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The Dissipative Linear Boltzmann Equation

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Abstract—We introduce and discuss a linear Boltzmann equation describing dissipative interactions of a gas of test particles with a fixed background. For a pseudo-Maxwellian collision kernel, it is shown that, if the initial distribution has finite temperature, the solution converges exponentially for large time to a Maxwellian profile drifting at the same velocity as field particles and with a universal nonzero temperature which is lower than the given background temperature. © 2004 Elsevier Ltd. All rights reserved.

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1. INTRODUCTION

In recent times, the study of the large-time behavior of dissipative granular gases has received a lot of attention. Essential progress has been made for simplified models, in particular thanks to the consideration of the nonlinear Boltzmann equation for inelastic Maxwell particles, both for the free case without energy input [1,2], for the driven case [3–5], and for mixtures [6]. Despite their importance in practical applications, however, the study of the evolution of a gas colliding inelastically with a fixed background has not been considered so far, at least to the authors' knowledge. By this mechanism, granular particles exchange momentum and energy with the field particles, and simultaneously some of the kinetic energy of the colliding pair is dissipated (transferred to nonparticipating degrees of freedom).

In this paper, we introduce and study a linear dissipative Boltzmann model under the assumption that the collision kernel corresponds to the so-called Maxwellian molecules interaction. Inelastic Maxwell models share with elastic Maxwell molecules the property that the collision rate

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in the Boltzmann equation is independent of the relative velocity of the colliding pair. In granular gases, it is usual to consider Boltzmann-like equations for partially inelastic hard spheres. This choice relies on the physical fact that the grains must be cohesionless, which implies hard-sphere interaction only, and no long-range forces of any kind. Hence, in the inelastic case, a constant collision rate can be considered just as a matter of mathematical convenience. In spite of that, these models are of great interest for spatially homogeneous granular fluids because of the resulting mathematical simplifications [7,8], which lead even to exact analytical results.

We introduce and discuss the inelastic linear Boltzmann model in Section 2, together with the relevant moment equations and with its representation in Fourier space. The background is supposed to be in thermodynamical equilibrium at given temperature and drift velocity. In Section 3, we will discuss the existence of a Maxwellian equilibrium for the granular gas, showing that this equilibrium has a well-defined temperature, lower than the temperature of the background, and that it attracts exponentially in time any initial data which has finite temperature. The last section is devoted to possible extensions and applications of the techniques.

2. THE DISSIPATIVE LINEAR BOLTZMANN EQUATION

We consider a linear Boltzmann-type equation for a granular gas subject to dissipative collisions against the field particles of a fixed background, labeled by a subscript 1 in the sequel. This simple kinetic model, which can be viewed as the dissipative generalization of the linear Maxwellian gas, originates from the inelastic hard-sphere Boltzmann equation with constant restitution coefficient e (with $0 < e < 1$)

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = \frac{1}{2\pi\lambda} \int_{\mathbb{R}^3 \times S^2} |q \cdot n| \left[\frac{1}{e^2} f(v_*) M_1(w_*) - f(v) M_1(w) \right] dw\, dn. \tag{1}$$

Here λ denotes the (constant) mean free path, q the relative velocity $v - w$, and (v_*, w_*) are the precollisional velocities of the so-called inverse collision, which results in (v, w) as postcollisional velocities. M_1 stands for the normalized field particle Maxwellian distribution function, characterized by given mass velocity u_1 and temperature T_1, in symbols, $M_1 = M(v; m_1, u_1, T_1)$. Test particles exchange momentum and energy with the background even in the elastic case $e = 1$, and this effect depends on the mass ratios appearing in the precollisional velocities. Mass ratios and inelasticity will be described by the dimensionless parameters

$$\alpha = \frac{m_1}{m + m_1}, \qquad \beta = \frac{1 - e}{2}, \tag{2}$$

where $0 < \alpha < 1$ (excluding thus the peculiarities of the limiting cases of Lorentz and Rayleigh gas) and $0 < \beta < 1/2$.

The pseudo-Maxwellian approximation consists in replacing the relative velocity q in the collision kernel $|q \cdot n|$ by a different vector $\tilde{q}\,\Omega$, where Ω is the unit vector in the direction of $v - w$, whereas $\tilde{q}$ is a parameter, possibly dependent on some macroscopic variable of the gas, like temperature, but independent of the integration variables. In this approximation, upon using $\lambda/\tilde{q}$ as time scale, we will consider the dimensionless equation in space homogeneous conditions

$$\frac{\partial f}{\partial t}(v, t) = Q(f)(v, t), \tag{3}$$

where

$$Q(f)(v) = \frac{1}{2\pi} \int_{\mathbb{R}^3 \times S^2} |\Omega \cdot n| \left[\frac{1}{e^2} f(v_*) M_1(w_*) - f(v) M_1(w) \right] dw\, dn. \tag{4}$$

Taking any smooth test function $\psi(v)$, it is useful to consider the weak form of either (1) or (3), in which the collision terms may be written as

$$(\psi, Q) = \frac{1}{2\pi} \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times S^2} |\Omega \cdot n| f(v) M_1(w) \left[\psi(v^*) - \psi(v) \right] dv\, dw\, dn, \tag{5}$$

where the postcollisional velocity v^* is defined by

$$v^* = v - 2\alpha(1-\beta)(q\cdot n)n. \tag{6}$$

Clearly, $\psi = 1$ is a collision invariant (mass conservation), whereas $\psi = v$ and $\psi = v^2$ are not. The equation being linear, normalization can be chosen so that

$$\int_{\mathbb{R}^3} f_0(v)\,dv = 1, \tag{7}$$

f_0 standing for the initial condition associated to (3).

One of the most important properties of Maxwell models is that moment equations are closed with respect to the moments of the distribution function. With constant (and unity) number density, self-consistent explicit equations are then in order for drift velocity and granular temperature of the gas

$$u = \int_{\mathbb{R}^3} v f(v)\,dv, \qquad T = \frac{1}{3}m\int_{\mathbb{R}^3}(v-u)^2 f(v)\,dv. \tag{8}$$

A little algebra yields

$$\begin{aligned}\frac{du}{dt} &= -\alpha(1-\beta)(u-u_1),\\ \frac{dT}{dt} &= \frac{2}{3}m\alpha^2(1-\beta)^2(u-u_1)^2 - 2\alpha(1-\beta)\{[1-\alpha(1-\beta)]T - (1-\alpha)(1-\beta)T_1\}.\end{aligned} \tag{9}$$

This set of ordinary differential equations exhibits, already at first glance, a unique equilibrium point $u = u_1$ and $T = T^{\#}$, where

$$T^{\#} = \frac{(1-\alpha)(1-\beta)}{1-\alpha(1-\beta)}T_1, \tag{10}$$

with $0 < T^{\#} < T_1$, is always proportional to, and lower than, the background temperature, the ratio being determined by the characteristic parameters (2). This effect disappears, of course, in the elastic limit $\beta \to 0$. Time evolution corresponds, as is easily checked, to an exponential relaxation of u and T to their equilibrium values

$$\begin{aligned} u &= u_1\left(1-\exp[-\alpha(1-\beta)t]\right) + u_0\exp[-\alpha(1-\beta)t],\\ T &= T^{\#}\left(1-\exp\{-2\alpha(1-\beta)[1-\alpha(1-\beta)]t\}\right) + T_0\exp\{-2\alpha(1-\beta)[1-\alpha(1-\beta)]t\}\\ &\quad + \frac{m(u_1-u_0)^2}{3}\left(\exp\{-2\alpha(1-\beta)[1-\alpha(1-\beta)]t\} - \exp[-2\alpha(1-\beta)t]\right).\end{aligned} \tag{11}$$

The asymptotic temperature ratio $T^{\#}/T_1$ is monotonically decreasing from $1-\beta$ to 0 with respect to α, and from 1 to $(1-\alpha)/(2-\alpha)$ with respect to β.

Another important feature of the pseudo-Maxwellian model is that it lends itself to a convenient Fourier analysis. After introducing the Fourier transform $\hat{f}$ of f,

$$\hat{f}(\xi) = \int_{\mathbb{R}^3} e^{-i\xi\cdot v} f(v)\,dv,$$

it is clear that the Fourier transform of the dissipative kinetic equation (3) is nothing but its weak form relevant to the test function $\exp(-i\xi\cdot v)$. It is also known [9] that the transformed equation can be made explicit in terms of $\hat{f}$, with a collision term involving only a two-dimensional integral. More precisely, we get

$$\frac{\partial\hat{f}}{\partial t}(\xi,t) = \hat{Q}\left(\hat{f}\right)(\xi,t), \tag{12}$$

where

$$\hat{Q}\left(\hat{f}\right)(\xi) = \frac{1}{2\pi}\int_{S^2} |\omega\cdot n| \left[\hat{f}\left(\xi^+\right)\hat{M}_1(\xi^-) - \hat{f}(\xi)\hat{M}_1(0)\right] dn. \tag{13}$$

In (13), we have set $\omega = \xi/|\xi|$ and

$$\begin{aligned}\xi^+ &= \xi - 2\alpha(1-\beta)(\xi\cdot n)n,\\ \xi^- &= 2\alpha(1-\beta)(\xi\cdot n)n,\end{aligned} \tag{14}$$

while $\hat{M}_1(\xi)$ is the Fourier transform of the background Maxwellian distribution

$$\hat{M}_1(\xi) = \exp\left\{-iu_1\cdot\xi - \frac{2T_1}{m_1}\frac{|\xi|^2}{4}\right\}.$$

Note that $\xi^+ + \xi^- = \xi$, and that initial condition (7) and mass conservation translate into $\hat{f}(0,t) = 1$.

3. TREND TO EQUILIBRIUM

The analysis of the previous section shows that mass velocity and temperature reach well-defined equilibrium values. We look now for possible nontrivial equilibrium distributions. A stationary solution to the Boltzmann equation (12) solves

$$\hat{f}(\xi) = \frac{1}{2\pi}\int_{S^2} |\omega\cdot n|\hat{f}\left(\xi^+\right)\hat{M}_1(\xi_-)\, dn. \tag{15}$$

If we set $\hat{f}(\xi) = \hat{F}(\xi)\exp\{-iu_1\cdot\xi\}$, $\hat{f}(\xi)$ is a solution to (15) if there exists $\hat{F}(\xi)$ such that, $\forall\, n\in S^2$,

$$\hat{F}(\xi) = \hat{F}\left(\xi - 2\alpha(1-\beta)(\xi\cdot n)n\right)\exp\left\{-\frac{2T_1}{m_1}|\alpha(1-\beta)(\xi\cdot n)|^2\right\}. \tag{16}$$

The structure of (16) suggests that an isotropic Maxwellian at a given temperature could fulfill the requirement. Since $\alpha \neq 1$ ($m \neq 0$) and number density is unity, the tentative solution is

$$F(v) = M\left(v; m, 0, T^{\#}\right), \qquad \hat{F}(\xi) = \exp\left\{-\frac{2T^{\#}}{m}\frac{|\xi|^2}{4}\right\}.$$

Inserting $\hat{F}(\xi)$ into (16) yields, for all ξ and n,

$$\exp\left\{-\frac{4}{m_1}\alpha^2(1-\beta)^2(\xi\cdot n)^2\left[(1-\beta)T_1 - \frac{1-\alpha(1-\beta)}{1-\alpha}T^{\#}\right]\right\} = 1, \tag{17}$$

which is identically satisfied. Actually, the Maxwellian is such that $\hat{Q} = 0$ for all ξ, and then $Q = 0$ for all $v\in\mathbb{R}^3$.

In order to investigate uniqueness and trend to equilibrium, now denote by $P_s(\mathbb{R}^3)$, $s > 0$, the class of all probability densities f on $\mathbb{R}^3$, such that

$$\int_{\mathbb{R}^3} |v|^s f(v)\, dv < \infty.$$

We introduce a metric on $P_s(\mathbb{R}^3)$ by

$$d_s(f,g) = \sup_{\xi\in\mathbb{R}^3}\frac{\left|\hat{f}(\xi) - \hat{g}(\xi)\right|}{|\xi|^s}. \tag{18}$$

Let us write $s = k + \delta$, where k is an integer and $0 \le \delta < 1$. The crucial property of this metric is that, for $d_s(f,g)$ to be finite, it suffices that f and g have the same moments up to order k. The norm (18) has been introduced in [10] to investigate the trend to equilibrium of the solutions to the Boltzmann equation for Maxwell molecules. Further studies showed the variety of applications of this metric both to kinetic theory [11–13] and to probability theory [14].

The existence of a solution to equation (3) can be seen easily using the same methods available for the elastic linear Boltzmann equation. Let f and g be two solutions of the Boltzmann equation (3) corresponding to initial densities with finite temperature, and $\hat{f}$, $\hat{g}$ their Fourier transforms. Then, given any positive constant $s > 0$, we may write

$$\frac{\partial}{\partial t}\frac{\left(\hat{f}-\hat{g}\right)}{|\xi|^s} + \frac{\hat{f}(\xi)-\hat{g}(\xi)}{|\xi|^s} = \frac{1}{2\pi}\int_{S^2} |\omega\cdot n| \frac{\hat{f}(\xi^+)-\hat{g}(\xi^+)}{|\xi|^s}\hat{M}_1(\xi^-)\,dn. \tag{19}$$

Now, we have the bound

$$\left|\frac{\hat{f}(\xi^+)-\hat{g}(\xi^+)}{|\xi|^s}\right|\left|\hat{M}_1(\xi)^-\right| \le \left|\frac{\hat{f}(\xi^+)-\hat{g}(\xi^+)}{|\xi^+|^s}\right| \frac{|\xi^+|^s}{|\xi|^s}$$
$$= \left|\frac{\hat{f}(\xi^+)-\hat{g}(\xi^+)}{|\xi^+|^s}\right| \left(1-4\alpha(1-\beta)|\omega\cdot n|^2\left(1-(1-\beta)\alpha\right)\right)^{s/2},$$

while, if $s \le 2$, a simple Taylor formula and a convexity argument give

$$\frac{1}{2\pi}\int_{S^2} |\omega\cdot n|\left(1-4\alpha(1-\beta)|\omega\cdot n|^2\left(1-\alpha(1-\beta)\right)\right)^{s/2} dn \le 1 - s\alpha(1-\beta)(1-\alpha(1-\beta)).$$

Upon defining

$$\gamma_s = 1 - s\alpha(1-\beta)(1-\alpha(1-\beta)), \qquad 1-\frac{s}{4} < \gamma_s < 1, \tag{20}$$

we obtain then

$$\left|\frac{\partial}{\partial t}\frac{\left(\hat{f}-\hat{g}\right)}{|\xi|^s} + \frac{\hat{f}(\xi)-\hat{g}(\xi)}{|\xi|^s}\right| \le \gamma_s \sup_{\xi\in\mathbb{R}^3}\frac{\left|\hat{f}-\hat{g}\right|}{|\xi|^s}, \tag{21}$$

and, if we set $h(\xi) = [\hat{f}(\xi)-\hat{g}(\xi)]/|\xi|^s$, the preceding computation shows that

$$\left|\frac{\partial h}{\partial t} + h\right| \le \gamma_s \|h\|_\infty. \tag{22}$$

Gronwall's lemma proves then that $\|h(t)\|_\infty$ is nonincreasing, and, provided $\|h(0)\|_\infty$ is bounded,

$$\|h(t)\|_\infty \le \|h(0)\|_\infty \exp\left\{-(1-\gamma_s)t\right\}. \tag{23}$$

If now we consider the initial-value problem for the dissipative Boltzmann equation with initial values $f_0(v)$, $g_0(v)$ of finite temperature and identical mass, thanks to the definition of $d_s(\cdot,\cdot)$, it follows that $d_s(f_0,g_0)$ is bounded at least if $s < 1$. Hence, we proved the following.

THEOREM 3.1. *Let $f(t)$ and $g(t)$ be two solutions of the dissipative Boltzmann equation (3), corresponding to initial values f_0 and g_0 of finite temperature and identical mass $\rho = 1$. Then, if $s < 1$, for all times $t \ge 0$,*

$$d_s(f(t),g(t)) \le d_s(f_0,g_0)\exp\left\{-(1-\gamma_s)t\right\}, \tag{24}$$

with γ_s provided by (20), and the same holds for $s = 1$ if the initial distributions have the same drift velocity. In particular, let f_0 be a nonnegative density with finite moments of order 2. Then, there exists a unique weak solution $f(t)$ of the dissipative Boltzmann equation (3), such that $f(0) = f_0$. This solution converges to the (unique) equilibrium $M^\# = M(v;m,u_1,T^\#)$ as time goes to infinity, where $T^\#$ is given by (10), and u_1 and T_1 are drift velocity and temperature of the background medium. The following decay estimate is in order for $0 < s < 1$:

$$d_s\left(f(t),M^\#\right) \le d_s\left(f_0,M^\#\right)\exp\left\{-\left(1-\gamma_s\right)t\right\}, \tag{25}$$

and it can be extended to $s = 1$ when the initial drift velocity u_0 is equal to u_1.

4. CONCLUSIONS

We have proved existence and uniqueness of a collision equilibrium for the pseudo-Maxwellian model of the dissipative linear Boltzmann equation, and exponential relaxation to such equilibrium in space homogeneous conditions for any initial datum with at least finite temperature. The equilibrium turns out to be Gaussian, and shares the same drift velocity with the background, whereas its temperature is always greater than zero and smaller than the background temperature, depending on mass ratio and inelasticity. This is the result of the combined effects of momentum and energy exchange with field particles, on one side, and of energy dissipation in the scattering collisions, on the other. We were not able to extend these results beyond the pseudo-Maxwellian model. Work is in progress in this direction.

Availability of a Maxwellian equilibrium at nonzero temperature allows us to construct hydrodynamic equations for the considered granular flow. If a macroscopic scale is used for time and space variable, we may rewrite (1) as

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = \frac{1}{\epsilon} Q(f), \tag{26}$$

where ϵ is the Knudsen number. With $\psi = 1$ as a unique collision invariant, the only conservation equation we can rely on is the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla_x \cdot (\rho u) = 0, \tag{27}$$

and ρ is the unique hydrodynamic variable. The Euler-type equation for this problem consists in expressing the higher-order moment u by using the equilibrium distribution $\rho M^{\#}$ instead of f. This amounts simply to substituting u_1 for u in (27). More significant is the Navier-Stokes approximation, which takes into account first-order corrections with respect to the small parameter ϵ. It can be obtained by a Chapman-Enskog asymptotic procedure in which the unknown f is expanded in powers of ϵ, but ρ is left unexpanded. Skipping all technical details, the zero-order distribution is $\rho M(v; m, u_1, T^{\#})$, and Fick's law is recovered for the first-order correction to the drift velocity. The resulting limiting equation is of convection-diffusion type,

$$\frac{\partial \rho}{\partial t} + \nabla_x \cdot (u_1 \rho) = \epsilon \nabla_x \cdot \left(\frac{T^{\#}}{(1-\beta)\mu} \nabla_x \rho \right), \tag{28}$$

with temperature dependent diffusion coefficient, and with $\mu = \alpha m$ standing for the reduced mass of the colliding pairs. The elastic case would correspond to the limit $\beta \to 0$ in (28), so that it is easily seen that inelasticity tends to slow down the diffusive process.

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