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A new look into the kinematics and dynamics of finite rigid body rotations using Lie group theory

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ABSTRACT

A systematic theoretical approach is presented, in an effort to provide a complete and illuminating study on kinematics and dynamics of rigid bodies rotating about a fixed point. Specifically, this approach is based on some fundamental concepts of differential geometry, with particular reference to Lie group theory. This treatment is motivated by the form of the configuration space corresponding to large rigid body rotation, which is a differentiable manifold possessing group properties. First, the basic steps of the classical approach on the subject are briefly summarized. Then, some geometrical tools are presented, which are essential for supporting and illustrating the steps and findings of the new approach. Finally, the emphasis is placed on a thorough investigation of the problem of finite rotations. A key idea is the introduction of a canonical connection, matching the manifold and group properties of the configuration space. This proves to be sufficient and effective for performing the kinematics. Next, following the selection of an appropriate metric, the dynamics is also carried over. The present approach is theoretically more demanding than the traditional treatments in engineering but brings substantial benefits. In particular, an elegant interpretation is provided for all the quantities with fundamental importance in both rigid body kinematics and dynamics. Most importantly, this also leads to a correction of some misconceptions and geometrical inconsistencies in the field. Among other things, the deeper understanding of the theoretical concepts provides powerful insight and a strong basis for the development of efficient numerical techniques in problems of solid and structural mechanics involving large rotations.

1. Introduction

The study of rigid body kinematics and dynamics has been in the epicenter of many previous investigations due to both the great theoretical importance and the large practical significance of the subject. Based on the authors' background, the previous work in this research area can be split in two major categories. The first includes work performed by mathematicians or physicists, where the emphasis was placed on the theoretical aspects mainly, while the problem of large rotation was treated mostly as an example fitting the theory (e.g., Sattinger and Weaver, 1986; Arnold, 1989; Marsden and Ratiu, 1999). On the other hand, the same subject has also been extensively treated by engineers, with the attention shifted on applications and on development of effective problem solving methods (e.g., Greenwood, 1988; Nikravesh, 1988; Geradin and Cardona, 2001; Shabana, 2005).

The authors of the present work take an intermediate position, trying to bridge the gap between these two schools of thought with seemingly different objectives. A similar point of view has also been adopted by previous researchers in the field and proved beneficial in bringing new and useful theoretical concepts and ideas in order to help and support the efforts of attacking and solving challenging engineering problems involving large rotation (e.g., Argyris, 1982; Simo and Vu-Quoc, 1991; Argyris and Poterasu, 1993; Papastavridis, 1999). Also, a similar approach was adopted and proved fruitful in other related fields of mechanics and materials, in systems involving components undergoing large rotations. For instance, this is the case encountered in finite element models of rods, plates and shells exhibiting small strains but large deformations induced by large rotations. In addition, similar approaches have also found successful application in other more complex deformable structures like ground and space vehicles, mechatronics and biosystems (e.g., Wempner, 1969; Lu and Papadopoulos, 1998; Parry, 2001; Vassilev and Djondjorov, 2003). In particular, there exists an extensive body of literature in robotics, using tools of geometrical mechanics and Lie group theory (Murray et al., 1994; Choset et al., 2005; Selig, 2005).

The main objective of this work is to present a new look into the old but practically significant and still challenging mechanics problem of finite rotations, based on sound geometrical concepts. These concepts are known and have been employed in the mathematics and physics literature for a long time. However, with a few exceptions (e.g., Park et al., 1995; Ostrowski and Burdick, 1998; Haller

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and Mezic, 1998; Iserles et al., 2000; Liu, 2004), they are still not widely known to or fully explored by the engineering community, despite their large value and usefulness. Here, an effort is made to use these concepts in order to create a clear and complete geometrical picture of the kinematics and dynamics of large rigid body rotation. In this way, more light is thrown into the meaning of some of the most commonly employed quantities in describing rigid body motion. This, in turn, provides an alternative view, clarification and better interpretation of the related formulas employed frequently in the classical engineering literature. At the same time, this helps to identify and correct some common misconceptions in the field and achieve another objective of the present work. The latter refers to providing the means for building a reliable theoretical basis for developing better and more effective numerical integration methodologies for studying and investigating dynamics of single or multiple rigid bodies. In particular, this is expected to support ongoing efforts in the field of nonlinear solid and structural mechanics, focusing on the development of geometrically exact temporal discretization numerical schemes (Simo and Vu-Quoc, 1991; Crouch and Grossman, 1993; Munthe-Kaas, 1998; Brüls et al., 2012).

The utilization of ideas from Lie group theory is of great importance in nonlinear structural dynamics problems. This is due to the fact that Lie groups possess a simpler structure (fewer requirements) than classical vector spaces. Consequently, they are appropriate for studying nonlinear problems, while the latter are mostly employed in studying linear problems. In fact, application of suitable procedures leads to the creation of a vector space at each point of a manifold, allowing the study of infinitesimal motions around the state of the system represented by the specific point, which are governed by linearized equations (Arnold, 1989).

The geometrical route chosen in the present study deviates significantly from that taken in previous studies of mechanics problems. Specifically, the underlying manifold structure in static problems is Riemannian, possessing a symmetric and positive definite metric. A typical path is to introduce a set of coordinates and employ a natural coordinate basis in order to define a metric and then produce a metric compatible connection, having as components the classical Christoffel symbols (e.g., Flugge, 1972; Fung and Tong, 2001; Wempner and Talaslidis, 2003). Usually, the same route is also chosen even for dynamics problems (e.g., Sattinger and Weaver, 1986; Zefran et al., 1999). However, a more primitive and natural path is more beneficial and followed in this work. Namely, after creating the manifold corresponding to the configuration space of the motion, a connection operator is first established in an appropriate manner. This proves to be sufficient for performing a complete study of the kinematics. In the present study, a canonical connection is selected (Bertram, 2008), so that one can fully exploit the benefits associated to matching the special curves related to the manifold and the group properties of the configuration space. Then, a study of the dynamics requires the introduction of a metric. In this work, a suitable metric is chosen, which is not compatible with the connection, but allows for a complete, simple and concise treatment of the dynamics.

The organization of this paper is as follows. First, the classical approach, referring to the study of spherical motion in the three dimensional Euclidean space, is briefly summarized in the following section. Then, some fundamental concepts of differential geometry are presented in Section 3, which are essential in providing the necessary theoretical background to an engineering audience. Some supplementary material on the same subject is also presented in Appendices A and B. In Section 4, a complete picture of rigid body kinematics is presented, based on the geometry of an appropriate three dimensional manifold. The basic properties of this manifold are extracted by defining an isomorphism with the well known special orthogonal group $SO(3)$. The latter group is shown to present certain serious defects in describing both the kinematics and the dynamics of a rigid body. For this reason, the new manifold proposed in this study is generated through application of a group representation (Hall, 2003). First, the emphasis is placed in introducing a suitable connection, which leads to an illuminating and thorough study of the kinematics. Moreover, in conjunction with the introduction of an appropriate metric, it also provides a useful tool for examining the dynamics of a rigid body, in a simple and effective manner, as illustrated by the material included in Section 5. Finally, the most important conclusions are summarized in the last section.

2. Classical approach

The main concepts referring to kinematics and dynamics of a rigid body undergoing large rotation about a fixed point (also known as spherical motion) are briefly presented in this section. This will provide the necessary reference for the material presented in the following sections.

Study of the spherical motion is typically performed in the ordinary Euclidean space $\mathbb{R}^3$ (Geradin and Cardona, 2001). The basic geometrical tools for this are shown in Fig. 1. In particular, a basis $\mathfrak{B}$ is introduced, consisting of three fixed orthonormal vectors $\vec{E}_i$ (with $i = 1, 2, 3$), having point O as origin. These vectors form a right-handed Cartesian inertial (or absolute or spatial) frame of reference. This basis will also be denoted by $\{\vec{E}_i\}$. On the other hand, another basis, $\mathfrak{B}'$ or $\{\vec{e}_i(t)\}$, is formed by considering a new set of three orthonormal vectors $\vec{e}_i(t)$, having O as origin, but rigidly attached to and following the motion of the rigid body. These vectors form the so-called body (or convective or corotational) frame of reference (Marsden and Ratiu, 1999).

The two bases $\mathfrak{B}$ and $\mathfrak{B}'$ coincide originally. In addition, during the subsequent motion, the basis vectors are related through a linear mapping, as follows

$$\vec{e}_i(t) = \mathfrak{R}(t)\vec{E}_i, \quad i = 1, 2, 3. \tag{1}$$

Then, the position vector of an arbitrary point of the body is expressed in the form

$$\vec{x}(t) = \sum_{i=1}^{3} x_i(t)\vec{E}_i = \sum_{i=1}^{3} X_i\vec{e}_i(t),$$

where x_i and X_i represent the components of the position vector in the spatial and the body frame, respectively. Adopting the notational convention of dropping the sum operator for products involving repeated indices (Papastavridis, 1999), the last relations can be rewritten in the simpler form

$$\vec{x}(t) = x_i(t)\vec{E}_i = X_i\vec{e}_i(t). \tag{2}$$

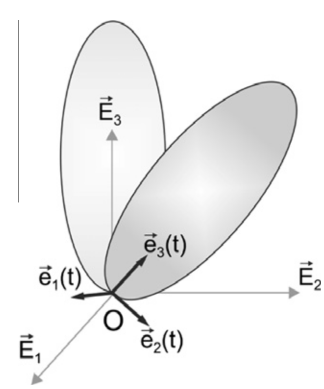

Fig. 1. A rigid body undergoing large rotation about a fixed point O.

Next, if the components of the transformation $\mathfrak{R}$ in the basis $\mathfrak{B}$ are expressed by matrix R, that is

$$M_{\mathfrak{B}}^{\mathfrak{B}}(\mathfrak{R}) = R = [r_{ij}],$$

then, by definition (Bowen and Wang, 2008), Eq. (1) can be rewritten in the form

$$\vec{e}_i(t) = r_{ji}(t)\vec{E}_j.$$

Therefore, from Eqs. (1) and (2) one gets

$$x_i(t) = r_{ij}(t)X_j \quad \text{or} \quad \underline{x}(t) = R(t)\underline{X}, \tag{3}$$

with

$$\underline{x} = (x_1 \quad x_2 \quad x_3)^T \quad \text{and} \quad \underline{X} = (X_1 \quad X_2 \quad X_3)^T.$$

Then, by employing the rigidity assumption of the body, expressed in the form $\underline{x} \cdot \underline{x} = \underline{X} \cdot \underline{X}$, where $\cdot$ represents the ordinary scalar (dot) product of $\mathbb{R}^3$, in conjunction with Eq. (3), leads to

$$R^T R = I, \tag{4}$$

where I is the 3×3 identity matrix. This implies that $R(t)$ is orthogonal for all times t. Moreover, by differentiating both sides of the last relation with respect to time, it turns out that the matrix

$$\widetilde{\Omega}(t) \equiv R^T(t)\dot{R}(t) \tag{5}$$

is skew-symmetric, with general form

$$\widetilde{\Omega} = spin(\underline{\Omega}) \equiv \begin{bmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{bmatrix}, \tag{6}$$

where the vector

$$\underline{\Omega} = vect(\widetilde{\Omega}) \equiv (\Omega_1 \quad \Omega_2 \quad \Omega_3)^T$$

is known as the axial vector associated with the skew-symmetric matrix $\widetilde{\Omega}$. From hereon, the symbol $\sim$ over a quantity will be reserved for denoting a 3×3 skew-symmetric matrix. By definition, $\widetilde{\Omega}\underline{x} \equiv \underline{\Omega} \times \underline{x}$, where $\times$ stands for the classical vector (cross) product in $\mathbb{R}^3$. This provides an explanation for the action of the spin operation defined by (6).

In a similar manner one can show that $RR^T = I$ and by differentiation obtain the new matrix

$$\tilde{\omega}(t) \equiv \dot{R}(t)R^T(t), \tag{7}$$

which is also skew-symmetric. Therefore, combination of Eqs. (4), (5) and (7) yields eventually

$$\dot{R} = \tilde{\omega}R = R\widetilde{\Omega}. \tag{8}$$

Next, direct differentiation of Eq. (3) with a simultaneous application of Eqs. (7) and (8) yields

$$\underline{v} = \dot{\underline{x}} = \dot{R}\underline{X} = \dot{R}R^T\underline{x} = \tilde{\omega}\underline{x}. \tag{9}$$

Finally, Eqs. (7), (8) and (4) lead to

$$\tilde{\omega} = R\widetilde{\Omega}R^T \quad \Rightarrow \quad \underline{\omega} = R\underline{\Omega}. \tag{10}$$

Having established the necessary kinematics, one can evaluate the angular momentum of the rigid body relative to the origin O. Specifically, this quantity is defined by

$$\underline{h}_O \equiv \int_m \underline{x} \times \underline{v}\, dm,$$

where m is the mass of the body. Then, by performing standard operations, it easily turns out that

$$\underline{h}_O(t) = I_O(t)\underline{\omega}(t), \tag{11}$$

with

$$I_O(t) = R(t)J_O R^T(t) \quad \text{and} \quad J_O = \int_m \widetilde{X}^T\widetilde{X}\, dm$$
$$= \int_m [(\underline{X} \cdot \underline{X})I - \underline{X}\underline{X}^T]dm, \tag{12}$$

where J_O is the mass moment of inertia matrix of the rigid body with respect to the origin O and the basis $\mathfrak{B}'$. Alternatively, introducing the body (or convective) angular momentum by

$$\underline{H}_O(t) = J_O\underline{\Omega}(t), \tag{13}$$

it turns out by combining the last three equations that

$$\underline{h}_O = RJ_O R^T\underline{\omega} \quad \Rightarrow \quad \underline{h}_O = R\underline{H}_O. \tag{14}$$

Then, the dynamics of the rigid body is expressed by Euler's law in the spatial form

$$\dot{\underline{h}}_O = \underline{m}_O.$$

After employing Eqs. (11)–(14), the last equation can be put in the convective form

$$J_O\dot{\underline{\Omega}} + \underline{\Omega} \times \underline{H}_O = \underline{M}_O, \tag{15}$$

where $\underline{m}_O$ and $\underline{M}_O$ are the spatial and body components of the resultant external moment about O.

Finally, the mass moment of inertia is also useful in evaluating the kinetic energy, defined by

$$T = \frac{1}{2}\int_m \dot{\underline{x}} \cdot \dot{\underline{x}}\, dm.$$

By employing relations established by the kinematic analysis and performing successive and straightforward operations the kinetic energy of the motion can be expressed in the convective form

$$T = \frac{1}{2}\underline{\Omega}^T J_O\underline{\Omega} = \frac{1}{2}\underline{H}_O \cdot \underline{\Omega}. \tag{16}$$

3. Some useful elements of differential geometry and Lie group theory

In this section, some fundamental concepts of differential geometry are briefly introduced. These concepts are necessary for comprehending the material presented in Sections 4 and 5.

3.1. Lie groups and subgroups

A group is a pair $(G, *)$, where G is a set of elements related by a binary operation $*$, known as the product of the group (Bowen and Wang, 2008). The most important property of a group is that if p and q are elements of G, then $p * q$ belongs to G, too. Moreover, there exists a special element e of G, known as the identity element, such that $e * p = p * e = e$ for any element p of G. Frequently, a group is denoted simply by G. In addition, the product $*$ is omitted usually.

The elements of a set can be considered as points in a geometrical space, with position determined by a system of coordinates. In particular, a differentiable manifold with a smooth group product and inverse operation is known as a Lie group (Marsden and Ratiu, 1999). This differentiability property permits a natural extension of differential and integral calculus techniques used in a Euclidean space and facilitates their application on a general manifold. A useful tool for accomplishing this task is the concept of the tangent space. This is a geometrical object, defined at each point of a smooth manifold and includes all the vectors which are tangent to all the curves of the manifold passing through the point. It is a vector space with dimension equal to the number of coordinates of the manifold (Papastavridis, 1999). To compare vectors belonging to different tangent spaces on a manifold, the so called connection operator is needed (see Appendix B). Finally, more structure can be added to a manifold by defining a metric tensor on each tangent space (Dodson and Poston, 1991).

Lie groups present some extra mathematical structure. For instance, if g and h are elements of a Lie group G, then one can define the left translation by g through the $L_g : G \to G$ mapping

$$L_g(h) = gh. \tag{17}$$

Likewise, the right translation by g is defined as a mapping R_g: $G \to G$, with

$$R_g(h) = hg. \tag{18}$$

These operations are smooth mappings and give rise to useful gradients, which are linear transformations between tangent spaces at points of G (Bowen and Wang, 2008).

Specifically, let $h(t)$ be a curve on G, with $h(0) = h$ and tangent vector $\dot{h}(0)$ at $t = 0$. This vector is denoted by $\underline{X}_h$ and belongs to the tangent space to G at h, denoted by T_hG. Then, according to Eq. (17), the set of points $p(t) = L_g(h(t))$ represents the image curve of $h(t)$ on G, obtained with a left translation by g. Then, the velocity vector of $p(t)$ at point $L_g(h)$ is given by

$$\dot{p} = \frac{d}{dt}\{L_g(h(t))\}|_{t=0} = L_{g*}\dot{h}(0), \tag{19}$$

where the quantity L_{g*} defines a linear transformation from T_hG to T_pG, known as the differential of L_g at h (Frankel, 1997). Consequently, Eq. (19) can be set in the form

$$\underline{X}_p = \underline{X}_{gh} = L_{g*}\underline{X}_h. \tag{20}$$

A vector field X on G satisfying the last equation is called left invariant. The mappings expressed by Eqs. (17) and (20) are independent on the path joining points h and p. Also, if $\underline{X}_e$ is a tangent vector of T_eG, then operation (20) can be employed to left translate $\underline{X}_e$ to all points of G by

$$\underline{X}_g = L_{g*}\underline{X}_e. \tag{21}$$

In this way, one can take a basis $\{\underline{e}_i(e)\}$ of $T_e\,G$ and left translate it to a basis $\{\underline{e}_i(p)\}$ of T_pG, with

$$\underline{e}_i(p) = L_{p*}\underline{e}_i(e), \tag{22}$$

on all of G. Then, any vector of the vector space T_pG, of dimension n, can be expressed in the form

$$\underline{v}(p) = \sum_{i=1}^{n} \underline{v}^i(p)\underline{e}_i(p) = v^i(p)\underline{e}_i(p), \tag{23}$$

where upper indices are chosen for the components of $\underline{v}$. The necessity for this will become clearer in Section 5. Moreover, if $\underline{v}$ is a left invariant vector field on G, then Eqs. (21) and (22) yield

$$\underline{v}(p) = L_{p*}\underline{v}(e) = L_{p*}[v^i(e)\underline{e}_i(e)] = v^i(e)L_{p*}\underline{e}_i(e) = v^i(e)\underline{e}_i(p).$$

Eventually, direct comparison of the last result with Eq. (23) yields

$$v^i(p) = v^i(e), \tag{24}$$

showing that the components of a left invariant vector field in a left invariant basis remain constant.

Left or right translation on a Lie group is important in the study of the one parameter subgroups of a group G. These are specific curves, say $g(t) : \mathbb{R} \to G$, satisfying the group homomorphism

$$g(t+s) = g(t)g(s) = g(s)g(t). \tag{25}$$

The second equality means that these subgroups are commutative (or Abelian). It can easily be shown (Frankel, 1997) that the most general monoparametric subgroup of G must pass from the identity e and is determined by the exponential map, acting from T_eG to G, with

$$g(t) = \exp[tg'(0)]. \tag{26}$$

This implies that its points are uniquely located by its tangent vector $g'(0)$ at the identity. Also, it can be shown by using Eq. (25) that

$$g'(t) = L_{g(t)*}g'(0) = R_{g(t)*}g'(0), \tag{27}$$

which demonstrates that the tangent vector $g'(t)$ to a monoparametric Lie subgroup undergoes a left or right translation along the subgroup. Put it in another way, Eq. (27) shows that the one parameter subgroup of G, with tangent vector $g'(0)$ at the identity, coincides with the integral curve through the identity of the vector field, which results by a left translation of $g'(0)$ over all of G.

3.2. Lie algebra and canonical connections

Given a Lie group G, one can construct its Lie algebra. This consists of the vector space T_eG, equipped with a special operator, known as Lie bracket. This operator is bilinear, skew-symmetric and satisfies Jacobi's identity (Warner, 1983). For a general manifold, it represents a map $[\cdot,\cdot]$, taking two vector fields $\underline{X}$ and $\underline{Y}$ to a new vector field (see Appendix A). For Lie groups, in particular, the Lie bracket can be defined in several ways. One of them employs the idea of left (or right) invariant vector fields. For instance, consider two vectors $\underline{X}_e$ and $\underline{Y}_e$ of T_eG and extend them by left translation to the vector fields $\underline{X}^L$ and $\underline{Y}^L$ on all of G. Then, their Lie bracket is defined by

$$[\underline{X}_e, \underline{Y}_e] \equiv [\underline{X}^L, \underline{Y}^L]_e. \tag{28}$$

For a Lie group, the Lie bracket is useful in defining an appropriate connection operator. In fact, it has been proved that there exist three such canonical connections (Bertram, 2008). The first two of them, known as left and right invariant canonical connection, are defined by

$$\nabla^R_{\underline{X}}\underline{Y} = [\underline{X}^L, \underline{Y}] \tag{29}$$

and

$$\nabla^L_{\underline{X}}\underline{Y} = [\underline{X}^R, \underline{Y}], \tag{30}$$

where $\underline{X}^L$ and $\underline{X}^R$ is a left and a right invariant vector field, respectively, extending an element $\underline{X}$ of the tangent space at a point of a Lie group G to all of G, while $\underline{Y}$ is a general vector field on G. Likewise, the symmetric canonical connection is defined by

$$\nabla^S_{\underline{X}}\underline{Y} = \frac{1}{2}[\underline{X}^L + \underline{X}^R, \underline{Y}]. \tag{31}$$

Clearly, all these connections are described completely by the Lie bracket and left/right translation. In Appendix A it is shown that the Lie bracket of the vector fields $\underline{X}$ and $\underline{Y}$ is given by Eq. (A6)

$$[\underline{X}, \underline{Y}] = \left(X^j\partial_jY^i - Y^j\partial_jX^i + c^i_{jk}X^jY^k\right)\underline{e}_i. \tag{32}$$

These three connections possess some important properties, which are summarized in the sequel.

First, let $\{\underline{e}^L_i(p)\}$ (or simply $\{\underline{e}_i(p)\}$) be a basis created at point p by left translating a basis $\{\underline{e}_i(e)\}$ at the identity, using Eq. (22). Then, $\underline{e}_i(p)$ is L_g-related to $\underline{e}_i(e)$ (Warner, 1983). Therefore

$$L_{g*}[\underline{e}_i(e), \underline{e}_j(e)] = [L_{g*}\underline{e}_i(e), L_{g*}\underline{e}_j(e)] = [\underline{e}_i(p), \underline{e}_i(p)],$$

which by Eq. (A8) implies that

$$L_{g*}\left\{c^k_{ij}(e)\underline{e}_k(e)\right\} = c^k_{ij}(e)\{L_{g*}\underline{e}_k(e)\} = c^k_{ij}(e)\underline{e}_k(p) = c^k_{ij}(p)\underline{e}_k(p)$$
$$\Rightarrow \quad c^k_{ij}(p) = c^k_{ij}(e). \tag{33}$$

This justifies the terminology 'structure constants' of the left invariant basis $\{\underline{e}_i(p)\}$. In addition, if both $\underline{X}$ and $\underline{Y}$ are left invariant vector fields on G, that is $\partial_jX^i = 0 = \partial_jY^i$, Eq. (32) implies that

$$[\underline{X},\underline{Y}] = \left(c^i_{jk}X^jY^k\right)\underline{e}_i \equiv Z^i\underline{e}_i.$$

Therefore, since the quantity $Z^i = c^i_{jk}X^jY^k$ remains constant everywhere on the manifold, this shows that the Lie bracket of two left invariant vector fields is a new left invariant vector field. That is

$$[\underline{X}^L,\underline{Y}^L] = \underline{Z}^L. \tag{34}$$

In a similar manner, if $\underline{X}$ is a left invariant vector field while $\underline{Y}$ is a right invariant vector field on G, it can be shown that their Lie bracket vanishes identically (Frankel, 1997), that is

$$[\underline{X}^L,\underline{Y}^R] = \underline{0}. \tag{35}$$

Then, it becomes apparent from Eq. (29) that

$$\nabla^R_{\underline{X}}Y^R = \underline{0}, \tag{36}$$

for any right invariant vector field $\underline{Y}$ on G. This means that the parallel translation of a vector on a manifold equipped with a left invariant canonical connection is equivalent to a right translation of it.

Next, by employing the definition of the left invariant canonical connection one arrives at

$$\nabla^R_{\underline{X}}\underline{Y} = [\underline{X}^L,\underline{Y}] = \left(X^j\partial_jY^i + c^i_{jk}X^jY^k\right)\underline{e}_i \quad \Rightarrow \quad \nabla^R_{\underline{X}}\underline{Y} = \left(\partial_jY^i + c^i_{jk}Y^k\right)X^j\underline{e}_i, \tag{37}$$

Geometrical interpretation since then $\underline{X} = \underline{X}^L$ and consequently $\partial_jX^i = 0$. On the other hand, by employing Eq. (B2), the covariant differential of $\underline{Y}$ along $\underline{X}$ is evaluated in the form

$$\nabla_{\underline{X}}\underline{Y} = \left(\partial_jY^i + \Lambda^i_{jk}Y^k\right)X^j\underline{e}_i.$$

Therefore, direct comparison of the last relation with Eq. (37) reveals that the structure constants of the left invariant basis must be equal to the affinities of the left invariant canonical connection, that is

$$c^i_{jk} = \Lambda^i_{jk}. \tag{38}$$

Consequently, by substitution in Eq. (B10) it follows immediately that the components of the torsion tensor associated to this connection must be given by $\tau^i_{jk} = -\Lambda^i_{kj}$. This indicates that the left invariant canonical connection possesses torsion when $\Lambda^i_{jk} \neq 0$. On the other hand, direct substitution of the last relation in Eq. (B14) shows that in such a case, all the components of the curvature tensor vanish ($R^i_{jkl} = 0$), which means that the curvature tensor of this connection is zero (Bertram, 2008).

Similar results are also available for the right invariant canonical connection $\nabla^L_{\underline{X}}\underline{Y}$. However, the picture obtained for $\nabla^S_{\underline{X}}\underline{Y}$ is different. First, if $\{\underline{e}^R_i(p)\}$ is a basis created at p by a right translation of the basis $\{\underline{e}_i(e)\}$, then application of the definitions expressed by Eqs. (31) and (B1) leads to

$$\nabla^S_{\underline{e}_j}\underline{e}_k = \frac{1}{2}\left[\underline{e}^L_j + \underline{e}^R_j, \underline{e}^L_k\right] \quad \Rightarrow \quad \Lambda^i_{jk}\underline{e}_i = \frac{1}{2}\left[\underline{e}^L_j, \underline{e}^L_k\right] + \frac{1}{2}\left[\underline{e}^R_j, \underline{e}^L_k\right].$$

In addition, taking into account Eq. (35), in conjunction with the definition (A8), it turns out that

$$\Lambda^i_{jk}\underline{e}_i = \frac{1}{2}\left[\underline{e}^L_j, \underline{e}^L_k\right] = \frac{1}{2}\left[\underline{e}_j, \underline{e}_k\right] = \frac{1}{2}c^i_{jk}\underline{e}_i,$$

or eventually

$$c^i_{jk} = 2\Lambda^i_{jk}. \tag{39}$$

Therefore, by direct substitution in Eq. (B10) it follows immediately that

$$\tau^i_{jk} = -\Lambda^i_{jk} - \Lambda^i_{kj}.$$

Also, based on Eqs. (A10) and (39), the affinities of the connection are anti-symmetric in their lower indices, i.e.,

$$\Lambda^i_{jk} = -\Lambda^i_{kj}. \tag{40}$$

This implies eventually that $\tau^i_{jk} = 0$, meaning that the symmetric canonical connection possesses no torsion. On the other hand, direct substitution of Eq. (39) in Eq. (B14) shows that some components of the curvature tensor are non-zero. Therefore, this connection possesses curvature.

For a Lie group, the choice of a canonical connection is a natural way to match the special integral curves associated with its properties as a manifold and as a group. Specifically, from Eq. (A10) it is obvious that the structure constants are always anti-symmetric. Consequently, Eqs. (38) and (39) reveal that the canonical connections considered lead to anti-symmetric affinities, satisfying Eq. (40). Then, Eq. (B7) shows that the tangent vector at each point of an autoparallel curve has constant components on a local frame produced by a left translation. Therefore, each canonical connection relates the affinities Λ^i_{jk} (which are defined by Eq. (B1) and express a manifold property) to the structure constants c^i_{jk} of the basis, so that the one parameter Lie subgroups and the curves resulting by their left translation (which are related to the group properties only) coincide with autoparallel curves (which are related to the manifold properties only). Moreover, these Lie subgroups and their left translations are conveniently captured by the exponential map, as was shown in Section 3.1. All these results will be shown to have remarkable implications in rigid body kinematics and dynamics, examined in the following two sections.

4. Rigid body kinematics by using Lie group theory

In this section, rigid body rotation about a fixed point is reconsidered, by employing concepts of differential geometry. A new manifold is first introduced, drawing its basic properties through a group representation on the classical $SO(3)$ group (Kobayashi and Nomizu, 1963). Study of this manifold, called $M(3)$, offers a strong basis for a complete and clear interpretation of rigid body kinematics.

4.1. Introduction of manifold M(3) for the description of rigid body rotation

The set of orthogonal matrices $R(t)$ introduced in Section 2 forms a Lie group. Specifically, each matrix $R(t)$ represents a point in the space of 3×3 matrices, coinciding with the Euclidean space $\mathbb{R}^9$. Considering the orthogonality condition (4), this point lies on a three dimensional subset of $\mathbb{R}^9$ (Frankel, 1997). Since these conditions are nonlinear, this subset is not a vector subspace of $\mathbb{R}^9$ but it forms a three dimensional manifold, instead, which can be viewed as a surface in $\mathbb{R}^9$. In addition, since composite rotations are represented by products of orthogonal matrices (Shabana, 2005), which are also orthogonal matrices, this subset is a Lie group, having as product the matrix multiplication and as identity element the 3×3 identity matrix I. Moreover, since $R(0) = I$, its determinant at $t = 0$ is equal to $+1$ and because there can occur no jump to the value -1 during the subsequent motion, this matrix belongs to a group known as the special orthogonal group of order three, denoted by $SO(3)$.

As will be shown next, the geometry of $SO(3)$ fails to predict both the kinematics and the dynamics of a rigid body. Therefore, a new manifold is introduced in this work for the correct description of rigid body rotation. This manifold belongs to the same abstract group as $SO(3)$, but possesses different geometrical properties. More specifically, its configuration space is isomorphic to that of $SO(3)$, but it possesses different connection and metric.

For this reason, this manifold is named $M(3)$. Next, the emphasis is placed in identifying the critical geometrical properties of $M(3)$.

First, the orientation of a rigid body can be represented by a point, say p, on the manifold formed by the newly introduced rotation group $M(3)$. Of large importance is also the definition of the vector space $T_pM(3)$, the tangent space of $M(3)$ at p. In general, the location of points and the evaluation of the components of vectors and other important geometrical objects on a manifold depends on the choice of a local coordinate system and a basis of the tangent space at each point of the manifold. For example, the holonomic set of coordinates corresponding to the classical Euler angles provides a full picture of rigid body kinematics, except at certain singular points (Bauchau, 2011). However, using holonomic coordinates for $M(3)$ and a non-natural basis for $T_pM(3)$ leads to certain advantages.

After choosing an appropriate local coordinate system, each point on $M(3)$ is described by three coordinates (or rotation parameters). Then, the spherical motion of a rigid body can be viewed as a motion of a point on a single parameter curve, say $\gamma(t) : \mathbb{R} \to M(3)$. Moreover, the angular velocity of the body is expressed by the tangent vector to $\gamma(t)$, defined by

$$\underline{w}(t) = \sum_{i=1}^{3} w^i(t)\underline{e}_i(t) = w^i(t)\underline{e}_i(t), \tag{41}$$

where $w^i(t)$ are the components of the tangent vector $w(t)$ in a local basis $\{\underline{e}_i(t)\}$ (with $i = 1,2,3$) of the tangent space $T_pM(3)$ at the current position, represented by point p of the manifold.

A useful concept in detecting changes of a vector field $\underline{v}(t)$ on $M(3)$ is the covariant differential of this field along the direction defined by vector w (see Appendix B). This quantity is determined by

$$\nabla_{\underline{w}}\underline{v}(t) = \left(\dot{v}^i + \Lambda^i_{jk} w^j v^k\right)\underline{e}_i.$$

Then, a parallel translation of vector $\underline{v}$ along the curve $\gamma(t)$, with tangent vector $\underline{w}$, is defined by

$$\nabla_{\underline{w}}\underline{v} = 0 \quad \Rightarrow \quad \dot{v}^i + \Lambda^i_{jk} w^j v^k = 0. \tag{42}$$

This represents a set of three coupled linear ordinary differential equations in $v^j(t)$, possessing a unique solution for a given set of initial conditions $v^j(0)$ (Nayfeh and Balachandran, 1995). However, in the general case, the affinities Λ^i_{jk} of the connection ∇ depend on position. The same is also true for the tangent vector $\underline{w}$. This means that the system of equations represented by Eq. (42) has variable coefficients in those cases. Nevertheless, there exists a special occasion where one can select the affinities so that the solution $\underline{v}(t)$ can be obtained in a convenient closed form, in terms of an exponential matrix, as explained next.

First, in the particular case with $\underline{v} = \underline{w} \equiv \underline{n}$, satisfaction of the condition of parallel translation leads to special curves on the manifold, known as autoparallel curves (Shabanov, 1998; Marsden and Ratiu, 1999). If, in addition, the affinities of the connection are antisymmetric, as in Eq. (40), it is shown in Appendix B (see Eq. (B8)) that the components of the tangent vector $\underline{n}(t)$ to the autoparallel curve in the local frame remain constant on the whole curve. That is

$$n^i(t) = n^i(0) \equiv n^i. \tag{43}$$

Therefore, if the affinities Λ^i_{jk} are also constant everywhere on the manifold and such that

$$\left[\Lambda^i_{jk} n^j\right] = [\tilde{n}^i_k] = \tilde{n} = spin(\underline{n}), \tag{44}$$

the parallel translation of any vector $\underline{u}(t)$ of $T_pM(3)$ along the autoparallel curve of $M(3)$ connecting point $p(t)$ to any other point of $M(3)$ is described by the following system of linear ordinary differential equations

$$\begin{pmatrix} \dot{u}^1 \\ \dot{u}^2 \\ \dot{u}^3 \end{pmatrix} = -\begin{bmatrix} 0 & -n^3 & n^2 \\ n^3 & 0 & -n^1 \\ -n^2 & n^1 & 0 \end{bmatrix} \begin{pmatrix} u^1 \\ u^2 \\ u^3 \end{pmatrix} \quad \text{or} \quad \dot{\underline{u}} = -\tilde{n}\underline{u}. \tag{45}$$

Simple inspection verifies that Eq. (44), together with condition (40), can be fulfilled simultaneously, indeed, provided that the non-zero affinities take the following constant values

$$\Lambda^1_{23} = -\Lambda^1_{32} = \Lambda^2_{31} = -\Lambda^2_{13} = \Lambda^3_{12} = -\Lambda^3_{21} = 1, \tag{46}$$

on all of $M(3)$. Since this set of affinities takes constant values, it can not be obtained by any natural basis, resulting by any of the classical sets of holonomic coordinates. Therefore, there appears the need for the selection of an appropriate non-natural (or anholonomic) basis. Since $M(3)$ forms a Lie group, this basis can conveniently be obtained by extending a basis $\{\underline{e}_i(e)\}$, defined in $m(3) \equiv T_eM(3)$, on all of $M(3)$ through a right or left translation. Taking into account Eq. (1), the latter choice is preferred, leading to

$$\underline{e}_i(p(t)) = L_{p(t)*}\underline{e}_i(e), \quad i = 1,2,3, \tag{47}$$

so that each basis vector $\underline{e}_i(p(t))$ is part of a left invariant vector field on $M(3)$. In essence, this is identical to Eq. (22), since for a given point p these vectors are fixed and their dependence on t is only implicit. To stress this, the basis vector $\underline{e}_i(p(t))$ will next be denoted simply by $\underline{e}_i(p)$. An immediate consequence of this basis choice is that if $\underline{v}(t)$ is any left invariant vector field on $M(3)$, then its representative vector at point p can be expressed in the form

$$\underline{v}_p(t) = v^i_p(t)\underline{e}_i(p).$$

Moreover, application of Eq. (24) yields

$$v^i_p(t) = v^i_e(t) \equiv v^i(t). \tag{48}$$

Then, Eq. (43) can be rewritten in the form

$$n^i_p(t) = n^i_e(t) \equiv n^i(t),$$

illustrating that the tangent vector to an autoparallel curve of $M(3)$ is part of a left invariant vector field on $M(3)$. Therefore, taking into account Eqs. (40) and (46), it is convenient to choose the left invariant canonical connection, expressed by Eq. (29), as most appropriate for $M(3)$. An immediate consequence of this, with enormous significance, is that the autoparallels of $M(3)$ will coincide with its one parameter Lie subgroups and their left translations (see end of Section 3.2). In this respect, Eq. (45) can be seen as a means of determining the components of any vector $\underline{u}_p(t)$ of $T_pM(3)$, obtained by parallel translation of a vector $\underline{u}_e(t)$ of $T_eM(3)$ along the autoparallel curve of $M(3)$ connecting point $p(t)$ to the identity e. In fact, since the coefficient matrix $\tilde{n}$ in Eq. (45) is constant, this solution can be expressed in the following form

$$\underline{u}_p(t) = B(t)\underline{u}_e(t) \quad \Rightarrow \quad \underline{u}_e(t) = A(t)\underline{u}_p(t), \tag{49}$$

with

$$A(t) = \exp(t\tilde{n}) \tag{50}$$

and

$$B(t) = A^{-1}(t) = \exp(-t\tilde{n}). \tag{51}$$

Among the infinity of available choices, the specific selection of the affinities expressed by Eq. (46) leads to a 3×3 matrix $A(t)$, given by Eq. (50), which resembles the map of Eq. (26). Here, however, this matrix is not an element of $M(3)$ and it appears in Eq. (49) as a linear transformation from $T_pM(3)$ to $T_eM(3)$, instead. In fact, it will be shown in Section 4.2 that $A(t)$ belongs to $SO(3)$.

The ideas presented above can be used to provide a complete and clear geometric interpretation of rigid body kinematics. For instance, the basis $\{\underline{e}_i(p)\}$ obtained by left translation of a basis $\{\underline{e}_i(e)\}$ of the tangent space $T_eM(3)$ on all of $M(3)$, as specified by Eq. (47), corresponds to a basis which remains fixed on the rigid body during its motion. For this reason, it is known as a body (or corotational) frame. This basis has as an advantage that it depends on the current orientation of the body only and not on its previous motion. In addition, the components of the tangent vector $\underline{w}(t)$ to the path $\gamma(t)$, representing the motion of the body, are components of the angular velocity of the body in the local basis $\{\underline{e}_i(p)\}$. That is

$$\underline{w}(t) = w^i(t)\underline{e}_i(p). \tag{52}$$

Obviously, this vector belongs to the tangent space $T_pM(3)$ at the current point p. Therefore, it can be viewed as the outcome of the left translation of a vector of $T_eM(3)$, which is expressed in the form

$$\underline{\Omega}(t) = \Omega^i(t)\underline{e}_i(e). \tag{53}$$

This vector is known as the convective angular velocity in the engineering literature (Simo and Wong, 1991). Specifically, based on Eq. (48), the following choice is made for its components

$$w^i(t) = \Omega^i(t). \tag{54}$$

Then, it is straightforward to prove with the help of Eqs. (53), (47) and (54) that

$$L_{p(t)*}\underline{\Omega}(t) = L_{p(t)*}[\Omega^i(t)\underline{e}_i(e)] = \Omega^i(t)L_{p(t)*}\underline{e}_i(e) = w^i(t)\underline{e}_i(p).$$

Direct comparison of the last result with Eq. (52) yields

$$\underline{w}(t) = L_{p(t)*}\underline{\Omega}(t). \tag{55}$$

Alternatively, vector $\underline{w}(t)$ can also be obtained by a parallel transfer of another vector belonging to the tangent space $T_eM(3)$, through the autoparallel curve of $M(3)$ joining the identity to the current point p. If this new vector appears in the form

$$\underline{\omega}(t) = \omega^i(t)\underline{e}_i(e), \tag{56}$$

then, according to Eq. (49), its components are interrelated to those of $\underline{w}(t)$ with

$$w^i(t) = B^i_j(t)\omega^j(t) \quad \text{and} \quad \omega^i(t) = A^i_j(t)w^j(t).$$

Taking into account Eq. (54), these lead to

$$\Omega^i(t) = B^i_j(t)\omega^j(t) \quad \text{and} \quad \omega^i(t) = A^i_j(t)\Omega^j(t), \tag{57}$$

where $A^i_j(t)$ and $B^i_j(t)$ are the components of matrices $A(t)$ and $B(t)$, given by Eqs. (50) and (51), respectively. In essence, the relations in Eq. (57) are based on the condition

$$\nabla_{\underline{n}}\hat{\underline{\omega}} = \underline{0}, \tag{58}$$

where $\hat{\underline{\omega}}$ is a vector field generated by a parallel translation of $\underline{\omega}(t)$ along the autoparallels of $M(3)$ starting from its identity. Since $\underline{n}(t)$ is part of a left invariant vector field and the left invariant canonical connection has been selected for $M(3)$, Eq. (58) in combination with Eq. (36) implies that $\underline{\omega}(t)$ generates a right invariant vector field on $M(3)$. Therefore, for a given $\underline{w}(t)$ in $T_pM(3)$, one can find an $\underline{\omega}(t)$ in $T_eM(3)$ with

$$\underline{w}(t) = R_{p(t)*}\underline{\omega}(t). \tag{59}$$

The results expressed by Eqs. (55) and (59) demonstrate that the vector $\underline{w}(t)$, belonging to the tangent space $T_pM(3)$, has two important images in $T_eM(3)$. Namely, $\underline{w}(t)$ can be viewed as the outcome of a left or right translation of vector $\underline{\Omega}(t)$ or $\underline{\omega}(t)$, respectively, both belonging to $T_eM(3)$. In the special case with $\underline{\omega}(t) = \underline{\Omega}(t)$, these vectors coincide and have a collinear direction with vector $\underline{n}(t)$, which is tangent to the autoparallel curve joining the identity with the current point of $M(3)$. In such a case, the 3×3 exponential matrix $A(t)$, expressed by Eq. (50), represents pure rotation of the rigid body about the axis defined by $\underline{n}(t)$.

Next, the most important of the results obtained in the present section are illustrated by Fig. 2. The actual motion of the body is described by a path $\gamma(t)$ on $M(3)$, starting from the identity element at $t = 0$. Then, for a fixed time t, points on the autoparallel curve of $M(3)$ connecting the current point $p(t)$ to the origin e, say $\eta_p(s)$, are located by another parameter, say s, related to the length of this path. In this case, in order to distinguish the true path $\gamma(t)$ from the autoparallel $\gamma_p(s)$, it is more appropriate to replace matrix $A(t)$ in Eq. (50) by the more complex but more accurate form

$$Q(s, \underline{n}(t)) = \exp(s\tilde{n}(t)), \tag{60}$$

for $0 \leqslant s \leqslant t$, together with the scaling $Q(t,\underline{n}(t)) = A(t) = R(t)$. This means that at any given time t, matrices A and Q coincide with the rotation matrix R, describing the orientation of the body with respect to its original position. This is a manifestation of the well known Euler's theorem, stating that one can move a rigid body, rotating about a fixed point, from an initial to any final position, through a pure rotation about a fixed axis (Greenwood, 1988). Likewise, matrix B coincides with its transpose R^T. In addition, the tangent vector $\underline{w}(t)$ to the actual path $\gamma(t)$, given by Eq. (52), represents the angular velocity of the body and creates two images, $\underline{\Omega}(t)$ and $\underline{\omega}(t)$, in $T_eM(3)$, given by Eqs. (53) and (56), respectively, at any time t. The components of these two vectors are related by Eq. (57), which is equivalent to Eq. (10) of the classical approach. In fact, $\underline{w}(t)$ can be reproduced by a left translation of $\underline{\Omega}(t)$ or a right translation of $\underline{\omega}(t)$. The latter translation is also equivalent to a parallel translation along the autoparallel curve connecting the identity to the current point of $M(3)$.

The selection of the left invariant canonical connection for $M(3)$ identifies special curves related to its group and manifold nature. Then, Eq. (38) implies that the affinities Λ^i_{jk} must be equal to the structure constants c^i_{jk} of the left translated basis. Also, according to material presented in Section 3.2, $M(3)$ possesses torsion but has zero curvature. This explains why the parallel translation of a vector can be equivalent to its right translation, which depends only on the initial and final point and not on the actual path on $M(3)$.

Finally, note that there exist several closed form expressions for matrix $A(t)$ in terms of rotation parameters. Among them, the most commonly known is probably the Rodrigues formula

$$A(t) = \exp[\widetilde{\Psi}(t)] = I + \frac{\sin\|\underline{\Psi}\|}{\|\underline{\Psi}\|}\,\widetilde{\Psi} + \frac{1}{2}\,\frac{\sin^2\left(\frac{1}{2}\|\underline{\Psi}\|\right)}{\left(\frac{1}{2}\|\underline{\Psi}\|\right)^2}\,\widetilde{\Psi}\,\widetilde{\Psi}.$$

Comparison with Eq. (50) shows that the quantity $\underline{\Psi}$, known as the Cartesian rotation vector (Geradin and Cardona, 2001), is defined by

$$\underline{\Psi}(t) = t\underline{n}(t). \tag{61}$$

4.2. Geometrical properties of SO(3)

The vector space $so(3) \equiv T_ISO(3)$, the tangent space of the manifold formed by the elements of rotation group $SO(3)$ at its identity element I, corresponds to infinitesimal rotations. This space is known as the Lie algebra of $SO(3)$ and includes as elements all the 3×3 skew-symmetric matrices, like $\tilde{\omega}$ and $\widetilde{\Omega}$, defined in Section 2. Therefore, a standard basis for $so(3)$ is usually represented by

$$\tilde{e}_1(I) = spin\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{bmatrix}0&0&0\\0&0&-1\\0&1&0\end{bmatrix}, \quad \tilde{e}_2(I) = \begin{bmatrix}0&0&1\\0&0&0\\-1&0&0\end{bmatrix}$$
$$\text{and} \quad \tilde{e}_3(I) = \begin{bmatrix}0&-1&0\\1&0&0\\0&0&0\end{bmatrix}. \tag{62}$$

Moreover, the basis of the tangent space at any point $R(t)$ of $SO(3)$ can be obtained by a left translation of this basis, according to

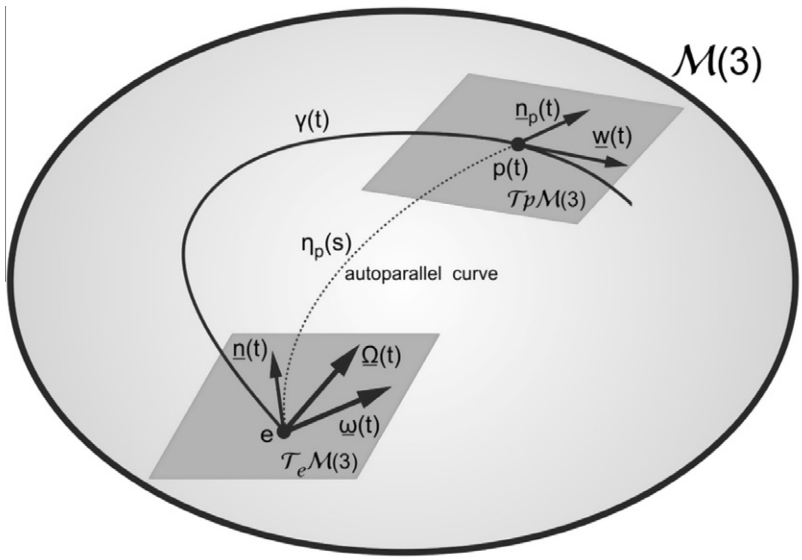

Fig. 2. Geometrical interpretation of spherical motion in $M(3)$ (vectors $\underline{w}(t)$, $\underline{\Omega}(t)$ and $\underline{\omega}(t)$ are defined by Eqs. (52), (53) and (56), respectively, while $\underline{n}(t) = n^i(t)\underline{e}_i(e)$ and $\underline{n}_p(t) = n^i(t)\underline{e}_i(p)$).

$$\hat{e}_i(R) = R(t)\tilde{e}_i(I), \quad i = 1,2,3. \tag{63}$$

Then, by employing the definition (A8), the corresponding structure constants of the basis $\{\tilde{e}_i(I)\}$ can be determined in a straightforward way from the definition of the Lie bracket. Specifically, all the non-zero structure constants in $so(3)$ are determined in the form

$$\tilde{c}^1_{23} = -\tilde{c}^1_{32} = \tilde{c}^2_{31} = -\tilde{c}^2_{13} = \tilde{c}^3_{12} = -\tilde{c}^3_{21} = 1. \tag{64}$$

Also, taking into account Eq. (63), application of Eq. (33) implies that the quantities $\tilde{\underline{c}}^i_{jk}$ retain the constant values they possess at the identity element on all of $SO(3)$.

Next, these structure constants will provide the foundation needed for evaluating the components of the connection on the same bases. According to the classical view, $SO(3)$ is a manifold with non-zero curvature and zero torsion (Sattinger and Weaver, 1986; Simo and Wong, 1991). Based on the results of Section 3.2, this can be achieved by employing the symmetric canonical connection, defined by Eq. (31). Therefore, direct application of Eq. (39) in combination with Eq. (64) yields the corresponding non-zero affinities in the form

$$\Lambda^1_{23} = -\Lambda^1_{32} = \Lambda^2_{31} = -\Lambda^2_{13} = \Lambda^3_{12} = -\Lambda^3_{21} = 1/2. \tag{65}$$

Consequently, from Eq. (B10) it turns out that $\tau^i_{jk} = 0$, which verifies that this choice renders $SO(3)$ torsionless. In addition, the components of the curvature tensor of $SO(3)$ in the standard basis can also be found by direct application of Eq. (B4). In particular, since the affinities are constant everywhere on $SO(3)$, the components of the curvature tensor will also be constant everywhere on $SO(3)$ and given by

$$R^i_{jkl} = \Lambda^i_{km}\Lambda^m_{lj} - \Lambda^i_{lm}\Lambda^m_{kj} - \tilde{c}^m_{kl}\Lambda^i_{mj}.$$

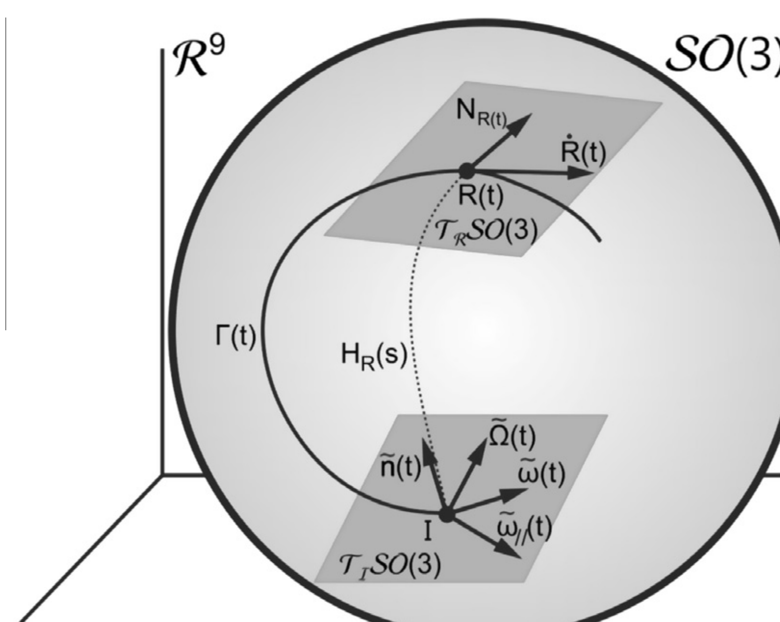

Fig. 3. Geometrical interpretation of a motion on $SO(3)$ (tangent vectors $\dot{R}(t)$, $\widetilde{\Omega}(t)$ and $\tilde{\omega}(t)$ are defined by Eqs. (66) and (67)).

In fact, by employing Eqs. (64) and (65) and performing direct calculation it turns out that

$$R^1_{212} = -R^1_{221} = R^1_{313} = -R^1_{331} = R^2_{121} = -R^2_{112} = R^2_{323} = -R^2_{332} = R^3_{131} = -R^3_{113} = R^3_{232} = -R^3_{223} = 1/4,$$

while all the remaining components of the curvature tensor are equal to zero.

In Fig. 3 are presented results referring to the geometry of a motion on $SO(3)$, similar to those presented for $M(3)$ in Fig. 2. In particular, this motion can be represented by a curve of $SO(3)$, say $\Gamma(t)$, with a tangent vector $\dot{R}(t)$ at the current position $R(t)$. Then, in analogy to Eqs. (55) and (59), the "tangent vector" $\dot{R}(t)$ can be viewed as the outcome of a left translation of a vector $\widetilde{\Omega}(t)$ and a right translation of another vector $\tilde{\omega}(t)$, both of $so(3)$, so that

$$\dot{R}(t) = R(t)\widetilde{\Omega}(t) \quad \text{and} \quad \dot{R}(t) = \tilde{\omega}(t)R(t), \tag{66}$$

respectively, in accordance to Eq. (21). Obviously, these results reproduce Eq. (8). Also, based on Eqs. (53) and (56), these two specific vectors of $so(3)$ are expressed in the form

$$\widetilde{\Omega}(t) = \Omega^i(t)\tilde{e}_i(I) \quad \text{and} \quad \tilde{\omega}(t) = \omega^i(t)\tilde{e}_i(I). \tag{67}$$

Then, Eq. (66a) in conjunction with Eq. (63), leads directly to

$$\dot{R}(t) = R(t)[\Omega^i(t)\tilde{e}_i(I)] = \Omega^i(t)[R(t)\tilde{e}_i(I)] \quad \Rightarrow \quad \dot{R}(t) = \Omega^i(t)\hat{e}_i(R), \tag{68}$$

while a similar treatment of Eq. (66b) leads to

$$\dot{R}(t) = [\omega^i(t)\tilde{e}_i(I)]R(t) = \omega^i(t)[\tilde{e}_i(I)R(t)] \quad \Rightarrow \quad \dot{R}(t) = \omega^i(t)\bar{e}_i(R), \tag{69}$$

where the vectors

$$\bar{e}_i(R) = \tilde{e}_i(I)R(t)$$

form the elements of a new basis, generated by right translating the standard basis $\{\tilde{e}_i(I)\}$ of $so(3)$. Also, since the connection chosen is canonical, the autoparallels of $SO(3)$ passing from its origin coincide with its one parameter Lie subgroups. Therefore, they are represented by the corresponding exponential map, expressed by Eq. (26). In fact, at any given time t, the points of the autoparallel curve starting at the identity and passing from the current element of $SO(3)$ are determined by the exponential matrix given by Eq. (50), or equivalently by Eq. (60). Based on Eq. (43), the tangent vector of the autoparallel at the identity is expressed in the form

$$\tilde{n}(t) = n^i(t)\tilde{e}_i(I) \equiv N_I(t).$$

Moreover, the tangent vector of the same curve at the current point $R(t) = Q(t, \underline{n}(t)) = A(t)$ is given by

$$N_R(t) = n^i(t)\hat{\underline{e}}_i(R).$$

Since the 3×3 matrices $A(t)$ and $Q(t, \underline{n}(t))$ are orthogonal, like $R(t)$, they belong to $SO(3)$. Finally, direct differentiation of Eq. (60) with respect to s leads to

$$Q' = Q\tilde{n} = \tilde{n}Q, \tag{70}$$

which resembles Eq. (27). In addition, direct comparison with Eq. (66) reveals that the tangent vector $\tilde{\underline{n}}(t)$ is part of both a left and a right invariant vector field on $SO(3)$.

The results presented in this section demonstrate that all the structure constants, the affinities and the components of the curvature tensor in the standard basis remain constant, while all the components of the torsion tensor are zero throughout $SO(3)$. Also, the values of the affinities given by Eq. (65) deviate from those required by Eq. (46). As a consequence, a factor of 1/2 enters the argument of the exponential matrices in Eqs. (50) and (51), governing parallel transfer of an arbitrary tangent vector along an autoparallel curve $H_R(s)$. More specifically, the form of those equations remains the same, but the transformation matrix appearing in them takes the following form in $SO(3)$

$$A_{SO(3)}(t) = \exp[t\tilde{n}(t)/2]. \tag{71}$$

Due to the 1/2 factor in the argument, the parallel transfer of a vector along the autoparallel is not equivalent to a right translation, with the exception of the tangent to the autoparallel. As a result, the vector of $T_I SO(3)$ resulting by a parallel translation of $\dot{R}(t)$, say $\tilde{\omega}_{II}(t)$, is not equal to vector $\tilde{\omega}(t)$, which produces $\dot{R}(t)$ through a right translation by $R(t)$, according to Eq. (66b). This renders the classical $SO(3)$ rotation group as not suitable for describing rigid body kinematics.

4.3. Group representation of manifold M(3) on SO(3)

In the present subsection, a group representation of $M(3)$ on $SO(3)$ is performed in order to extract all the important group properties, like the group product, identity element, structure constants and Lie bracket of $M(3)$ from those of $SO(3)$, through an appropriate differentiable map (Hall, 2003). This map, say Φ from $M(3)$ to $SO(3)$, is first selected to be one to one and onto, so that both of these groups belong to the same abstract group. That is, the map is invertible, while the two groups have the same dimension and each element of one is mapped uniquely to an element of the other. Moreover, this map must also be a homomorphism. In particular, if $*$ and $\cdot$ represent the product operations in group $M(3)$ and $SO(3)$, respectively, then

$$\Phi(p * q) = \Phi(p) \cdot \Phi(q), \quad \forall p, q \in M(3). \tag{72}$$

This implies that if p and q are elements of an Abelian subgroup of $M(3)$, then

$$p * q = q * p \quad \Rightarrow \quad \Phi(p) \cdot \Phi(q) = \Phi(q) \cdot \Phi(p),$$

which means that $\Phi(p)$ and $\Phi(q)$ are also elements of a corresponding Abelian subgroup of $SO(3)$. Thus, Φ preserves the structure of the subgroups of $SO(3)$. In fact, this map is then an isomorphism (Frankel, 1997). Since the group operation in $SO(3)$ is the ordinary multiplication of 3×3 matrices, an immediate consequence is that the group operation in $M(3)$ is defined by

$$p * q = \Phi^{-1}(\Phi(p)\Phi(q)). \tag{73}$$

Next, in order to derive an explicit form of the mapping Φ, appropriate coordinate systems on the manifolds and bases on the tangent spaces should first be selected for both $M(3)$ and $SO(3)$. For the latter, a suitable coordinate system and a basis can easily be obtained by embedding its manifold into the space of the 3×3 real matrices, which is equivalent to the Euclidean space $\mathbb{R}^9$. However, a more sophisticated choice needs to be made for both the coordinate system of $M(3)$ and the bases of the tangent spaces at any point of $M(3)$, as explained next.

In many occasions, it is beneficial to employ a special set of local coordinates on a general manifold, known as canonical or (more frequently but less accurately) as normal coordinates, which simplify the subsequent analysis (Murray et al., 1994). Specifically, let $\eta(s)$ be the autoparallel curve of a manifold M^n emanating from a point p of the manifold at $s = 0$, with a tangent vector n on the tangent space $T_p M^n$. If the coordinates of the origin (or pole) p are selected as

$$p^i = 0, \quad i = 1, \ldots, n$$

and the tangent vector $\underline{n}$ is expressed over a basis $\{\underline{e}_i\}$ of $T_p M^n$ in the form

$$\underline{n} = n^i \underline{e}_i,$$

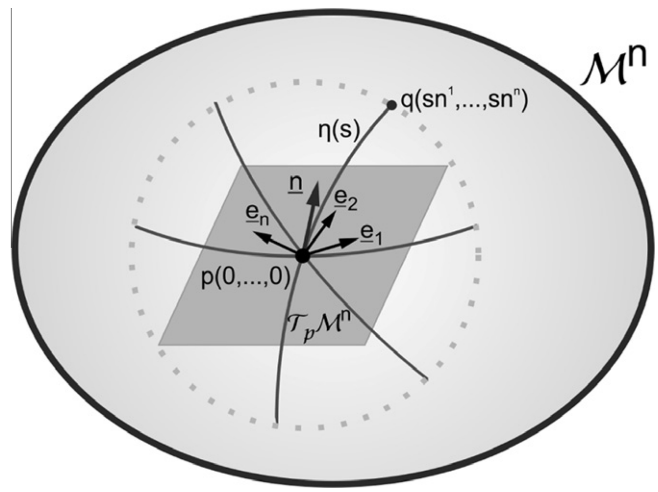

Fig. 4. Definition of canonical coordinates on a manifold.

then the canonical coordinates of any point q on the curve $\eta(s)$ are uniquely specified to be

$$q^i = sn^i. \tag{74}$$

Namely, the coordinates of points on an autoparallel curve are linear in s. Comparison with Eq. (26) reveals that the corresponding exponential map coincides with the identity. Moreover, according to Eq. (B7), the autoparallel curve originating from p with a tangent vector $n^i = \dot{q}^i$ and passing from point q is given by

$$\frac{d^2q^i}{ds^2} + \Lambda^i_{jk}(q)\frac{dq^j}{ds}\frac{dq^k}{ds} = 0.$$

Then, taking Eq. (74) into account this yields

$$\Lambda^i_{jk}(q)n^jn^k = 0.$$

Since the last condition must hold for arbitrary n^i at the origin p, the following condition must be satisfied by the symmetric part of the affinities

$$\Lambda^i_{(jk)}(p) \equiv \frac{1}{2}\left[\Lambda^i_{jk}(p) + \Lambda^i_{kj}(p)\right] = 0. \tag{75}$$

On the other hand, $\Lambda^i_{(jk)}(q) \neq 0$ for $q \neq p$, since the components n^i are fixed at point q.

Canonical coordinates specify uniquely any point q of M^n in the vicinity of the pole p. A geometrical picture of a local canonical coordinate patch is shown in Fig. 4. Since the autoparallels starting from p can cross on another point of M^n, again, several canonical coordinate patches may be selected to cover it completely. However, despite the fact that the canonical coordinates are holonomic, the natural coordinate basis they provide is not useful in applications, since it depends on the local coordinates. For this reason, the (holonomic) canonical coordinate system is employed in locating the coordinates of a point in a unique way, while another non-natural coordinate frame is used for establishing a more convenient basis of the tangent space at any point of M^n.

In the special case of a Lie group, it was shown at the end of Section 3.2 that the affinities Λ^i_{jk} can be obtained from the structure constants c^i_{jk}, so that the autoparallel curves of the manifold coincide with the one parameter subgroups and their left translations. In the particular case of $M(3)$, it was demonstrated in Section 4.1 that the left invariant canonical connection is the most natural choice for it, with affinities given by Eq. (46). Then, its autoparallel curves are identified by Eq. (43), which leads to Eq. (74), with $n = 3$. Therefore, Eq. (74) defines a canonical coordinate system on $M(3)$ with origin at its identity element e. Moreover, direct comparison of Eq. (74) with Eq. (61) reveals that the canonical coordinates coincide with the components of the so called Cartesian rotation vector. In fact, it can be shown that these coordinates also fulfill condition (75). In this respect, the quantity defined by Eq. (61) is actually not a vector but represents canonical coordinates, instead.

Based on the above, a canonical coordinate system is first placed on $M(3)$ with origin at e for locating its points (see Fig. 5). Next, a basis $\{\underline{e}_i(e)\}$ is selected at $m(3)$, according to conditions that will be stated more explicitly in Section 5. This basis is then extended to all points q of $M(3)$ by left translation, according to

$$\underline{e}_i(q) = q\underline{e}_i(e). \tag{76}$$

This is a non-natural basis and, consequently, it is not a convenient system for the coordinates of points on the manifold, since they then depend on the path (Papastavridis, 1999). Its main advantage is that it depends on the current position only.

According to material presented in Sections 3.2 and 4.1, an autoparallel curve emanating from the identity element e of $M(3)$ coincides with a one parameter Lie subgroup, which corresponds to pure rigid body rotation about an axis determined by the tangent vector to the curve at the identity. Conversely, given any point q on the manifold, one can find a vector on the tangent space at the identity element of $M(3)$, representing an axis of rotation of the body, which is tangent to the unique one parameter subgroup emanating from e and passing from q. Moreover, these special curves are captured by the corresponding exponential map of the group. In fact, the exponential map of a group G is a local diffeomorphism (i.e., it is one to one, onto and possesses a differentiable inverse) from a neighborhood of zero in T_eG onto a neighborhood of e in G (Marsden and Ratiu, 1999). In addition, this map can be

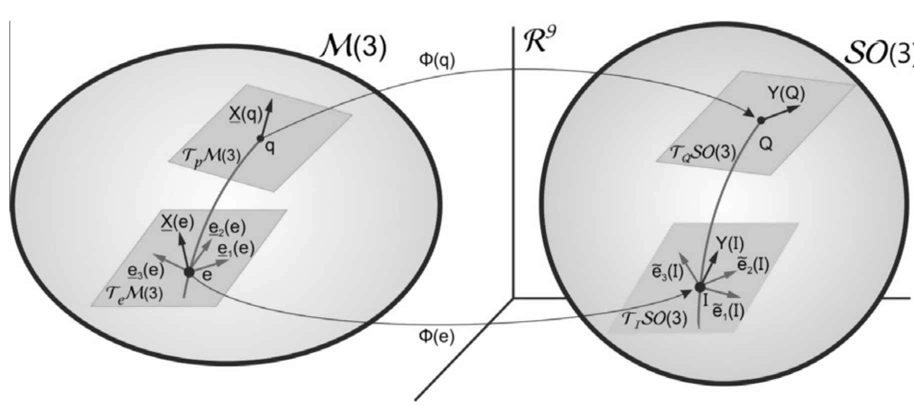

Fig. 5. Definition of mapping Φ for a group representation of $M(3)$ on $SO(3)$.

extended over all of G through a left translation. This implies that the exponential map provides a convenient coordinate system on all of G. Therefore, guided by Eq. (50), the mapping Φ from $M(3)$ to $SO(3)$ is selected in the following form

$$\Phi(q) = \exp(s\tilde{n}), \tag{77}$$

where $\tilde{\underline{n}} = spin(\underline{n})$ and $\underline{n}$ is the vector of $m(3)$ which is tangent to the autoparallel curve starting from the identity e and passing from point q of $M(3)$ (see Fig. 4). Based on Eq. (74), the components of q depend on the parameters s and $\underline{n}$. Therefore, the quantity

$$Q(s,\underline{n}) = \Phi(q(s,\tilde{n})) \tag{78}$$

represents a 3×3 exponential matrix. Since matrix Q is orthogonal, it is an element of $SO(3)$, indeed. In view of Eq. (74), this matrix represents a map from the canonical coordinates of point q to $\mathbb{R}^9$. Moreover, since the identity element e of $M(3)$ has canonical coordinates (0,0,0), it turns out from Eqs. (77) and (78) that

$$\Phi(e) = Q(0,\underline{n}) = I, \tag{79}$$

which verifies that Φ maps the identity element e of $M(3)$ to the identity element I of $SO(3)$.

Next, in order to determine the Lie bracket of $m(3)$, the tangent mapping of Φ needs to first be obtained. In general, this quantity is defined by

$$Y(Q) = \frac{d}{dt}\{\Phi(q(s,n(t)))\}|_{t=0} = \Phi(q)_*\underline{X}(q), \tag{80}$$

where $\Phi(q)_*$ is the differential of Φ at q. This is a linear transformation, relating any vector $\underline{X}(q)$ of $T_qM(3)$ to a vector (which in fact is a 3×3 matrix) $Y(Q)$ of $T_QSO(3)$, shown in Fig. 5.

In the special case where Eq. (80) is applied at the identity element of $M(3)$, it yields a relation between vectors $\underline{X}(e)$ of $m(3)$ to vectors (i.e., 3×3 matrices) $Y(I)$ of $so(3)$ with form

$$Y(I) = \Phi(e)_*\underline{X}(e). \tag{81}$$

By taking Eq. (70) into account and applying Eqs. (79) and (81), it can easily be shown that

$$Y(I) = \widetilde{X}(e) = spin(\underline{X}(e)), \tag{82}$$

which means that the tangent mapping at the identity point e of $M(3)$ is defined by

$$\Phi(e)_*(\cdot) = spin(\cdot). \tag{83}$$

This can be seen as a formal definition of the spin operator in $SO(3)$. Moreover, according to Eq. (81), $Y(I)$ is Φ-related to $\underline{X}(e)$ (Warner, 1983). Then, if $\underline{X}_1$ and $\underline{X}_2$ are two vector fields on $M(3)$, with corresponding vector fields Y_1 and Y_2 on $SO(3)$, it turns out that

$$\Phi(e)_*[\underline{X}_1(e), \underline{X}_2(e)] = [\Phi(e)_*\underline{X}_1(e), \Phi(e)_*\underline{X}_2(e)] = [Y_1(I), Y_2(I)]. \tag{84}$$

Therefore, by taking the inverse mapping of the last relation, in conjunction with Eq. (82), the Lie bracket of the Lie algebra $m(3)$ can be defined by

$$[\underline{X}_1(e), \underline{X}_2(e)] = spin^{-1}([Y_1(I), Y_2(I)]) \quad \Rightarrow \quad [\underline{X}_1(e), \underline{X}_2(e)] = vect([Y_1(I), Y_2(I)]). \tag{85}$$

This leads naturally to the classical vector product in $\mathbb{R}^3$.

Next, in order to determine the tangent mapping of Φ at an arbitrary point q of $M(3)$, consider first a point g in the neighborhood of point q, so that $g = q * h$. This implies that h represents a point in an infinitesimal neighborhood of e. Then, direct application of Eq. (72) yields

$$\Phi(g) = \Phi(q * h) = \Phi(q)\Phi(h),$$

which in conjunction with Eqs. (80) and (81) gives

$$Y(Q) = \Phi(q)\frac{d}{dt}\{\Phi(h(t))\}|_{t=0} = \Phi(q)[\Phi(e)_*\underline{X}(e)],$$

or eventually, by employing Eqs. (78) and (83), it turns out that

$$Y(Q) = Q\widetilde{X}(e), \tag{86}$$

which is identical to Eqs. (8b) and (66a). This implies that the tangent mapping at an arbitrary point q of $M(3)$ is defined by

$$\Phi(q)_* = \Phi(q)\Phi(e)_* = Q\Phi(e)_*$$

and completes the geometrical picture of the mapping between $M(3)$ and $SO(3)$, shown in Fig. 5.

Finally, the connection in $M(3)$ can be established by considering the structure constants through its Lie bracket. First, by definition (A8), for a basis $\{\underline{e}_i(e)\}$ on $m(3)$ it is true that

$$[\underline{e}_i(e), \underline{e}_j(e)] \equiv c_{ij}^k \underline{e}_k(e), \tag{87}$$

while for the corresponding basis $\{\tilde{e}_i(I)\}$ of $so(3)$, related to $\{\underline{e}_i(e)\}$ by Eq. (82), with

$$\tilde{e}_i(I) = spin(\underline{e}_i(e)), \tag{88}$$

it holds that

$$[\tilde{e}_i(I), \tilde{e}_j(I)] \equiv \tilde{c}_{ij}^k \tilde{e}_k(I). \tag{89}$$

Then, by employing Eqs. (87) and (88) and performing straightforward operations one arrives at

$$spin([\underline{e}_i(e), \underline{e}_j(e)]) = spin\left(c_{ij}^k \underline{e}_k(e)\right) = c_{ij}^k spin(\underline{e}_k(e)) = c_{ij}^k \tilde{e}_k(I).$$

Likewise, using Eqs. (84) and (89) leads to

$$spin([\underline{e}_i(e), \underline{e}_j(e)]) = [spin\{\underline{e}_i(e)\}, spin\{\underline{e}_j(e)\}] = [\tilde{e}_i(I), \tilde{e}_j(I)] = \tilde{c}_{ij}^k \tilde{e}_k(I).$$

Therefore, direct comparison of the last two relations yields

$$c_{ij}^k = \tilde{c}_{ij}^k, \tag{90}$$

which determines the structure constants in $m(3)$ and consequently completes the definition of its Lie bracket, since the structure constants $\tilde{c}_{ij}^k$ of the basis in $so(3)$ are taken from Eq. (64). At the same time, this result establishes the affinities of the connection on all of $M(3)$, through Eqs. (33), (38) and (46). This also verifies that the left translated basis $\{\underline{e}_i(p)\}$ is non-natural, since its structure constants are non-zero. Finally, the basis vectors of $m(3)$ corresponding to the standard basis of $so(3)$, defined by Eq. (62), are obtained through an inversion of Eq. (88) in the form

$$\underline{e}_i(e) = vect(\tilde{e}_i(I)),$$

which represent the standard base vectors of $\mathbb{R}^3$.

5. Application to rigid body dynamics

In this section, the study of spherical motion of a rigid body is completed, by applying principles of dynamics. This makes necessary the consideration of a new vector space, known as the dual space to the tangent space $T_pM(3)$ and denoted by $T_pM(3)^*$. This space is defined at every point p of $M(3)$ and includes elements known as covectors (or covariant vectors or one-forms), which are linear functionals on $T_pM(3)$ (Frankel, 1997). More specifically, if $\underline{v}$ is a vector of $T_pM(3)$, then there exists a covector $\underset{\sim}{v}^*$ of $T_pM(3)^*$, such that

$$\underset{\sim}{v}^*(\underline{u}) \equiv \langle \underline{v}, \underline{u} \rangle, \quad \forall \underline{u} \in T_pM(3), \tag{91}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product of $T_pM(3)$. In addition, a basis $\{\underset{\sim}{e}^i\}$ (with $i = 1,2,3$) of $T_p M(3)^*$ is called dual basis to the basis $\{\underline{e}_i\}$ of $T_pM(3)$, provided that it satisfies the condition

$$\underset{\sim}{e}^i(\underline{e}_j) = \delta^i_j, \tag{92}$$

where δ^i_j is a Kronecker delta symbol. Then, the covector $\underset{\sim}{v}^*$ can be expressed in the form

$$\underset{\sim}{v}^* = v_i \underset{\sim}{e}^i, \tag{93}$$

where its components v_i are now denoted by a lower index, in order to distinguish them from (tangent) vector components. Therefore, for a general $\underline{u} = u^i\underline{e}_i$, it turns out from Eqs. (91)–(93) that

$$\underset{\sim}{v}^*(\underline{u}) = (v_i \underset{\sim}{e}^i)(u^j\underline{e}_j) = v_i u^j \underset{\sim}{e}^i(\underline{e}_j) = v_i u^j \delta^i_j = v_i u^i. \tag{94}$$

Moreover, another straightforward calculation, based on the definition (91), yields

$$\underset{\sim}{v}^*(\underline{u}) = \langle \underline{v}, \underline{u}\rangle = \langle v^i\underline{e}_i, u^j\underline{e}_j\rangle = v^i u^j \langle \underline{e}_i, \underline{e}_j\rangle = v^i u^j g_{ij}, \tag{95}$$

where the scalars

$$g_{ij} = \langle \underline{e}_i, \underline{e}_j\rangle \tag{96}$$

give the components of the so called metric tensor in the basis $\{\underline{e}_i\}$ selected for $T_pM(3)$. Then, for an arbitrary vector $\underline{u}$, direct comparison of Eqs. (94) and (95) yields the useful relation

$$v_i = g_{ij} v^j. \tag{97}$$

The choice of the body frame can now be made, based on the selection of an appropriate metric. This selection is performed in a way that fits the dynamics (Papastavridis, 1999). First, by reconsidering Eq. (95) with $\underline{u} = \underline{v} = \underline{w}$ and taking into account Eq. (54), it turns out successively that

$$\underset{\sim}{w}^*(\underline{w}) = \langle \underline{w}, \underline{w}\rangle = w^i w^j g_{ij} = \Omega^i g_{ij} \Omega^j.$$

Therefore, direct comparison of the last result with Eq. (16) shows that the kinetic energy of the rigid body can be expressed in the form

$$T = \frac{1}{2}\langle \underline{w}, \underline{w}\rangle = \frac{1}{2}\Omega^i g_{ij}\Omega^j, \tag{98}$$

provided that the metric tensor is equal to the mass moment of inertia tensor of the body. That is

$$g_{ij} = J_{ij}, \tag{99}$$

with $J_O = [J_{ij}]$. Next, a basis $\{\underline{e}_i(e)\}$ can be selected for the tangent space $m(3)$ through Eq. (96), so that it satisfies the conditions

$$\langle \underline{e}_i(e), \underline{e}_j(e)\rangle = J_{ij}. \tag{100}$$

This basis is then left translated at any point p of $M(3)$, according to Eq. (76). Therefore, if the metric tensor is chosen to be left invariant, its components on this basis remain constant on all of $M(3)$, which means that

$$\langle \underline{e}_i(p), \underline{e}_j(p)\rangle = J_{ij}, \quad \forall p \in M(3). \tag{101}$$

At this point, a complete geometrical description of spherical motion can be provided. To achieve this in a way that can also be interpreted by employing the traditional approach as well, Fig. 6 is used. First, the orientation of the body at a given time t is described by a point, say p, of the three dimensional manifold $M(3)$, the configuration space of the body. In particular, the identity element e of $M(3)$ is chosen to represent the initial orientation. Then, a basis $\{\underline{e}_i(e)\}$ is selected by Eq. (100) and is left translated on all of $M(3)$ through Eq. (76), which has similar structure with Eq. (1). In this respect, these bases are related to the spatial and body frame, respectively, employed in Section 2 (Fig. 1). The affinities on the body frames on all of $M(3)$ are then chosen from Eq. (46). This permits the construction of all the autoparallel curves $\eta_p(s)$ of $M(3)$ starting from point e. Each of these curves represents pure rotation of the body about an axis $\underline{n}$, which is the tangent of the autoparallel at e. Then, considering this point as a pole, all other points are located uniquely by employing canonical coordinates. Here, these generalized coordinates coincide with the components of the Cartesian rotation vector. Finally, the motion of the body is viewed as a curve $\gamma(t)$ on $M(3)$. The tangent to this curve at each point p is the angular velocity vector $\underline{w}$ of the body and belongs to the tangent space at p. This explains why it is more convenient to express this vector in the body frame.

In general, the geometry of a manifold depends not only on its elements but on its connection and metric (if they are available) as well. Based on this, $M(3)$ presents some significant differences with the classical rotation group $SO(3)$. The first deviation appears in the selection of the connection. Specifically, it was shown in Section 4.1 that the most convenient choice for describing spherical motion on $M(3)$ is the left invariant canonical connection, defined by Eq. (29). This choice leads to a non-Riemannian manifold, with torsion and no curvature, which is in sharp contrast to the geometrical properties of the ordinary $SO(3)$.

Another important deviation between $M(3)$ and $SO(3)$ originates from the selection of the metric tensor as well. More specifically, by employing Eq. (63) and the orthogonality property of $R(t)$, as expressed by Eq. (4), the components of the metric tensor in $SO(3)$ can be determined from

$$\begin{aligned}\hat{g}_{ij}(R) &= \langle \hat{\underline{e}}_i(R), \hat{\underline{e}}_j(R)\rangle = \langle R(t)\tilde{e}_i(I), R(t)\tilde{e}_j(I)\rangle = \langle \tilde{e}_i(I), R^T(t)R(t)\tilde{e}_j(I)\rangle \\ &= \langle \tilde{e}_i(I), \tilde{e}_j(I)\rangle\end{aligned}$$

or eventually

$$\hat{g}_{ij}(R) = \hat{g}_{ij}(I).$$

This means that the components of the metric tensor remain constant at any point of $SO(3)$. In analogy to Eq. (24), this means that the metric of $SO(3)$ associated to the left invariant basis defined by Eq. (63) is also left invariant. Moreover, these components can be evaluated easily by direct application of the definitions in the standard basis of $so(3)$ and the Euclidean inner product in the space of 3×3 matrices. In fact, proceeding in this manner, it eventually turns out that

$$\hat{g}_{ij} = 2\delta_{ij}, \tag{102}$$

where δ_{ij} is a Kronecker delta. This implies that the selection of a basis for $so(3)$ by Eq. (62) leads to orthogonal bases on $SO(3)$. Also, this metric is symmetric and positive definite and can be shown that it is compatible with the connection chosen (Papastavridis, 1999). This verifies that $SO(3)$ is a Riemannian manifold, since it also possesses curvature and is torsionless, as shown in Section 4.2. However, the components of the metric tensor in $SO(3)$ are different than those of the inertia tensor in $M(3)$, defined by Eq. (99).

The picture describing the dynamics of a rigid body exhibiting spherical motion can now be completed by illuminating the role of $T_pM(3)^*$ and the selection of $\gamma(t)$. First, direct comparison of Eqs. (97), (99) and (13) demonstrates that the covector $\underset{\sim}{w}^*$ in Fig. 6, corresponding to the angular velocity vector $\underline{w}$, is the angular momentum vector. That is

$$\underset{\sim}{w}^* = \underset{\sim}{H}_O \quad \Rightarrow \quad H_i = J_{ij}\Omega^j. \tag{103}$$

Also, comparison of Eqs. (91) and (98) shows that the dual product of the angular velocity vector $\underline{w}$, belonging to the tangent space $T_pM(3)$, with the angular momentum $\underset{\sim}{H}_O$, which is the corresponding covector of the dual space $T_pM(3)^*$, yields the kinetic energy of the body. That is,

$$T = \frac{1}{2}\underset{\sim}{w}^*(\underline{w}) = \frac{1}{2}\underset{\sim}{H}_O(\underline{w}),$$

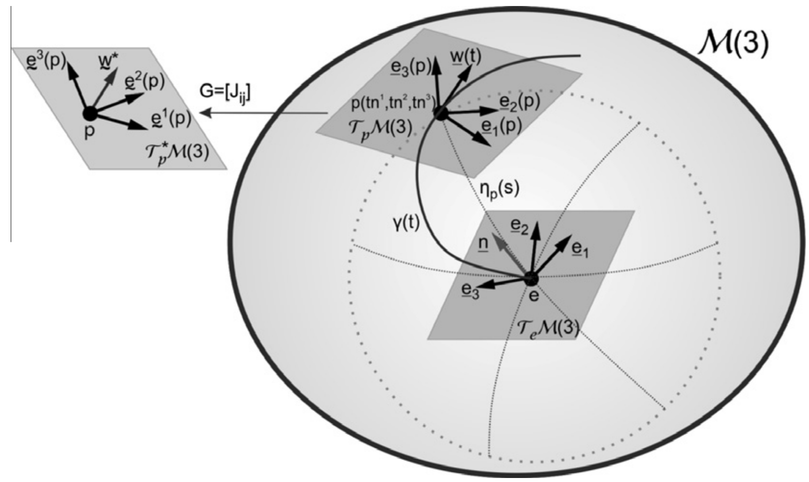

Fig. 6. Illustration of rigid body spherical motion and coordinate systems.

which is equivalent to Eq. (16). In addition, the selection of the solution curve $\gamma(t)$ on $M(3)$ is performed by applying Euler's law in the form

$$\dot{\underset{\sim}{H}}_O = \underset{\sim}{M}_O, \tag{104}$$

where $\underset{\sim}{M}_O = M_i \underset{\sim}{e}^i$ is the resultant moment with respect to point O, while

$$\dot{\underset{\sim}{H}}_O \equiv \frac{D}{Dt}\left(H_i \underset{\sim}{e}^i\right) = \frac{dH_i}{dt}\underset{\sim}{e}^i + H_i \frac{\partial \underset{\sim}{e}^i}{\partial \Theta^k}\frac{d\Theta^k}{dt} = \dot{H}_i \underset{\sim}{e}^i + H_j \frac{\partial \underset{\sim}{e}^j}{\partial \Theta^k}\Omega^k, \tag{105}$$

with $\dot{\Theta}^k = \Omega^k$. Moreover, Eq. (B1) in combination with Eq. (92) yields

$$\frac{\partial \underset{\sim}{e}^j}{\partial \Theta^k} = \nabla_{e_k}\underset{\sim}{e}^j = -\Lambda^j_{ki}\underset{\sim}{e}^i.$$

Therefore, by employing Eq. (103) and taking into account that the components J_{ij} are constant, Eq. (105) becomes

$$\dot{\underset{\sim}{H}}_O = \left[\dot{H}_i + \left(-\Lambda^j_{ki}\Omega^k\right)H_j\right]\underset{\sim}{e}^i = \left(J_{ij}\dot{\Omega}^j + \widetilde{\Omega}^j_i H_j\right)\underset{\sim}{e}^i, \tag{106}$$

with

$$\left[-\Lambda^j_{ki}\Omega^k\right] \equiv \left[\widetilde{\Omega}^j_i\right] = \widetilde{\Omega} = spin(\underline{\Omega}). \tag{107}$$

Substituting Eq. (106) into the underlying principle of motion, expressed by Eq. (104), leads to

$$J_{ij}\dot{\Omega}^j + \widetilde{\Omega}^j_i J_{jk}\Omega^k = M_i, \tag{108}$$

which matches Eq. (15) and furnishes the components Ω^i of the angular velocity in the local basis.

Note that choosing the symmetric canonical connection, associated to the rotation group $SO(3)$, would introduce a factor of 1/2 in the definition of $\widetilde{\Omega}$ in Eq. (107). This in turn would cause the appearance of the same erroneous factor in front of the second term in the equation of motion (108).

Finally, the affinities selected for $M(3)$, according to Eq. (46), match its group and manifold properties but are not compatible with its metric, expressed by Eq. (99). This implies that the metric (and consequently the kinetic energy) is not preserved under parallel transfer along an arbitrary path. For instance, differentiation of both sides of Eq. (98) with respect to time and simultaneous application of the symmetry condition of the inertia tensor $J_{ij} = J_{ji}$, yields eventually

$$\dot{T} = \Omega^i J_{ij}\dot{\Omega}^j. \tag{109}$$

Among all the possible paths, only on the real one it is true that $\dot{\Theta}^m = \Omega^m$. Therefore, for $\underset{\sim}{M}_O = \underset{\sim}{0}$, i.e., for torque-free motion, Eqs. (108) and (107) imply that

$$J_{ij}\dot{\Omega}^j = -\widetilde{\Omega}^j_i J_{jk}\Omega^k = -\left(-\Lambda^j_{mi}\Omega^m\right)J_{jk}\Omega^k = \Lambda^j_{mi}\Omega^m H_j,$$

which after substituting in Eq. (109) yields

$$\dot{T} = \Omega^i\left(\Lambda^j_{mi}\Omega^m H_j\right) = \left(\Lambda^j_{mi}\Omega^i\Omega^m\right)H_j = 0,$$

due to the anti-symmetry property of the affinities selected for $M(3)$. This shows that the kinetic energy is conserved along the path corresponding to the actual (torque-free) motion of the body.

6. Synopsis

The problem of finite rigid body rotation has been treated in some detail in this study. Borrowing ideas from Lie group theory provided a solid foundation for a thorough, clear and consistent investigation of both rigid body kinematics and dynamics. As a result of this study, the following elegant geometrical picture emerged on rigid body rotation about a fixed point.

First, the orientation of a rigid body was represented by a single point, while the motion over a finite time interval was described by a curve on a three dimensional manifold. The tangent vector at the current point of this curve is the angular velocity of the body. Then, it was demonstrated that, contrary to common belief, the well known special orthogonal group $SO(3)$ is not appropriate for describing either the kinematics or the dynamics of large rigid body rotation. In fact, a new manifold was introduced in a natural way, named $M(3)$, which is diffeomorphic to $SO(3)$. Specifically, a significant contribution of this work was the selection of a canonical connection for $M(3)$, so that its autoparallel curves, representing pure rotation of the body, coincide with its one parameter Lie subgroups, which are located conveniently by the exponential map. This led to a manifold possessing torsion and no curvature, in contrast to the classical rotational group $SO(3)$, which is a Riemannian manifold with curvature and no torsion. Moreover, the exponential map provided the ground for choosing a holonomic coordinate system in determining uniquely the points on $M(3)$. In particular, the components of the classical Cartesian rotation vector were picked up as (canonical) coordinates. However, a non-holonomic coordinate frame was selected for expressing the vectors of the tangent space at the current point. This frame is fixed on the body (body frame) and was obtained mathematically by a left translation of an appropriate basis at the identity. Moreover, all the important geometrical properties of $M(3)$, like its group product, identity element and structure constants, were also selected by a suitable representation on the rotation group $SO(3)$.

Among other things, it was shown that the angular velocity of the body has two special images in the tangent space at the identity point of $M(3)$. Namely, it can be viewed as the result of a left or a right translation of two vectors, $\underline{\Omega}$ and $\underline{\omega}$, known as the convective and spatial angular velocity, respectively, in the engineering literature. In addition, it was illustrated that the right translation of $\underline{\omega}$ is equivalent to a parallel translation from the identity to the current point along an autoparallel curve.

Next, the emphasis was put in the study of the dynamics. This made necessary the selection of a left invariant metric. Specifically, based on the expression of the kinetic energy of the body, the components of the metric tensor were chosen to be equal to the components of the mass moment of inertia tensor of the body. This provided the body frame at the identity. Moreover, the concept of the dual space was utilized for the complete dynamic description. Initially, it was demonstrated that the covector corresponding to the angular velocity is the angular momentum. Then, the equations of motion of the body were derived by application of Euler's law, which led eventually to the selection of the exact path on the manifold. Finally, as a consequence of the fact that the connection and metric chosen for $M(3)$ are not compatible, in contrast to the case of $SO(3)$, the inner product in the tangent space (representing kinetic energy) is not preserved along any arbitrary curve. However, it was proved that the kinetic energy of the body is conserved along the true path, during a torque-free motion.

Apart from clarifying the picture of rigid body kinematics and dynamics, the results presented in this study are expected to provide valuable tools for setting up the equations of motion and developing novel, robust and efficient techniques for their geometrically exact temporal discretization in all areas of mechanics, where the configuration space possesses group properties. For instance, this is the case in robotics, where the configuration space includes the special Euclidean group $SE(3)$, which involves a combination of $SO(3)$ and $\mathbb{R}^3$ for the rotational and translational part of the motion, respectively. Moreover, the approach presented lays the foundation for a more systematic treatment of constrained mechanical systems.

Appendix A. Evaluation of the components of a Lie bracket in a general basis

The concept of the Lie bracket arises naturally in the study of the Lie derivative, which is a based on a generalization of the classical directional derivative from a Euclidean manifold to a general non-flat manifold M^n and provides a measure of its non-commutativity. If $f = f(\theta^1, \dots, \theta^n)$ is a function expressed in a local coordinate system $\{\theta^i\}$ with $i = 1, \dots, n$ near a point p of M^n, while $\{\underline{e}_i\}$ is a basis on the tangent space T_pM^n to M^n at p, then the following definition

$$\underline{X}(f) = X^i \partial_i f, \tag{A1}$$

represents the derivative of f with respect to the vector $\underline{X} = X^i \underline{e}_i$ of T_pM^n (Frankel, 1997), with

$$\partial_i f \equiv \partial f / \partial \theta^i. \tag{A2}$$

Likewise, the Lie derivative $\mathfrak{L}_{\underline{X}}\underline{Y}$ at a point of a manifold determines the change of a vector field $\underline{Y}$ along the flow generated by another field $\underline{X}$ on M^n. This derivative is a vector field on M^n, giving

$$(\mathfrak{L}_X \underline{Y})(f) = \underline{X}\{\underline{Y}(f)\} - \underline{Y}\{\underline{X}(f)\}, \tag{A3}$$

when applied to a scalar function f (Warner, 1983). Therefore, it can be expressed in the form

$$\mathfrak{L}_X \underline{Y} = [\underline{X}, \underline{Y}], \tag{A4}$$

where the corresponding Lie bracket is defined by

$$[\underline{X}, \underline{Y}] \equiv \underline{XY} - \underline{YX}. \tag{A5}$$

In order to determine the components of this bracket on the basis $\{\underline{e}_i\}$, straightforward evaluation of the terms on the right hand side of Eq. (A3), taking into account the definition (A1), yields

$$[\underline{X}, \underline{Y}]f = \left(X^j \partial_j Y^i - Y^j \partial_j X^i + c^i_{jk} X^j Y^k \right) \underline{e}_i(f). \tag{A6}$$

Then, Eq. (A4) shows that the Lie derivative of a vector field $\underline{Y}$ with respect to field $\underline{X}$ is given by

$$\mathfrak{L}_{\underline{X}} \underline{Y} = [\underline{X}, \underline{Y}] = \left(X^j \partial_j Y^i - Y^j \partial_j X^i + c^i_{jk} X^j Y^k \right) \underline{e}_i. \tag{A7}$$

This is true in the general case, where the basis $\{\underline{e}_i\}$ is non-natural (or anholonomic or a frame), with

$$[\underline{e}_i, \underline{e}_j] = c^k_{ij} \underline{e}_k, \tag{A8}$$

where the terms c^k_{ij} are known as structure constants and their values depend on the basis $\{\underline{e}_i\}$ entirely. When these terms are multiplied by 1/2 become the components of the anholonomicity object (Papastavridis, 1999). In the special case where the basis $\{\underline{e}_i\}$ is natural (or holonomic or a coordinate frame) (Bowen and Wang, 2008), the basis vector $\underline{e}_i$ is tangent to the ith coordinate curve and

$$[\underline{e}_i, \underline{e}_j] = \underline{0}, \quad \forall i, j. \tag{A9}$$

Finally, from the anti-symmetry property of the Lie bracket, that is $[\underline{X}, \underline{Y}] = -[\underline{Y}, \underline{X}]$, resulting easily from the definition (A5), it turns out from Eq. (A8) that

$$[\underline{e}_j, \underline{e}_i] = -[\underline{e}_i, \underline{e}_j] \quad \Rightarrow \quad c^k_{ji} = -c^k_{ij}. \tag{A10}$$

This means that the structure constants are always anti-symmetric in their lower indices.

Appendix B. Connection and covariant differentiation on a manifold

A valuable geometrical tool in moving from a tangent space of a non-flat manifold to an adjacent tangent space is the so called affine connection of the manifold, represented by symbol ∇. For any manifold M^n, this leads to a mapping $\nabla_{\underline{w}} \underline{v}$ from $T_pM^n \times T_pM^n$ to T_pM^n, known as the covariant differential of $\underline{v}$ along $\underline{w}$ and provides the derivative of a vector field $\underline{v}(t)$ at a point p of the manifold along a direction specified by a vector $\underline{w}$ at point p (Frankel, 1997).

Evaluation of the components of this quantity depends on the choice of a local coordinate system and a basis of the tangent space at each point of the manifold. In particular, if $\{\theta^i\}$ with $i = 1, \dots, n$ is a set of coordinates of M^n and $\{\underline{e}_i\}$ is a basis on the tangent space T_pM^n, the following definition

$$\nabla_{\underline{e}_j} \underline{e}_k = \Lambda^i_{jk} \underline{e}_i \tag{B1}$$

introduces the components Λ^i_{jk} of the connection ∇ in the basis $\{\underline{e}_i\}$, known as affinities. Then, the covariant differential of a vector field $\underline{v}(t) = v^j(t) \underline{e}_j(t)$ along $\underline{w} = w^i \underline{e}_i$ is determined in the form

$$\nabla_{\underline{w}} \underline{v} = \left(\partial_i v^k + \Lambda^k_{ij} v^j \right) w^i \underline{e}_k. \tag{B2}$$

This means that $\nabla_{\underline{w}} \underline{v}(t)$ represents a vector on the tangent space T_pM^n, which does not depend on the derivatives of $\underline{w}$. In fact, if $\underline{w}$ is the tangent vector of a curve $\gamma(t)$ on the manifold, then

$$\nabla_{\underline{w}}\underline{v}(t) = \left(\frac{\partial v^k}{\partial \theta^i}\frac{d\theta^i}{dt} + \Lambda^k_{ij} v^j w^i\right)\underline{e}_k = \left(\frac{dv^k}{dt} + \Lambda^k_{ij} w^i v^j\right)\underline{e}_k$$
$$\equiv \frac{Dv^k}{Dt}\underline{e}_k = \frac{D\underline{v}}{Dt}, \tag{B3}$$

which leads to the classical definition of the covariant derivative (Papastavridis, 1999).

Next, a parallel translation of vector field $\underline{v}(t)$ along a curve $\gamma(t)$ of a manifold with tangent vector $\underline{w}$ is defined by

$$\nabla_{\underline{w}}\underline{v} = \underline{0}. \tag{B4}$$

From Eq. (B3), this implies that

$$\dot{v}^i + \Lambda^i_{jk} w^j v^k = 0, \tag{B5}$$

which represents a set of linear ordinary differential equations in $v^j(t)$. Therefore, for a given set of initial values, $v^j(0)$, this leads to a unique solution $\underline{v}(t)$, for any curve $\gamma(t)$ and any set of Λ^i_{jk}. This illustrates that the parallel displacement of a vector along a curve of a non-flat manifold is path dependent. Furthermore, in the special case with $\underline{v} = \underline{w} \equiv \underline{n}$, application of the condition

$$\nabla_{\underline{n}}\underline{n} = \underline{0}, \tag{B6}$$

which requires that the tangent vector to the path remains parallel to itself, yields a set of nonlinear (quadratic) ordinary differential equations in n^i, with form

$$\dot{n}^i + \Lambda^i_{jk} n^j n^k = 0. \tag{B7}$$

Solution of this set of equations leads to special curves on the manifold, known as autoparallels and representing its "straightest" curves (Shabanov, 1998). When the manifold possesses a metric, another special family of curves can be defined on it, known as geodesics and representing its "shortest" curves. Namely, the basic property of a geodesic curve is that it has the minimum length among all the curves joining two points of the manifold (Dodson and Poston, 1991). When the affinities Λ^i_{jk} are compatible with the metric and possesses no torsion, like in $SO(3)$, the geodesics and autoparallels coincide.

From Eq. (B7) it is apparent that the autoparallels are not affected by the anti-symmetric (in the two lower indices) part of the affinities. However, Eq. (B5) suggests that this part affects the parallel translation of an arbitrary vector. Moreover, in the special case with $\Lambda^i_{jk} = -\Lambda^i_{kj}$, Eq. (B7) leads to

$$\dot{n}^i = 0 \quad \Rightarrow \quad n^i(t) = n^i(0). \tag{B8}$$

This means that the components of vector $\underline{n}(t)$, which is tangent to an autoparallel curve of the manifold, remain constant. This has important implications in Mechanics.

In general, there exists an infinite number of affine connections on any manifold. For each of them, the main geometrical invariants of a connection are its torsion and curvature tensors. The definition of both of these objects involves the covariant derivative and the Lie bracket of the tangent space at each point of the manifold. Specifically, the torsion of the connection is defined by

$$\tau(\underline{X}, \underline{Y}) = \nabla_{\underline{X}}\underline{Y} - \nabla_{\underline{Y}}\underline{X} - [\underline{X}, \underline{Y}]. \tag{B9}$$

By assuming that $\underline{X} = X^i\underline{e}_i$ and $\underline{Y} = Y^i\underline{e}_i$, employing Eqs. (B2) and (A7) and performing direct calculation reveals that

$$\tau(\underline{X}, \underline{Y}) = \tau^i_{jk} X^j Y^k \underline{e}_i,$$

which furnishes the components of the torsion tensor in the basis $\{\underline{e}_i\}$ in the form

$$\tau^i_{jk} = \Lambda^i_{jk} - \Lambda^i_{kj} - c^i_{jk}. \tag{B10}$$

In view of Eq. (A10), this yields the anti-symmetry property $\tau^i_{jk} = -\tau^i_{kj}$. Obviously, these components depend both on the properties of the manifold (through the affinities Λ^i_{jk}) and the basis (through the structure constants c^i_{jk}). Therefore, the condition for a torsion free (or symmetric) connection is

$$\tau^i_{jk} = 0 \quad \Rightarrow \quad c^i_{jk} = \Lambda^i_{jk} - \Lambda^i_{kj}. \tag{B11}$$

Moreover, when the basis is natural to the coordinate system $\left(c^i_{jk} = 0\right)$ the last expression becomes

$$\Lambda^i_{jk} = \Lambda^i_{kj}. \tag{B12}$$

This shows that the condition for a torsionless connection implies symmetry in the lower two indices of its affinities, provided the basis is holonomic.

Finally, the curvature tensor of a connection is defined by

$$R(\underset{\sim}{w}^*, \underline{X}, \underline{Y})\underline{v} = \underset{\sim}{w}^*(\nabla_{\underline{X}}[\nabla_{\underline{Y}}\underline{v}] - \nabla_{\underline{Y}}[\nabla_{\underline{X}}\underline{v}] - \nabla_{[\underline{X},\underline{Y}]}\underline{v})$$
$$= \underset{\sim}{w}^*(\widehat{R}(\underline{X}, \underline{Y})\underline{v}), \tag{B13}$$

which represents a mapping from $T^*_pM^n \times T_pM^n \times T_pM^n \times T_pM^n$ to the set of real scalars $\mathbb{R}$. Again, direct evaluation of the components of the curvature tensor in the basis $\{\underline{e}_i\}$ (and its dual) yields

$$R^i_{jkl} = \Lambda^i_{lj,k} - \Lambda^i_{kj,l} + \Lambda^i_{km}\Lambda^m_{lj} - \Lambda^i_{lm}\Lambda^m_{kj} - c^m_{kl}\Lambda^i_{mj}. \tag{B14}$$

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