1\,du\\
&= \sum_{k=0}^{k^*-1} Q_{nk}^{(\alpha)}(x)+\varepsilon Q_{nk^*}^{(\alpha)}(x)\\
&= J_{n0}^{\alpha}(x)-J_{n1}^{\alpha}(x)+J_{n1}^{\alpha}(x)-J_{n2}^{\alpha}(x)+\cdots\\
&\quad + J_{n,k^*-1}^{\alpha}(x)-J_{n,k^*}^{\alpha}(x)+\varepsilon J_{n,k^*}^{\alpha}(x)-\varepsilon J_{n,k^*+1}^{\alpha}(x)\\
&= 1-(1-\varepsilon)\,J_{nk^*}^{\alpha}(x)-\varepsilon J_{n,k^*+1}^{\alpha}(x)\\
&\leqslant 1-(1-\varepsilon)\,J_{nk^*}(x)-\varepsilon J_{n,k^*+1}(x).
\end{aligned}$$

The last inequality holds due to the fact that $0\leqslant\alpha,\ J_{n,k^*}(x)\leqslant 1$. We observe that $1-(1-\varepsilon)\,J_{nk^*}(x)-\varepsilon J_{n,k^*+1}(x)$ is just $\int_0^t K_{n,1}^{(2)}(x,u)\,du$. Hence the left hand inequality of (17) is obtained. As $n>(3x(1-x))^{-1}$, the right hand inequality of (17) follows from Lemma 9 of [1]. The proof is complete.

LEMMA 6. *Let $l>2$ be fixed. Then there exist three positive numbers $r>0$, $s>0$, and $p>1$ such that $l=r+s$, $rp=2$, and $\frac{sp}{p-1}$ is a positive even integer.*

Proof. In fact, let $[l]$ denote the greatest integer not exceeding l. One can take $p=\frac{2[l]}{2[l]+2-l}>1$, $r=\frac{2}{p}$, and $s=l-r$. Then

$$rp=2 \qquad \text{and} \qquad \frac{sp}{p-1}=\frac{lp-rp}{p-1}=\frac{lp-2}{p-1}=2[l]+2.$$

LEMMA 7. *For* $0 < \alpha \leqslant 1$ *and* $0 \leqslant x < t < 1$ *we have*

$$H_{n,\alpha}(x,t) \leqslant A_\alpha \frac{(x(1-x))^\alpha}{n(x-t)^2}, \tag{18}$$

where A_α *is a positive constant depending only on* α.

Proof. For $\alpha = 1$ the conclusion is known from Lemma 8 of [1]. Below we consider the case $0 < \alpha < 1$. Since $0 \leqslant x < t < 1$, so $|\frac{k/n - x}{t-x}| \geqslant 1$ for $k \geqslant nt$. Thus

$$\begin{aligned} H_{n,\alpha}(x,t) &= 1 - K_{n,\alpha}^{(1)}(x, t-) = 1 - \sum_{k \leqslant nt-} Q_{nk}^{(\alpha)}(x) \leqslant \sum_{k \geqslant nt} Q_{nk}^{(\alpha)}(x) \\ &= \sum_{k \geqslant nt} (J_{nk}^\alpha(x) - J_{n,k+1}^\alpha(x)) = \left(\sum_{k \geqslant nt} P_{nk}(x)\right)^\alpha \\ &\leqslant \left(\sum_{k \geqslant nt} \frac{|k/n - x|^{2/\alpha}}{(t-x)^{2/\alpha}} P_{nk}(x)\right)^\alpha \\ &\leqslant \frac{1}{(t-x)^2} \left(\sum_{k=0}^n |k/n - x|^{2/\alpha} P_{nk}(x)\right)^\alpha. \end{aligned}$$

By Lemma 6 we choose $r > 0$, $s > 0$, and $p > 1$ such that $2/\alpha = r + s$, $rp = 2$, and $\frac{sp}{p-1}$ is a positive even integer. Let q be the conjugate exponent to p, i.e, $1/p + 1/q = 1$. Then by the Hölder inequality

$$\begin{aligned} &\left(\sum_{k=0}^n |k/n - x|^{2/\alpha} P_{nk}(x)\right)^\alpha \\ &\quad = \left(\sum_{k=0}^n |k/n - x|^r \, |k/n - x|^s \, P_{nk}^{1/p}(x) \, P_{nk}^{1/q}(x)\right)^\alpha \\ &\quad \leqslant \left(\sum_{k=0}^n |k/n - x|^{rp} P_{nk}(x)\right)^{\alpha/p} \left(\sum_{k=0}^n |k/n - x|^{sq} P_{nk}(x)\right)^{\alpha/q} \\ &\quad \leqslant (x(1-x)\, n^{-rp/2})^{\alpha/p} \, (x(1-x)\, An^{-sq/2})^{\alpha/q} \\ &\quad = A_\alpha (x(1-x))^\alpha \, n^{-1}. \end{aligned} \tag{19}$$

The second inequality in (19) is known from [4, p. 14, Theorem 1.5.1], and $A_\alpha = A^{\alpha/q}$ is a positive constant depending only on α. The proof is complete.

In the same manner we get

LEMMA 8. *For* $0<\alpha\leqslant 1$ *and* $0\leqslant x+(1-x)/\sqrt{n}\leqslant t<1$, *as* $n>(3x(1-x))^{-1}$ *we have*

$$R_{n,\alpha}(x,t)\leqslant B_\alpha \frac{(x(1-x))^\alpha}{n(x-t)^2}, \tag{20}$$

where B_α *is a positive constant depending only on* α.

LEMMA 9. *For* $0<\alpha\leqslant 1$ *and* $n=2m$ $(m=1,2,3,...)$ *we have*

$$|J^\alpha_{2m,m+1}(1/2)-1/2^\alpha|\geqslant \alpha/(4\sqrt{n}). \tag{21}$$

Proof. From the proof of Lemma 13 of [1] we know that

$$\begin{aligned}|J^\alpha_{2m,m+1}(1/2)-1/2^\alpha| &= 1/2^\alpha - J^\alpha_{2m,m+1}(1/2)\\ &=(\alpha/2)\,\gamma_m^{\alpha-1}\,P_{2m,m}(1/2),\end{aligned}$$

where $1/4\leqslant J_{2m,m}(1/2)<\gamma_m<1/2$ and $P_{2m,m}(1/2)>1/(2\sqrt{n})$.
Now $\gamma_m^{\alpha-1}\geqslant 1$, hence

$$|J^\alpha_{2m,m+1}(1/2)-1/2^\alpha|=(\alpha/2)\,\gamma_m^{\alpha-1}\,P_{2m,m}(1/2)>\alpha/(4\sqrt{n}).$$

3. PROOFS OF THE THEOREMS AND THE REMARK

Proofs of Theorems 1 *and* 2. (We shall here refer to some computations already detailed in the study [1] of the case $\alpha\geqslant 1$.)

For any $f\in BV[0,1]$, we decompose $f(t)$ into four parts as (see [1, (30)])

$$\begin{aligned}f(t)=&\frac{1}{2^\alpha}f(x+)+\left(1-\frac{1}{2^\alpha}\right)f(x-)+g_x(t)+\frac{f(x+)-f(x-)}{2^\alpha}\,\hat{sgn}(t-x)\\ &+\delta_x(t)\left[f(x)-\frac{1}{2^\alpha}f(x+)-\left(1-\frac{1}{2^\alpha}\right)f(x-)\right],\end{aligned} \tag{22}$$

where $s\hat{g}n(t)$ and $\delta_x(t)$ are defined by

$$s\hat{g}n(t)=\begin{cases}2^{\alpha}-1, & t>0\\ 0, & t=0;\\ -1, & t<0\end{cases}\qquad \delta_x(t)=\begin{cases}1, & t=x\\ 0, & t\neq x.\end{cases}$$

Hence

$$\left|B_n^{(\alpha)}(f,x)-\frac{1}{2^{\alpha}}f(x+)-\left(1-\frac{1}{2^{\alpha}}\right)f(x-)\right|$$

$$\leqslant |B_n^{(\alpha)}(g_x,x)|+\left|\frac{f(x+)-f(x-)}{2^{\alpha}}B_n^{(\alpha)}(s\hat{g}n(t-x),x)\right.$$

$$\left.+\left[f(x)-\frac{1}{2^{\alpha}}f(x+)-\left(1-\frac{1}{2^{\alpha}}\right)f(x-)\right]B_n^{(\alpha)}(\delta_x,x)\right|. \tag{23}$$

Using Lemma 4, Lemma 7, and along the same lines of the proof of [1, Lemma 10], we get

$$|B_n^{(\alpha)}(g_x,x)|=\left|\int_0^1 g_x(t)\,d_tK_{n,\alpha}^{(1)}(x,t)\right|$$

$$\leqslant\frac{A_\alpha}{n(x(1-x))^{2-\alpha}}\sum_{k=1}^{n}\bigvee_{x-x/\sqrt{k}}^{x+(1-x)/\sqrt{k}}g(x), \tag{24}$$

where A_α is a positive constant depending only on α.

On the other hand, by direct calculation, we get (see [1, Lemma 4])

$$B_n^{(\alpha)}(\delta_x,x)=\varepsilon_n(x)\,Q_{nk'}^{(\alpha)}(x)$$

and

$$B_n^{(\alpha)}(s\hat{g}n(t-x),x)=2^{\alpha}\left(\sum_{nx<k\leqslant n}P_{nk}(x)\right)^{\alpha}-1+\varepsilon_n(x)\,Q_{nk'}^{(\alpha)}(x),$$

where

$$\varepsilon_n(x)=\begin{cases}1, & \text{if}\quad x=k'/n,\quad \text{for some } k'\in\mathbf{N}\\ 0, & \text{if}\quad x\neq k/n,\quad \text{for all } k\in\mathbf{N}.\end{cases}$$

Hence

$$\left|\frac{f(x+)-f(x-)}{2^\alpha} B_n^{(\alpha)}(s\hat{g}n(t-x), x)\right.$$

$$\left.+\left[f(x)-\frac{1}{2^\alpha} f(x+)-\left(1-\frac{1}{2^\alpha}\right) f(x-)\right] B_n^{(\alpha)}(\delta_x, x)\right|$$

$$=\left|\frac{f(x+)-f(x-)}{2^\alpha}\left[2^\alpha\left(\sum_{nx<k\leqslant n} P_{nk}(x)\right)^\alpha-1+\varepsilon_n(x)\, Q_{nk'}^{(\alpha)}(x)\right]\right.$$

$$\left.+\left[f(x)-\frac{1}{2^\alpha} f(x+)-\left(1-\frac{1}{2^\alpha}\right) f(x-)\right] \varepsilon_n(x)\, Q_{nk'}^{(\alpha)}(x)\right|$$

$$=\left|\frac{f(x+)-f(x-)}{2^\alpha}\left[2^\alpha\left(\sum_{nx<k\leqslant n} P_{nk}(x)\right)^\alpha-1\right]\right.$$

$$\left.+[f(x)-f(x-)]\, \varepsilon_n(x)\, Q_{nk'}^{(\alpha)}(x)\right|.$$

Now using Lemma 1 and Lemma 3, we get

$$\left|\frac{f(x+)-f(x-)}{2^\alpha}\left[2^\alpha\left(\sum_{nx<k\leqslant n} P_{nk}(x)\right)^\alpha-1\right]\right.$$

$$\left.+[f(x)-f(x-)]\, \varepsilon_n(x)\, Q_{nk'}^{(\alpha)}(x)\right|$$

$$\leqslant\frac{1}{\sqrt{nx(1-x)}}(|f(x+)-f(x-)|+\varepsilon_n(x)\,|f(x)-f(x-)|). \tag{25}$$

Theorem 1 now follows from (23)–(25).

For proving Theorem 2 we only need to note that by Lemma 5, Lemma 8, and along the same lines of the proof of [1, Lemma 11], we can obtain

$$|L_n^{(\alpha)}(g_x, x)|\leqslant\frac{B_\alpha}{n(x(1-x))^{2-\alpha}}\sum_{k=1}^{n}\bigvee_{x-x/\sqrt{k}}^{x+(1-x)/\sqrt{k}}(g_x),$$

where B_α is a positive constant depending only on α. The remainder estimates are similar to the proof of Theorem 1. Hence we omit the details of the proof.

Remark. We now prove that estimations (3) and (4) are asymptotically optimal for continuity points and discontinuity points of the function of bounded variation $f(t)$. If x is a continuity point of f, (3) becomes

$$|B_n^{(\alpha)}(f, x) - f(x)| \leqslant \frac{A_\alpha}{n(x(1-x))^{2-\alpha}} \sum_{k=1}^{n} \bigvee_{x - x/\sqrt{k}}^{x+(1-x)/\sqrt{k}} (f). \tag{26}$$

Consider the function $f(t) = |t - x|$ $(x \in (0, 1))$. From (26) we have

$$|B_n^{(\alpha)}(f, x) - f(x)| = B_n^{(\alpha)}(|t-x|, x) \leqslant \frac{A_\alpha}{n(x(1-x))^{2-\alpha}} \sum_{k=1}^{n} \frac{1}{\sqrt{k}}$$

$$< \frac{2A_\alpha}{\sqrt{n}\, x^2(1-x)^2}. \tag{27}$$

On the other hand, by Lemma 2 and a result of Cheng [2, p. 240], we have for $n > 2(x(1-x))^{-1}$

$$|B_n^{(\alpha)}(|t-x|, x)| = \sum_{k=0}^{n} |k/n - x|\ Q_{nk}^{(\alpha)}(x) \geqslant \sum_{k=0}^{n} |k/n - x|\ \alpha P_{nk}(x)$$

$$\geqslant \frac{\alpha (x(1-x))^{1/2}}{16 n^{1/2}}. \tag{28}$$

Therefore from (27) and (28) we deduce that (26) cannot be asymptotically improved.

For the discontinuity point of f, when $g_x \equiv 0$, (3) becomes

$$\left| B_n^{(\alpha)}(f, x) - \frac{1}{2^\alpha} f(x+) - \left(1 - \frac{1}{2^\alpha}\right) f(x-) \right|$$

$$\leqslant \frac{1}{\sqrt{nx(1-x)}} (|f(x+) - f(x-)| + \varepsilon_n(x)\ |f(x) - f(x-)|). \tag{29}$$

We consider the function

$$f(t) = \begin{cases} 0, & \text{if} \quad 0 \leqslant t \leqslant 1/2 \\ 1, & \text{if} \quad 1/2 < t \leqslant 1, \end{cases}$$

and $x=1/2$, $n=2m$ $(m=1, 2, 3, ...)$. By Lemma 9 and (29), it follows that

$$\alpha/(4\sqrt{n}) \leqslant |J^{\alpha}_{2m, m+1}(1/2) - 1/2^{\alpha}|$$
$$= |B_n^{(\alpha)}(f, x) - (1/2^{\alpha})\, f(x+) - (1-1/2^{\alpha})\, f(x-)| \leqslant 4/\sqrt{n}.$$

Therefore (29) cannot be asymptotically improved as $n \to +\infty$.

In the same way we can show that estimation (4) is asymptotically optimal for continuity points and discontinuity points of bounded variation functions.

4. MONOTONE FUNCTIONS AND FUNCTIONS OF BOUNDED VARIATION

In this section we give some interesting behaviors of the operators $B_n^{(\alpha)}(f, x)$ and $L_n^{(\alpha)}(f, x)$ $(\alpha > 0)$ for monotone functions and functions of bounded variation.

THEOREM 3. *If $f(x)$ is monotone non-decreasing (non-increasing), then $B_n^{(\alpha)}(f, x)$ and $L_n^{(\alpha)}(f, x)$ are non-decreasing (non-increasing) with variable $x \in [0, 1]$ and non-increasing (non-decreasing) with variable $\alpha > 0$. Moreover, let $\bigvee_0^1 f(x)$ denote the total variation of the function $f(x)$ in $[0, 1]$. We have*

$$\bigvee_0^1 B_n^{(\alpha)}(f, x) \leqslant \bigvee_0^1 f(x) \qquad \text{and} \qquad \bigvee_0^1 L_n^{(\alpha)}(f, x) \leqslant \bigvee_0^1 f(x). \tag{30}$$

In particular, if $f(x)$ is the monotone function, then $\bigvee_0^1 B_n^{(\alpha)}(f, x) = \bigvee_0^1 f(x)$.

(If taking $\alpha = 1$, from the first inequality of (30) we get a result of Lorentz [4, p. 23, 1.7(1)].)

Proof. From (1) we have

$$B_n^{(\alpha)}(f, x) = \sum_{k=0}^{n} f(k/n)\, Q_{nk}^{(\alpha)}(x)$$
$$= f(0) + \sum_{k=1}^{n} [f(k/n) - f((k-1)/n)]\, J_{nk}^{\alpha}(x). \tag{31}$$

Again, for $x \in [0, 1]$ and $\alpha > 0$

$$\frac{d}{dx} J_{nk}^{\alpha}(x) = \alpha J_{nk}^{\alpha-1}(x)\, J'_{nk}(x) = \alpha J_{nk}^{\alpha-1}(x) \sum_{j=k}^{n} P'_{nj}(x)$$
$$= \alpha J_{nk}^{\alpha-1}(x) \left(\sum_{j=k}^{n-1} n(P_{n-1, j-1}(x) - P_{n-1, j}(x)) + nP_{n-1, n-1}(x) \right)$$
$$= \alpha n J_{nk}^{\alpha-1}(x)\, P_{n-1, k-1}(x) \geqslant 0. \tag{32}$$

Hence if $f(x)$ is monotone non-decreasing (non-increasing), then $\frac{d}{dx} B_n^{(\alpha)}(f, x) \geqslant 0$ $(\leqslant 0)$, that is, $B_n^{(\alpha)}(f, x)$ is non-decreasing (non-increasing) with variable x.

On the other hand, let $\alpha_1 \geqslant \alpha_2$ and note that $0 \leqslant J_{nk}(x) \leqslant 1$. Thus from (31)

$$B_n^{(\alpha_1)}(f, x) - B_n^{(\alpha_2)}(f, x) = \sum_{k=1}^{n} [f(k/n) - f((k-1)/n)](J_{nk}^{\alpha_1}(x) - J_{nk}^{\alpha_2}(x))$$

$$\begin{cases} \leqslant 0, & \text{if } f(x) \text{ is monotone non-decreasing,} \\ \geqslant 0, & \text{if } f(x) \text{ is monotone non-increasing.} \end{cases}$$

In addition, from (31) and (32) we have

$$\begin{aligned}
&\bigvee_0^1 B_n^{(\alpha)}(f, x) \\
&\quad = \int_0^1 \left| \frac{d}{dx} B_n^{(\alpha)}(f, x) \right| dx \\
&\quad = \int_0^1 \left| \sum_{k=1}^{n} [f(k/n) - f((k-1)/n)] \, \alpha n J_{nk}^{\alpha-1}(x) \, P_{n-1, k-1}(x) \right| dx \\
&\quad \leqslant \sum_{k=1}^{n} |[f(k/n) - f((k-1)/n)]| \, (J_{nk}^{\alpha}(1) - J_{nk}^{\alpha}(0)) \\
&\quad = \sum_{k=1}^{n} |[f(k/n) - f((k-1)/n)]| \\
&\quad \leqslant \bigvee_0^1 f(x).
\end{aligned}$$

The above inequality becomes equality when $f(x)$ is the monotone function. An analogous property for the operators $L_n^{(\alpha)}(f, x)$ can be obtained in the same way. The proof is complete.

ACKNOWLEDGMENT

The author is thankful to the referees for their valuable comments and suggestions that helped improve this article.

REFERENCES

1. X. M. Zeng and A. Piriou, On the rate of convergence of two Bernstein–Bézier type operators for bounded variation functions, *J. Approx. Theory* **95** (1998), 369–387.
2. F. Cheng, On the rate of convergence of Bernstein polynomials of functions of bounded variation, *J. Approx. Theory* **39** (1983), 259–274.

3. X. M. Zeng, Bounds for Bernstein basis functions and Meyer–König and Zeller basis functions, *J. Math. Anal. Appl.* **219** (1998), 364–376.
4. G. G. Lorentz, “Bernstein Polynomials,” Univ. of Toronto Press, Toronto, 1953.
5. R. Bojanic and M. Vuilleumier, On the rate of convergence of Fourier–Legendre series of functions of bounded variation, *J. Approx. Theory* **31** (1981), 67–79.
6. P. Bézier, “Numerical Control-Mathematics and Applications,” Wiley, London, 1972.
7. G. Chang, Generalized Bernstein–Bézier polynomials, *J. Comput. Math.* **1**, No. 4 (1983), 322–327.
8. F. Cheng, On the rate of convergence of the Szász–Mirakyan operator for functions of bounded variation, *J. Approx. Theory* **40** (1984), 226–241.
9. R. Bojanic and F. Cheng, Rate of convergence of Bernstein polynomials for functions with derivatives of bounded variation, *J. Math. Anal. Appl.* **141** (1989), 136–151.
10. Z. Liu, Approximation of continuous functions by the generalized Bernstein–Bézier polynomials, *Approx. Theory. Appl.* **2**, No. 4 (1986), 105–130.
11. S. Guo and M. K. Khan, On the rate of convergence of some operators for functions of bounded variation, *J. Approx. Theory* **58** (1989), 90–101.
12. R. A. Devore, “The Approximation of Continuous Functions by Positive Linear Operators,” Lecture Notes in Mathematics, Springer-Verlag, Berlin/Heidelberg/New York, 1972.