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A rational basis for second-kind Abel integral equations

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Abstract

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A rational basis set is constructed using [1, 1] rational interpolation. The basis is used in a product integration method for solving second-kind Abel integral equations. Criteria are established for determining the amount of bias to be used in the basis set when the integral equation has a nonsmooth solution. Several test problems are solved to illustrate the performance of the approximation method.

Keywords: Integral equations, rational basis.

1. Introduction

This paper is concerned with the numerical solution of the weakly-singular Volterra-type integral equation of the second kind

$$y(t)=g(t)-\int_0^t(t-s)^{-\alpha}K(t,\ s,\ y(s))\,\mathrm{d}s, \tag{1.1}$$

where $0<\alpha<1$. Equation (1.1) is sometimes referred to as a second-kind Abel integral equation. It will be assumed that g is continuous and $K(t,\ s,\ y)$ is continuous with respect to s and t and uniformly Lipschitz continuous with respect to y,

$$|K(t,\ s,\ y_1)-K(t,\ s,\ y_2)|\leqslant L|y_1-y_2|, \tag{1.2}$$

and that there exists a unique solution $y(t)$ for $t\in[0,\ T]$. Equation (1.1) arises, for example, in problems of stereology (see [13, Chapter 2]).

A number of numerical techniques are available for finding approximate solutions of (1.1) [5,13]. Often these solutions have a nonsmooth behaviour and special techniques are required to solve the integral equation [7]. A nonpolynomial basis has been used in a collocation-type approach in [3,4,17], and in a Galerkin approach in [8]. A highly successful method using fractional powers of linear multistep methods has been pioneered by Lubich [14]. Other methods involve replacing the weak singularity by a polynomial [1]. In the present paper we shall consider a finite-element method based on product integration [6].

Rational functions have been studied as a finite-element base for space variables [20], and more recently also for the time variable in initial-value problems [19]. Each basis function is compactly supported.

The [1, 1] rational function to be considered in this paper has the form suggested in [19]:

$$r_0(t) = a + \frac{b}{1 + k(t/h)}, \tag{1.3}$$

$t \in [0, h]$. Coefficients a and b are determined from the conditions $r_0(0) = 1$ and $r_0(h) = 0$. This leaves one undetermined coefficient k, which may be used to regulate the amount of bias in the basis function. The interval $[0, h]$ is spanned by two basis functions

$$r_0(t) = \frac{(h-t)}{(h+kt)}, \tag{1.4}$$

$$r_1(t) = 1 - r_0(t), \tag{1.5}$$

where the second equation is a consistency requirement. The parameter k can take any value in the interval $(-1, \infty)$. Clearly $k = 0$ is the linear approximation, while $k > 0$ gives a rational function biased in the upwind direction and $k < 0$ in the downwind direction.

The aim of this paper is to investigate the utility of the rational basis for solving (1.1) in the case of a nonsmooth solution. The method of approximation is described in Section 2. Section 3 gives some convergence results. Practical implementation of the algorithm is discussed in Section 4. Section 5 presents numerical results for both linear and nonlinear test problems.

2. Approximation method

Let π_n be a partition of the interval $[0, T]$ defined by the points $0 = t_0 < t_1 < \cdots < t_n = T$. On this mesh we construct rational basis functions $\{\phi_j;\ j = 0, \ldots, n\}$, where $\phi_0(t) = r_0(t - t_0)$,

$$\phi_j(t) = \begin{cases} r_1(t - t_{j-1}), & t_{j-1} \leqslant t < t_j, \\ r_0(t - t_j), & t_j \leqslant t < t_{j+1}, \end{cases} \tag{2.1}$$

for $j = 1, \ldots, n-1$, and $\phi_n(t) = r_1(t - t_{n-1})$.

Discretizing (1.1) gives

$$y(t_i) = g(t_i) - \int_0^{t_i} (t_i - s)^{-\alpha} K(t_i, s, y(s))\, \mathrm{d}s. \tag{2.2}$$

The next step is to approximate the kernel $K(t_i, s, y(s))$ in a linear space of rational basis functions

$$\sum_{j=0}^{i} K\big(t_i, t_j, y(t_j)\big)\phi_j(s). \tag{2.3}$$

The values $y(t_i)$ can be approximated by coefficients $\{y_i\}$ which satisfy the algebraic equations

$$y_i = g(t_i) - h^{1-\alpha} \sum_{j=0}^{i} K\big(t_i, t_j, y_j\big) w_{ij}, \tag{2.4}$$

where the coefficients $\{w_{ij}\}$ are quadrature weights given by the product integrals

$$h^{1-\alpha} w_{ij} = \int_0^{t_i} (t_i - s)^{-\alpha} \phi_j(s)\, \mathrm{d}s, \tag{2.5}$$

with $j \leqslant i \leqslant n$.

Now consider a uniform mesh. Putting $s = t_j + ph$ in (2.5), and using the definition (2.1), the weights w_{ij} form a lower triangular matrix

$$\begin{pmatrix} 0 & & & & \\ a_0(1) & a_1(1) & & & \\ a_0(2) & a_0(1) + a_1(2) & a_1(1) & & \\ a_0(3) & a_0(2) + a_1(3) & a_0(1) + a_1(2) & a_1(1) & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{2.6}$$

For a case of practical interest $\alpha = \frac{1}{2}$, putting $\beta = i - j$,

$$a_\tau(\beta) = A_\tau\big[(\beta - 1)^{1/2} - \beta^{1/2}\big] + B_\tau \int_0^1 \frac{\mathrm{d}p}{(\beta - p)^{1/2}(1 + kp)}, \tag{2.7}$$

where $r = 0$ or 1. Here $A_0 = -1/k$, $B_0 = k/(1+k)$, $A_1 = -B_1 = (1+k)/k$.

The integral in (2.7) can be evaluated using standard tables of integrals [9]

$$\int \frac{\mathrm{d}p}{(\beta - p)^{1/2}(1 + kp)} = \begin{cases} \dfrac{1}{\sqrt{\sigma}} \ln\left[\dfrac{k(\beta - p)^{1/2} - \sqrt{\sigma}}{k(\beta - p)^{1/2} + \sqrt{\sigma}}\right], & \sigma > 0, \\ \dfrac{2(\beta - p)^{1/2}}{(1 + kp)}, & \dfrac{\sigma}{k} = 0, \\ \dfrac{2}{\sqrt{-\sigma}} \arctan\left[\dfrac{k(\beta - p)^{1/2}}{\sqrt{-\sigma}}\right], & \sigma < 0, \end{cases} \tag{2.8}$$

where $\sigma = k(\beta k + 1)$.

For the special case of $k = 0$,

$$a_\tau^{(0)}(\beta) = A_\tau^{(0)}\big[(\beta - 1)^{1/2} - \beta^{1/2}\big] + B_\tau^{(0)}\big[(\beta - 1)^{3/2} - \beta^{3/2}\big], \tag{2.9}$$

where $A_0^{(0)} = 2(\beta - 1)$, $A_1^{(0)} = -2\beta$, $B_0^{(0)} = -B_1^{(0)} = -\frac{2}{3}$.

3. Convergence

At present we are not able to establish optimal error bounds for rational interpolation. We can however provide a rough estimate of the rational approximation to a function u [18]. Divide $[a, b]$ into N subintervals with spacing h. We have the following theorem.

Theorem 1. *Let $u \in C^2[a, b]$, and assume $u^{(1)}$ and $u^{(2)}$ are bounded. The* $[1, 1]$ *rational interpolant u_N of u is convergent of order* 1.

Proof. Consider an interval $[t_j, t_{j+1}]$. The rational approximation to $u(t)$ is

$$u_N(t) = u(t_j)r_0(t - t_j) + u(t_{j+1})r_1(t - t_j) = \frac{P(t)}{Q(t)}, \tag{3.1}$$

where

$$\begin{aligned} P(t) &= \left[(1 + k)u(t_{j+1}) - u(t_j)\right](t - t_j) + hu(t_j), \\ Q(t) &= h + k(t - t_j). \end{aligned} \tag{3.2}$$

Define

$$L(t) = Q(t)u(t) - P(t), \tag{3.3}$$

where $L(t)$ satisfies the interpolation conditions $L(t_j) = L(t_{j+1}) = 0$. Define

$$\Pi(t) = \prod_{s=j}^{j+1} (t - t_s), \tag{3.4}$$

and write

$$L(t) = \Pi(t)A(t). \tag{3.5}$$

The function $A(t)$ is obtained by defining the auxiliary function

$$\Omega(z) = \Pi(z)A(t) - L(z), \tag{3.6}$$

where $t = t_s$. The function $\Omega(z)$ vanishes at three distinct points $t_j < t < t_{j+1}$. By the application of Rolle's theorem, $\Omega^{(2)}(z)$ vanishes at the point $\Gamma \in (t_j, t_{j+1})$. Evaluating $\Omega^{(2)}(z)$ at this point gives

$$A(t) = \frac{1}{2} \frac{\mathrm{d}^2}{\mathrm{d}z^2}[Q(z)u(z)]_{z=\Gamma}. \tag{3.7}$$

The error $e = u - u_N$ in the interval $[t_j, t_{j+1}]$ is therefore

$$e(t) = \frac{1}{2} \frac{\Pi(t)}{Q(t)} \frac{\mathrm{d}^2}{\mathrm{d}z^2}[Q(z)u(z)]_{z=\Gamma}, \tag{3.8}$$

where $\Gamma \in (t_j, t_{j+1})$. On substituting $t = t_j + hp$ we find

$$\| e \| \leqslant \tfrac{1}{4}kh\left[\frac{h(1 + k)}{2k} \| u^{(2)} \| + \| u^{(1)} \|\right], \tag{3.9}$$

where $\| \cdot \|$ denotes the supremum norm on the interval $[t_j, t_{j+1}]$. The result follows. □

Next we investigate convergence of the numerical method. The local consistency error is defined as

$$\delta(h,\ t_i)=\int_0^{t_i}K(t_i,\ s,\ y(s))\ \mathrm{d}s-h^{1-\alpha}\sum_{j=0}^{i}K\big(t_i,\ t_j,\ y(t_j)\big)w_{ij}. \tag{3.10}$$

The method is consistent of order p if, for some constant C independent of h,

$$\max_{0\leqslant i\leqslant n}|\delta(h,\ t_i)|\leqslant Ch^p. \tag{3.11}$$

The value of p depends on the nonsmooth behaviour of the solution at $t=0$ and is given by $p=1-\alpha$. We now have the following theorem.

Theorem 2. *The method is convergent of order* $1-\alpha$.

Proof. The discretization error is

$$e_i=y(t_i)-y_i=h^{1-\alpha}\sum_{j=0}^{i}\left\{K\big(t_i,\ t_j,\ y_j\big)-K\big(t_i,\ t_j,\ y(t_j)\big)\right\}w_{ij}-\delta(h,\ t_i). \tag{3.12}$$

From (3.12) we obtain

$$|e_i|\leqslant h^{1-\alpha}L\bar{w}\,|e_i|+h^{1-\alpha}L\bar{w}\sum_{j=0}^{i-1}\frac{|e_j|}{(i-j)^{\alpha}}+Ch^{1-\alpha}. \tag{3.13}$$

This follows from the fact that there exists a $\bar{w}>0$, independent of i and j, such that $|w_{ii}|\leqslant\bar{w}$ and

$$|w_{ij}|\leqslant\frac{\bar{w}}{(i-j)^{\alpha}},\quad i>j, \tag{3.14}$$

together with (1.2) and (3.11).

For $h\leqslant[(L\bar{w})^2\pi]^{-1}$ it is possible to apply the generalized Gronwall lemma of [15] (see also [6, Lemma 4.1]). The result follows directly. □

4. Choice of the parameter k

The choice of free parameter k is determined by the interpolation properties of the basis. Let $\mu(t)$ be chosen to mimic the nonsmooth behaviour of the kernel in equation (1.1). In general $\mu(t)$ may be piecewise continuous. In the spirit of exponential fitting [12], we consider a function $f_i(t)$ that interpolates to $\mu(t)$ over $t_{i-1}\leqslant t\leqslant t_i$,

$$f_i(t)=\mu(t_{i-1})r_0(t)+\mu(t_i)r_1(t). \tag{4.1}$$

One approach is to choose k so as to minimize

$$\int_{t_{i-1}}^{t_i}(f_i(t)-\mu(t))^2\ \mathrm{d}t. \tag{4.2}$$

Then k satisfies

$$\int_{t_{i-1}}^{t_i}\frac{(f_i(t)-\mu(t))(h-t)t}{(h+kt)^2}\ \mathrm{d}t=0. \tag{4.3}$$

The numerical performance of the method is expected to depend to a large extent on the appropriate choice of the auxiliary function $\mu(t)$. Two choices of this auxiliary function will be investigated.

Choice (A). $\mu(t) = K(\cdot, t, y(t))$ is the exact kernel of (1.1). This choice is made to test the rational basis for an optimal choice of k. This is only possible if the exact solution $y(t)$ of (1.1) is known.

Choice (B). $\mu(t)$ is a simple function that approximates and has the correct asymptotic behaviour of the kernel K. This situation may often arise in practice where the asymptotic behaviour of the solution $y(t)$ is known. Consider equation (1.1) with $\alpha = \frac{1}{2}$ that has a solution with the assumed form [7]

$$y(t) = X(t) + Y(t)\sqrt{t}, \tag{4.4}$$

where $X(t)$ and $Y(t)$ are smooth functions on $[0, T]$. We have chosen to implement the following algorithm. Given $y_0, y_1, \ldots, y_{i-1}$, we first calculate an intermediate value $y_i^{(0)}$ from the equation

$$\begin{aligned} y_i^{(0)} + h^{1/2}K(t_i, t_i, y_i^{(0)})a_1^{(0)}(1) = g(t_i) &- h^{1/2}K(t_i, t_{i-1}, y_{i-1})a_0^{(0)}(1) \\ &- h^{1/2}K(t_i, t_{i-1}, y_{i-1})a_1(2) \\ &- h^{1/2}\sum_{j=0}^{i-2} K(t_i, t_j, y_j)w_{ij}, \end{aligned} \tag{4.5}$$

where $a_0^{(0)}(1)$ and $a_1^{(0)}(1)$ are defined by (2.9). The next step is to calculate y_i using (2.4). In order to calculate the weights $a_0(1)$ and $a_1(1)$ defined by (2.7) the solution is assumed to have the form

$$y(t) \simeq X + Y\sqrt{t}, \quad t \in [t_{i-1}, t_i], \tag{4.6}$$

where coefficients X and Y are obtained from y_{i-1} and $y_i^{(0)}$. Then

$$\mu(t) = K(\cdot, t, X + Y\sqrt{t}), \tag{4.7}$$

and k is computed from equation (4.3). In this algorithm the intermediate values $y_i^{(0)}$ are "corrected" using the assumed form of the solution, namely (4.6). Henceforth we shall refer to this particular scheme as the choice (B).

5. Results and conclusions

In this section we solve a number of test problems to illustrate the numerical performance of the method. Before presenting the results it should be remarked that exponential fitting was originally introduced as a method for solving stiff equations on coarse meshes. Consequently, we anticipate that the method described here should work well for large values of the stepsize h.

A simple example of a linear equation with nonsmooth solution is [16]

$$y(t) = 1 - \int_0^t (t-s)^{-1/2} y(s)\, \mathrm{d}s, \quad t \in [0, 1], \tag{5.1}$$

which has the solution $y(t) = \exp(\pi t)\operatorname{erfc}(\sqrt{\pi t})$, where erfc is the complementary error function [2].

Table 1 shows results for the rational basis approximation of the linear equation (5.1) on the coarse mesh $h = 0.2$. For choices (A) and (B) the k-parameters in each interval $[t_{i-1}, t_i]$ were

Table 1
Errors $e_i = y_i - y(t_i)$ obtained for equation (5.1) using $h = 0.2$

i	t_i	$y(t_i)$	e_i ($k=0$)	Rational basis				
				e_i ($k=1$)	Choice (A)		Choice (B)	
					e_i	k	e_i	k
1	0.2	0.491648	$-5.2\cdot10^{-2}$	$-2.6\cdot10^{-2}$	$2.4\cdot10^{-3}$	4.756	$-1.5\cdot10^{-2}$	1.850
2	0.4	0.396653	$-1.6\cdot10^{-2}$	$-6.2\cdot10^{-3}$	$1.8\cdot10^{-4}$	0.436	$-6.0\cdot10^{-3}$	0.187
3	0.6	0.343677	$-9.7\cdot10^{-3}$	$-2.8\cdot10^{-3}$	$8.8\cdot10^{-5}$	0.259	$-3.7\cdot10^{-3}$	0.106
4	0.8	0.308158	$-6.7\cdot10^{-3}$	$-1.5\cdot10^{-3}$	$4.6\cdot10^{-5}$	0.187	$-2.6\cdot10^{-3}$	0.074
5	1.0	0.282059	$-5.0\cdot10^{-3}$	$8.0\cdot10^{-4}$	$2.8\cdot10^{-5}$	0.147	$-1.9\cdot10^{-3}$	0.057

obtained using (4.3). All the values of k are positive. We also show for comparison results using a linear basis ($k = 0$) and a rational basis with fixed k ($k = 1$). Choice (A) gives results that are between one and two orders of magnitude more accurate than the those using the linear basis. This improvement extends over the entire interval [0, 1]. Results for choice (B) are close to those obtained using a fixed value of k.

Next we consider two examples of nonlinear equations [17]:

$$y(t) = t^{1/2} + \tfrac{3}{8}\pi t^2 - \int_0^t (t-s)^{-1/2} y(s)^3 \, \mathrm{d}s, \qquad t \in [0, 2], \tag{5.2}$$

$$y(t) = t^{1/2} + \tfrac{16}{15} t^{5/2} - \int_0^t (t-s)^{-1/2} y(s)^4 \, \mathrm{d}s, \quad t \in [0, 3], \tag{5.3}$$

both of which have the solution $y(t) = t^{1/2}$. The resulting nonlinear equations were solved using the IMSL subroutine ZBREN.

Table 2 shows linear and rational basis (choice (B)) approximations of the two nonlinear equations (5.2) and (5.3) again on a mesh with $h = 0.2$. Both examples show improvement in the accuracy of the rational approximation over the linear approximation for the entire interval $[0, T]$.

Table 3 gives a list of results for the quantity $-\log_{10}|e_n|$ for a range of mesh sizes. In both the linear (5.2) and nonlinear examples (5.3), (5.4) we obtain improvement in the accuracy of the approximate solutions.

It is interesting to compare the results using a rational basis with results from a collocation method using a nonpolynomial basis, such as those found by te Riele [17]. In his approach, te Riele approximates the solution $y(t)$ by a linear combination of basis functions. Nonpolynomial functions of the form $t^{i/2}$ are used as basis functions in the first interval. Results obtained using the lowest-order basis $t^{1/2}$ do not show any improvement when compared with results using a linear basis for equations (5.2) and (5.3) [17, 10(b) and 10(c)]. The rational basis does, however, yield improvement over the linear basis. Further, it is stated by te Riele that extending the nonpolynomial basis over more intervals leads to a loss of accuracy. This is in stark contrast to the results we have obtained using the rational basis where a marked improvement in accuracy was obtained by extending the rational basis over all intervals.

Our final example is a nonlinear equation that arises in the theory of superfluidity [11]:

$$y(t) = -\sqrt{\pi} \int_0^t (t-s)^{-1/2} (y(s) - \sin s)^3 \, \mathrm{d}s, \quad t \in [0, 1]. \tag{5.4}$$

Table 2
Errors $e_i = y_i - y(t_i)$ obtained for equations (5.2) and (5.3) using $h = 0.2$

i	t_i	$y(t_i)$	Example (5.2)				Example (5.3)			
			e_i $(k=0)$	Rational basis			e_i $(k=0)$	Rational basis		
				e_i $(k=1)$	Choice (B)			e_i $(k=1)$	Choice (B)	
					e_i	k			e_i	k
1	0.2	0.447214	$-4.6\cdot10^{-3}$	$-9.4\cdot10^{-3}$	$1.4\cdot10^{-4}$	-0.440	$-3.9\cdot10^{-3}$	$-6.4\cdot10^{-3}$	$9.0\cdot10^{-5}$	-0.624
2	0.4	0.632456	$-3.0\cdot10^{-3}$	$-1.1\cdot10^{-2}$	$-1.1\cdot10^{-4}$	-0.150	$-4.1\cdot10^{-3}$	$-1.0\cdot10^{-2}$	$-1.1\cdot10^{-4}$	-0.276
3	0.6	0.774597	$-2.0\cdot10^{-3}$	$-1.1\cdot10^{-2}$	$-5.4\cdot10^{-5}$	-0.094	$-3.2\cdot10^{-3}$	$-1.1\cdot10^{-2}$	$-7.5\cdot10^{-5}$	-0.178
4	0.8	0.894427	$-1.3\cdot10^{-3}$	$-1.1\cdot10^{-2}$	$-3.4\cdot10^{-5}$	-0.068	$-2.3\cdot10^{-3}$	$-1.1\cdot10^{-2}$	$-5.3\cdot10^{-5}$	-0.131
5	1.0	1.000000	$-9.7\cdot10^{-4}$	$-1.0\cdot10^{-2}$	$-2.2\cdot10^{-5}$	-0.053	$-1.7\cdot10^{-3}$	$-1.0\cdot10^{-2}$	$-3.6\cdot10^{-5}$	-0.103
6	1.2	1.09545	$-7.3\cdot10^{-4}$	$-9.4\cdot10^{-3}$	$-1.5\cdot10^{-5}$	-0.044	$-1.3\cdot10^{-3}$	$-9.8\cdot10^{-3}$	$-2.5\cdot10^{-5}$	-0.085
7	1.4	1.18322	$-5.8\cdot10^{-4}$	$-8.8\cdot10^{-3}$	$-1.1\cdot10^{-5}$	-0.037	$-1.0\cdot10^{-3}$	$-9.1\cdot10^{-3}$	$-1.8\cdot10^{-5}$	-0.073
8	1.6	1.26491	$-4.7\cdot10^{-4}$	$-8.3\cdot10^{-3}$	$-7.9\cdot10^{-6}$	-0.033	$-8.3\cdot10^{-4}$	$-8.6\cdot10^{-3}$	$-1.4\cdot10^{-5}$	-0.064
9	1.8	1.34164	$-3.9\cdot10^{-4}$	$-7.9\cdot10^{-3}$	$-6.1\cdot10^{-6}$	-0.029	$-6.9\cdot10^{-4}$	$-8.1\cdot10^{-3}$	$-1.1\cdot10^{-5}$	-0.056
10	2.0	1.41421	$-3.3\cdot10^{-4}$	$-7.6\cdot10^{-3}$	$-4.8\cdot10^{-6}$	-0.026	$-5.9\cdot10^{-4}$	$-7.7\cdot10^{-3}$	$-8.3\cdot10^{-6}$	-0.051
11	2.2	1.48324	—	—	—	—	$-5.1\cdot10^{-4}$	$-7.3\cdot10^{-3}$	$-6.7\cdot10^{-6}$	-0.046
12	2.4	1.54919	—	—	—	—	$-4.5\cdot10^{-4}$	$-7.0\cdot10^{-3}$	$-5.5\cdot10^{-6}$	-0.042
13	2.6	1.61245	—	—	—	—	$-4.0\cdot10^{-4}$	$-6.8\cdot10^{-3}$	$4.5\cdot10^{-6}$	-0.039
14	2.8	1.67332	—	—	—	—	$-3.5\cdot10^{-4}$	$6.5\cdot10^{-3}$	$-3.8\cdot10^{-6}$	-0.036
15	3.0	1.73205	—	—	—	—	$-3.2\cdot10^{-4}$	$-6.3\cdot10^{-3}$	$-3.2\cdot10^{-6}$	-0.034

Table 3
Results for $-\log_{10}|e_n|$

Equation	h	Linear basis	Rational basis		te Riele [17]
			(A	(B)	
(5.1)	0.2	2.3	4.6	2.7	4.08
	0.1	2.8	5.2	3.3	4.30
	0.05	3.2	5.7	3.8	4.67
	0.025	3.7	6.2	4.3	—
(5.2)	0.2	3.5	5.7	5.3	—
	0.1	4.1	6.6	6.2	4.16
	0.05	4.7	7.3	7.2	4.75
	0.025	5.3	7.9	9.0	5.35
(5.3)	0.2	3.5	7.0	5.5	—
	0.1	4.1	7.4	6.4	4.12
	0.05	4.7	8.2	7.5	4.69
	0.025	5.3	7.7	8.6	5.27

Equation (5.4) has a solution with asymptotic behaviour

$$y(t) \sim O(t^{7/2}), \quad t \to 0^+. \tag{5.5}$$

We therefore replace the assumed form of the solution (4.6) by a solution of the form

$$y(t) \simeq X + Yt^{7/2}, \quad t \in [t_{i-1}, t_i]. \tag{5.6}$$

Table 4 shows results using the rational basis (choice (B)). The reference solution is obtained using the high-order convergence method of [14], and these calculations were performed using the algorithm of [10] with a fine mesh ($h = \frac{1}{70}$). For the purpose of comparison we also show the results obtained from a linear basis ($k = 0$) and a rational basis with $k = 1$.

Table 4
Approximate solutions y_i obtained for equation (5.4) using $h = 0.1$

i	t_i	Reference solution [10]	y_i ($k = 0$)	Rational basis		
				y_i ($k = 1$)	Choice (B)	
					y_i	k
1	0.1	$0.161755 \cdot 10^{-3}$	$0.235029 \cdot 10^{-3}$	$0.265415 \cdot 10^{-3}$	$0.160873 \cdot 10^{-3}$	-0.780
2	0.2	$0.177806 \cdot 10^{-2}$	$0.200415 \cdot 10^{-2}$	$0.221880 \cdot 10^{-2}$	$0.177697 \cdot 10^{-2}$	-0.456
3	0.3	$0.695735 \cdot 10^{-2}$	$0.732012 \cdot 10^{-2}$	$0.790462 \cdot 10^{-2}$	$0.695948 \cdot 10^{-2}$	-0.280
4	0.4	$0.175626 \cdot 10^{-1}$	$0.179857 \cdot 10^{-1}$	$0.190495 \cdot 10^{-1}$	$0.175756 \cdot 10^{-1}$	-0.167
5	0.5	$0.345440 \cdot 10^{-1}$	$0.349374 \cdot 10^{-1}$	$0.364680 \cdot 10^{-1}$	$0.345822 \cdot 10^{-1}$	-0.080
6	0.6	$0.577505 \cdot 10^{-1}$	$0.580508 \cdot 10^{-1}$	$0.599347 \cdot 10^{-1}$	$0.578323 \cdot 10^{-1}$	-0.003
7	0.7	$0.861496 \cdot 10^{-1}$	$0.863323 \cdot 10^{-1}$	$0.884093 \cdot 10^{-1}$	$0.862901 \cdot 10^{-1}$	0.072
8	0.8	0.118210	0.118279	0.120392	0.118415	0.151
9	0.9	0.152242	0.152218	0.154237	0.152509	0.244
10	1.0	0.186623	0.186527	0.188360	0.186941	0.374

It is clear that the amount of bias introduced into the rational basis can have a significant effect on the accuracy of the product integration method. Although the rational basis can be tailored to the asymptotic behaviour of the kernel, this does not guarantee that accurate results can be obtained over the entire domain of integration. In the examples that we have considered it is found that a nonzero k in each of the intervals of integration is required in order to yield high accuracy solutions.

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