Confounding in Relation to Duality of Finite Abelian Groups

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ABSTRACT

Using the duality in the theory of finite abelian groups, we give a precise description of confounded effects in fractional factorial design, when the fraction is a subgroup or a coset of a subgroup. The result works for a confounded block design, considered here as a fraction of a bigger design with complete blocks.

1. INTRODUCTION

Confounding is generally regarded as the division of a set of factorial treatments into blocks, in a way which allows interesting contrasts between treatments to be estimated within blocks whereas uninteresting or negligible contrasts, for instance high-order interactions, are confounded with differences between blocks.

Closely related to confounding is fractional replication. It is used when, because of the number of factors studied, only a fraction of the complete set of factorial treatments can be achieved. The art is to choose the fraction so that important effects, in general principal effects and low-order interactions, are confounded with negligible effects, such as high-order interactions.

We shall not make any distinction between these two subjects. Indeed, every block design can be considered as a fraction of a bigger one for which each block receives the whole set of treatments. Confounding can thus be considered as a consequence of fraction taking and studied in the same way as confounded effects in fractional replication. The roles of blocks and treatments are naturally very different in the randomization and the statistical analysis of the data. However, in the construction of the design, it is

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unnecessary and sometimes troublesome to distinguish block from treatment factors, and in the rest of the paper we shall not distinguish them.

Therefore, we shall study here how effects are confounded when physical impossibility, such as limited size of blocks or financial limitation, requires restricting the investigation to a fraction, called the *actual* design, of a *reference* design, whose units are usually all the combinations of factor levels. Moreover, after Section 3.2, we shall confine our investigations to the special following case:

(1) The units of the reference design form a commutative group H, and factors are group morphisms defined on it.

(2) The actual units also form a commutative group G, and the combination of levels of factors on a unit u of G is $\phi(u) = w + \theta(u)$, where $\theta: G \to H$ is a group morphism and w a given element of H.

This framework was used by Finney [8] to introduce fractional replication in the particular case of 2^n and 3^n factorial experiments. It covers a wide class of orthogonal designs, including the classical symmetrical t^m designs (Bose [5]) where t is a power of a prime, as well as "asymmetrical" or "mixed" designs like those defined by White and Hultquist [16] or Dean and John [6].

The DSIGN method (Patterson [14]) can be considered as an extension of this framework. It defines the function θ by a matrix which is not necessarily a morphism, thus allowing the construction of interesting nonorthogonal design, as pointed out by Patterson. However, there is no simple way to derive the efficiency of estimation of the different contrasts and no simple rule to choose the function ϕ from specified objectives in the general case, and so we shall not consider it here.

Generalized cyclic designs (John [11]) cannot be brought into this precise framework. But they can be presented as a juxtaposition of this kind of fraction, having the same θ, but different w. Hence, the results given here suggest an interesting approach to generalized cyclic design.

In the theory developed by Bose, reference and actual designs are vectorial spaces over a finite field and morphisms are linear applications. However, many important results of the theory rest only on additive group structure. By restating them in the frame of group theory, Bailey [1, 2] gave a general method of construction for confounding design involving factors with arbitrary numbers of levels. Her approach, very well suited to the confounding problem, does not clearly show what parts of the effects are confounded together in the case of fractional replication. Nor does it clearly give the parts of block-treatment interactions which are confounded with a particular effect, hence falsely increasing it.

Using the same algebraic tools as Bailey, in particular duality in commutative groups, we give here simple rules for determining confounded effects in the framework described above. We first precisely define what we call confounded effects.

2. SIMPLE CONFOUNDING SITUATION

We denote by G the actual design and by H the reference design. ϕ is the map giving for each actual unit u of G the corresponding combination of levels of factors $v = \phi(u)$ in H. For the moment, we do not assume that G, H, and ϕ have any special structure.

Let μ be the vector of R^H whose coordinate μ_v gives the expectation of an observation made at the levels of factors defined by v. The expectation of the observation y_u made on the actual unit u is thus $\mu_{\phi(u)}$.

The expectation of the vector y of observations belonging to R^G can then be written $\psi(\mu)$, where ψ is the linear map from R^H to R^G, induced by ϕ, defined by $\psi(\mu) = (\mu_{\phi(u)})_{u \in G}$. If μ is regarded as a function from the reference design to R, we can also write $\psi(\mu)$ as the composition map $\psi(\mu) = \mu \circ \phi$.

Now, suppose there are orthogonal bases $(f_h)_{h \in H}$ of R^H and $(e_g)_{g \in G}$ of R^G such that ψ maps each f_h precisely on one e_g. If the f_h are of norm 1, we have

$$\mu = \sum_{h \in H} (f_h' \mu) f_h,$$

$$E(y) = \psi(\mu) = \sum_{h \in H} (f_h' \mu) \psi(f_h) = \sum_{g \in G} \left(\sum_{\psi(f_h) = e_g} f_h' \mu \right) e_g.$$

It follows from the last equality that the linear functions $f_h' \mu$ corresponding to vectors f_h having the same image e_g by ψ cannot be estimated separately from the vector y of observations, which depends on them only through their sum. Such functions will be said to be confounded or aliased by the actual design.

We shall show in Section 4 that in the framework of group theory described in the introduction, bases like (f_h) and (e_g) can always be found, and moreover that the orthogonal decomposition given by (f_h) can be chosen compatible with the usual orthogonal decomposition in factorial effects (see Proposition 11). Before proving these results, we give the following example of a simple situation of confounding.

TABLE 1

CONFOUNDING IN A HALF FRACTION OF A 2^3 FACTORIAL DESIGN

	Treatment			Column vector							
	A	B	C	O	A	B	C	AB	AC	BC	ABC
Half fraction performed	0	0	0	1	1	1	1	1	1	1	1
	0	1	1	1	1	−1	−1	−1	−1	1	1
	1	0	1	1	−1	1	−1	−1	1	−1	1
	1	1	0	1	−1	−1	1	1	−1	−1	1
	1	0	0	1	−1	1	1	−1	−1	1	−1
	0	1	0	1	1	−1	1	−1	1	−1	−1
	0	0	1	1	1	1	−1	1	−1	−1	−1
	1	1	1	1	−1	−1	−1	1	1	1	−1
				f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8

EXAMPLE 1. A, B, C are three factors having two levels, 0 and 1, each. The reference design H is formed by the eight combinations of factor levels, and the actual design G is the half fraction of those combinations for which the total of levels is congruent to 0, modulo 2.

Table 1 gives the eight treatments of the reference design. The fraction performed is composed of the four treatments above the dashed line. The basis $f_1, \ldots, f_8$ is constituted by the eight columns on the right. If they are correctly normalized, they define the eight factorial effects.

The mapping ψ is the natural projection on the subspace of R^8 spanned by the first four coordinates. It maps each f_h on one of the four distinct orthogonal column vectors appearing above the dashed line on the right of Table 1. Since f_1 and f_8 have the same image, the general mean O is confounded with the three-factor interaction ABC. Similarly, Table 1 shows that A is confounded with BC, B with AC, and C with AB.

That half fraction of a 2^3 factorial experiment was one of the examples used by Finney [8] in his introduction to fractional replication. In Finney's notation, the eight treatments are represented by the symbols $1, a, b, ab, c, ac, bc, abc$. A multiplicative abelian group structure is defined on this set of symbols by using the relations $a^2 = b^2 = c^2 = 1$, and the fraction actually experimented on is chosen as a subgroup. Since we use matricial representation of morphisms in the following, it will be more convenient to use an additive notation instead of the multiplicative one, using a row vector (α, β, γ) or a column vector $(\alpha, \beta, \gamma)'$ instead of $a^\alpha b^\beta c^\gamma$ to represent a treatment having levels α of factor A, β of B, γ of C, and defining the group operation by

$$(\alpha_1, \beta_1, \gamma_1) + (\alpha_2, \beta_2, \gamma_2) = (\alpha_1 + \alpha_2, \beta_1 + \beta_2, \gamma_1 + \gamma_2).$$

3. ORTHOGONALITY OF GROUP MORPHISMS

From any family of morphisms on a finite abelian group we can get an orthogonal design. This is the main result of this section. Before proving it, we first give some formal mathematical definitions of basic concepts related to experimental designs. We then define an orthogonal design in the general case and give its main properties. The reader interested by a more detailed account on orthogonal designs can refer to a paper by Tjur [15].

3.1. *Factors and Designs*

Given a set of units G, we define a factor A on G as a mapping $A: G \to L_A$ from G into some other finite set L_A. For each $u \in G$, $A(u)$ is called the *level* of factor A on unit u. The *set of levels of* A is therefore the image of A.

Two factors play a special role as extremes, the *constant factor* $O: G \to L_O$, where L_O is an arbitrary set with a single element, and the *unit factor* $I: G \to G$, equal to the identity Id_G of G.

The *product* of two factors A and B is the mapping $A \times B: G \to L_A \times L_B$, defined by $A \times B(u) = (A(u), B(u))$.

A factor A is said to be *nested in* B, and we write $B \leqslant A$, if we have the implication $A(u) = A(v) \Rightarrow B(u) = B(v)$, or equivalently if there is a mapping $C: L_A \to L_B$ such that $B = C \circ A$.

The *partition induced by a factor* $A: G \to L_A$ on G is the set of nonempty reciprocal images $A^{-1}(a)$ of the elements of L_A. A is nested in B iff the partition induced by A is finer than that induced by B. Two factors A and B inducing the same partition are called *equivalent*.

There are generally two steps in the planning of an experiment. The first is the research in a standard textbook, in a catalog, or by any mathematical method to find an appropriate design. Units and factor levels of this design are elements of some abstract set (1,2,3,..., or $\alpha, \beta, \ldots$, or anything else) having no relation with our particular experiment. The second step then assigns to these abstract units and levels the concrete, physical ones. For instance, suppose we want to investigate the effect of small variations of pH (factor A), temperature (B), and oxygen (C) on the growth rate (Y) of a given bacterium. On a little domain of variation, the relation between Y and

A, B, C can reasonably be approximated by a linear relation, and the half fraction of Example 1 (repeated twice) provides an appropriate design to start with. However, there is generally no reason to assign the abstract level 1 of a given factor systematically to the upper concrete level. In most cases, it is better to make a random assignment. Moreover, if only one experimental condition can be carried out at a time, the order of realization of each of the four units appearing at the top left of Table 1 will also be chosen randomly.

In the first step, the only one considered here, a factor can be replaced by any equivalent one without modifying the design. Therefore, we shall sometimes not distinguish between equivalent factors. Nevertheless, it must be kept in mind that each of the abstract factors constructed has its concrete counterpart, and consequently equivalence does not imply identity. For instance, pH and temperature will never be the same factor, even if the design chosen is a fraction of the 2^3 factorial design of Table 1 on which they are equivalent.

For each factor A on G, we define the *subspace associated with* A, S_A, as the subspace of $E = R^G$ consisting of vectors having equal coordinates for each level of A. More formally, S_A is the space of all composition maps of the form $f \circ A$, where f is any function from L_A to R. S_A is also the space spanned by the columns of 0 and 1 usually associated to levels of A in the incidence matrix of the linear model of the design. Clearly A is nested in B iff $S_B \subset S_A$, and A and B are equivalent iff $S_B = S_A$.

To a design G is associated an *analysis-of-variance model*, which can be defined as a family $\mathscr{E}$ of factors on G. Besides the factors corresponding to principal effects, $\mathscr{E}$ generally contains the product factors corresponding to interactions, the constant factor associated with the general mean, and the identity of G, which corresponds to the residual. We do not exclude *a priori* that $\mathscr{E}$ contains several equivalent factors, corresponding to different concrete factors but inducing the same partition of the set of units.

EXAMPLE 2. Consider the complete 2^3 factorial design used as a reference design in Example 1 (the set of units H contains the eight triplets (α, β, γ) appearing on the left of Table 1). The analysis-of-variance model can be defined by the family $\mathscr{F} = \{O, A, B, C, A \times B, A \times C, B \times C, I\}$. The factors appearing in this model can be defined as projections. For instance A is the projection $(\alpha, \beta, \gamma) \mapsto \alpha$, and $A \times B$ the projection $(\alpha, \beta, \gamma) \mapsto (\alpha, \beta)$. The partition induced by A contains the two subsets of four units each having respectively 0 and 1 as levels of A. S_A is the subspace of R^H of dimension 2 generated by the two vectors $(0\ 0\ 1\ 1\ 1\ 0\ 0\ 1)'$ and $(1\ 1\ 0\ 0\ 0\ 1\ 1\ 0)'$, or equivalently by the sum and the difference between these two vectors, which are the columns labeled O and A on the right of Table 1. Similarly, $S_{A \times B}$ is the subspace of dimension 4 spanned by the four vectors (1 0 0 0 0 0 1 0)′, (0 1 0 0 0 1 0 0)′, (0 0 1 0 1 0 0 0)′, and (0 0 0 1 0 0 0 1)′, or equivalently by the columns O, A, B, AB on the right of Table 1. S_O is of dimension 1, spanned by column O, and $S_I = R^H$.

3.2. *Orthogonal Design*

DEFINITION 1 (Orthogonal design). A design G is called orthogonal if for any two factors A and B of the model $\mathscr{E}$, the associated subspaces S_A and S_B are distinct and there exists a third factor C in $\mathscr{E}$ such that:

(i) $S_C = S_A \cap S_B$,

(ii) the orthogonal supplementary spaces of S_C in S_A and S_B respectively are mutually orthogonal (for the usual scalar product of R^G).

S_A is distinct from S_B iff A and B induce distinct partitions of G. Hence, the model of an orthogonal design cannot contain several equivalent factors. Then conditions (i) and (ii) are satisfied if A and B are both nested in C and if their cross-table for a given level of C has proportional rows.

Let G be an orthogonal design, whose analysis of variance is given by a family $\mathscr{E}$ of factors. We define, for each $A \in \mathscr{E}$, the *subspace associated with the effect of* A, $\bar{S}_A$, as the orthogonal supplementary space in S_A of all spaces S_B, $B \in \mathscr{E}$, strictly included in S_A. If μ is the expectation of the vector of observations on G, the set of linear functions $\{\mu \mapsto c'\mu / c \in \bar{S}_A\}$ is, when A is different from the constant factor, the space of contrasts traditionally associated to the term A of the model. Definition 1 implies the orthogonality of these spaces. More precisely, we have:

PROPOSITION 1. *Let $\mathscr{E}$ be the family of factors defining the model of an orthogonal design. For each $B \in \mathscr{E}$, S_B is the direct orthogonal sum of subspaces $\bar{S}_A$, $A \in \mathscr{E}$, such that S_A is included in S_B. In particular, if $\mathscr{E}$ contains the identity I of G, then $R^G = S_I$ is the direct orthogonal sum of the $\bar{S}_A$ for $A \in \mathscr{E}$.*

We shall call this direct orthogonal sum the *decomposition induced by the family $\mathscr{E}$.*

For each $B \in \mathscr{E}$, we have

$$\dim S_B = \sum_{S_A \subset S_B} \dim \bar{S}_A = \dim \bar{S}_B + \sum_{S_A \subsetneq S_B} \dim \bar{S}_A.$$

From these equalities, we can easily obtain by recurrence the dimension of each $\bar{S}_B$, which is the number of degrees of freedom of the term B of the model.

EXAMPLE 3. Consider again the complete 2^3 factorial design H of Example 1, with the analysis-of-variance model defined by the family: $\mathcal{F} = \{O, A, B, C, A\times B, A\times C, B\times C, I\}$. It is easy to check that H is an orthogonal design. $\bar{S}_O = S_O$ is the subspace spanned by column O on the right of Table 1. $\bar{S}_A$ is the orthogonal of S_O in S_A and is spanned by column A on the right of Table 1. Similarly $\bar{S}_B$ is spanned by column B, and from these results we infer that $\bar{S}_{A\times B}$ is spanned by column AB. Continuing this process, we see that $\bar{S}_O, \bar{S}_A, \bar{S}_B, \bar{S}_C, \bar{S}_{A\times B}, \bar{S}_{A\times C}, \bar{S}_{B\times C}, \bar{S}_I$, are the eight spaces of dimension 1 spanned by the eight columns on the right of Table 1.

Consider now the half fraction G formed by the four units above the dashed line in Table 1. If ϕ is the canonical injection from G into H, the factors $O, A, B,\dots$ of $\mathcal{F}$ induce factors $O\circ\phi, A\circ\phi, B\circ\phi,\dots$ on G. If $\mathcal{E}$ is the family of the eight factors thus induced by the factors of $\mathcal{F}$, then the design defined by G and the family $\mathcal{E}$ is not orthogonal, because the partitions induced by $I\circ\phi$, $(A\times B)\circ\phi$, $(A\times C)\circ\phi$, and $(B\times C)\circ\phi$ are identical. However, if the last three factors are taken off so that $\mathcal{E} = \{O\circ\phi, A\circ\phi, B\circ\phi, C\circ\phi, I\circ\phi\}$, then G together with $\mathcal{E}$ defined an orthogonal design.

3.3. *Orthogonal Family of Morphisms*

From now on, G will be an additive abelian finite group and $A, B,\dots$, the factors defined on it, will be group morphisms, i.e. Z linear maps. We shall prove that they define an orthogonal design if the family $\mathcal{E}$ giving the terms of the model satisfies one not too restrictive condition.

The fundamental result is the following theorem, where $\operatorname{Ker} A$ is the kernel of A, i.e. the subgroup of elements in G mapped by A on zero.

PROPOSITION 2. *Let G be a commutative additive finite group, and A and B two group morphisms defined on it. The number of elements in G having levels $A(u)$ of A and $B(v)$ of B is equal to the order of* $\operatorname{Ker} A \cap \operatorname{Ker} B$ *when u and v are in the same coset of* $\operatorname{Ker} A + \operatorname{Ker} B$, *and equal to* 0 *otherwise.*

Proof. If w has levels $A(u)$ of A and $B(v)$ of B, we have $w - u \in \operatorname{Ker} A$, $w - v \in \operatorname{Ker} B$, and by difference $v - u \in \operatorname{Ker} A + \operatorname{Ker} B$. Conversely, if $v - u = a + b$, with $a \in \operatorname{Ker} A$ and $b \in \operatorname{Ker} B$, the element $w = v - b = u + a$ has levels $A(u)$ of A and $B(v)$ of B. Thus, the set of elements having level $A(u)$ of A and $B(v)$ of B is not empty if and only if u and v are in the same coset of $\operatorname{Ker} A + \operatorname{Ker} B$.

Now, the set of elements w' having the same A and B levels as w is clearly the coset $w + \operatorname{Ker} A \cap \operatorname{Ker} B$, which proves the proposition. ∎

DEFINITION 2 (Infimum of two factors). The infimum $A \wedge B$ of two factors A and B which are morphisms defined on G is the canonical projection from G on $G/(\operatorname{Ker} A + \operatorname{Ker} B)$.

The use of the word infimum will be justified in Section 4.3. A general definition of the infimum of two arbitrary factors can be found in Tjur [15] (he calls it the minimum, which is not quite correct).

Factors A and B are both nested in $A \wedge B$, and for a fixed level of $A \wedge B$, the proposition shows that their cross-table has all the numbers in its cells equal to the order of $\operatorname{Ker} A \cap \operatorname{Ker} B$.

The following example is an illustration of Proposition 2.

EXAMPLE 4. Let $G = Z_6 \times Z_6$, where $Z_6 = Z/6Z$ is the cyclic group of order 6. A and B are morphisms from G to Z_6 defined, for an element $u = (u_1, u_2)$ of G, by

$$A(u) = u_2, \quad B(u) = 3u_1 + 2u_2 \pmod 6.$$

It can be proved, using Proposition 7 (Section 4.3), and the reader can easily check, that the morphism $u \mapsto 4u_2$ from G into Z_6 has the 12 elements of $\operatorname{Ker} A + \operatorname{Ker} B$ as kernel. We can thus identify $A \wedge B$ with this morphism. It is then clear that $(A \wedge B)(u) = 4u_2 = 4A(u) = 2B(u)$, which shows that A and B are nested in $A \wedge B$. Moreover, the number of elements $(u_1, u_2) \in G$ having levels α of A, β of B [i.e. such that $u_2 = \alpha$, $3u_1 + 2u_2 = \beta \pmod 6$] is 3 if $4\alpha = 2\beta \pmod 6$, and 0 if not.

A and B being nested in $A \wedge B$, $S_{A \wedge B}$ is included in S_A and S_B. Proposition 2 then implies:

COROLLARY 1. *With the hypothesis of Proposition 2, the orthogonal supplementary spaces of $S_{A \wedge B}$ in S_A and S_B respectively are mutually orthogonal (for the usual scalar product of R^G).*

From this corollary and Definition 1, we get:

COROLLARY 2. *Let $\mathscr{E}$ be a family of group morphisms on G with distinct kernels. If for any two morphisms A and B in $\mathscr{E}$, there is a third one in $\mathscr{E}$ having same kernel as $A \wedge B$, then $\mathscr{E}$ is the model of an orthogonal design.*

DEFINITION 3 (Orthogonal family of morphisms). A family $\mathscr{E}$ satisfying the previous condition will be called an orthogonal family of morphisms.

Indeed, from any family of morphisms, we can always get an orthogonal one by adding morphisms like $A \wedge B$ and removing redundant morphisms, i.e. those with a kernel identical to others. For instance, in Example 4, we simply have to add $A \wedge B$ to the family (O, A, B, I) to get an orthogonal design.

4. IRREDUCIBLE CHARACTERS AND DUALITY

By rotating each pair of conjugate irreducible characters of G, Bailey [2] obtained an orthogonal basis of R^G compatible with any of the decompositions of R^G induced by an orthogonal family of morphisms. Since we shall use this kind of basis to specify the nature of confounding, we need some results on finite abelian groups and character theory. These will be expounded in the next two sections. The contents of these two sections are part of the classical theory which can be found in books such as Hall [9], Lang [12, Chapter I], and Ledermann [13, Section 2.4]. However, we give unusual matricial representations of all results, which make clearer the link between pragmatic constructions of design, by methods such as DSIGN, and the abstract theory.

4.1. *Matricial Representations*

It is well known that any abelian finite group G can be represented as a product $Z_{m_1} \times \cdots \times Z_{m_r}$ of cyclic groups (we denote by Z_m the cyclic group of order m). Once such a representation has been chosen, the elements of G can be written as column vectors of dimension r.

If $\theta: G \to H$ is a group morphism from $G = Z_{m_1} \times \cdots \times Z_{m_r}$ into $H = Z_{n_1} \times \cdots \times Z_{n_s}$, we can represent it by an $s \times r$ matrix, denoted by the same letter θ. The successive columns of this matrix are the image by θ of the elements $a_1 = (1,0,\ldots,0)'$, $a_2 = (0,1,\ldots,0)', \ldots,$ $a_r = (0,0,\ldots,1)'$ of G. The image of the vector $u \in G$ by the morphism θ is obtained by matricial multiplication, as usual. However, it must be noticed that the elements of the matrix θ do not belong to the same ring, unless we have $n_1 = \cdots = n_s$, so that the developments of standard algebra books on matricial representations must be modified a little to deal with this special case.

Since a_j, the vector of G having a 1 in position j and 0 in the other positions, is cyclic of order m_j, the jth column θ^j of the matrix θ must verify $m_j \theta^j = 0$. Conversely, if the columns of an $s \times r$ matrix are in H and verify these equalities, it can be shown that it is the matrix of a morphism. Thus we have:

PROPOSITION 3. *The $s \times r$ matrix (θ_{ij}), where $\theta_{ij} \in Z_{n_i}$, defines a morphism from $G = Z_{m_1} \times \cdots \times Z_{m_r}$ in $H = Z_{n_1} \times \cdots \times Z_{n_s}$ if and only if*

$$m_j \theta_{ij} = 0 \ (\text{mod } n_i) \qquad \textit{for every } i \textit{ and } j.$$

As a particular important case, we see that $\theta_{ij} = 0 \pmod{n_i}$ when m_j and n_i are coprime.

EXAMPLE 5. The matrices of morphisms from $G = Z_2 \times Z_4 \times Z_3$ into itself (endomorphisms) are the $2^3 \times 4 \times 3$ matrices of the form

$$\begin{bmatrix} a & c & 0 \\ 2b & d & 0 \\ 0 & 0 & e \end{bmatrix}$$

where $a = 0,1$; $b = 0,1$; $c = 0,1$; $d = 0,1,2,3$; $e = 0,1,2$.

4.2. *Duality*

The *irreducible characters* of an abelian finite group G are the morphisms from G into the multiplicative group $C_\times$ of the field C. The set of such morphisms, denoted $\mathrm{Mor}(G, C_\times)$, together with the multiplication induced by the multiplication of $C_\times$, form a multiplicative group, called the *dual group* of G and denoted by G^*.

A clear exposition of dual groups may be found in Section V.6 of Huppert [10], quoted by Bailey [2]. Chapter 13 of Hall [9] and Section I.11 of Lang [12] are also devoted to duality, but their definition of a character and of the dual is given in a slightly different form.

If $G = Z_{m_1} \times \cdots \times Z_{m_r}$ and p is a common multiple of $m_1, \ldots, m_r$, we have for every character z in G^* and every u in G

$$z(u)^p = z(pu) = z(0) = 1.$$

Hence the image of any character is in the multiplicative group of pth roots of unity in $C_\times$. Since this group is isomorphic to the cyclic additive group Z_p, we could just as well define the dual as the additive group $\mathrm{Mor}(G, Z_p)$ of morphisms from G into Z_p, and this is in fact Lang's definition. The interest of this definition is to allow the use of matricial representations. A morphism $\sigma: G \to Z_p$ can be represented by its matrix $\sigma = (\sigma_1, \ldots, \sigma_r)$.

It follows from Proposition 3 that σ must be of the form $g'D_G$, where g is an element of $Z_{m_1} \times \cdots \times Z_{m_r}$ and D_G is the diagonal matrix with entries $p/m_1, \ldots, p/m_r$ on its diagonal:

$$D_G = \mathrm{diag}\left(\frac{p}{m_1}, \ldots, \frac{p}{m_r}\right).$$

Moreover, it is easy to show that the mapping $g \mapsto g'D_G$ is injective. Hence, it defines an isomorphism from $Z_{m_1} \times \cdots \times Z_{m_r}$ on $\mathrm{Mor}(G, Z_p)$ (see Hall [9, Theorem 13.2.1] for a more detailed proof). So we have:

PROPOSITION 4. *The three groups* $Z_{m_1} \times \cdots \times Z_{m_r}$, $\mathrm{Mor}(G, Z_p)$, *and* $G^* = \mathrm{Mor}(G, C_\times)$ *are isomorphic. If* g *is an element of* $Z_{m_1} \times \cdots \times Z_{m_r}$, *then the associated morphism in* $\mathrm{Mor}(G, Z_p)$ *has matrix* $g'D_G$, *and the corresponding character* z_g *is defined by* $u \mapsto \exp(2\pi i g' D_G u/p)$.

In view of this proposition, we shall represent G^* by the same product $Z_{m_1} \times \cdots \times Z_{m_r}$ as G, this product being denoted by $G^\times$ when it represents the dual. Thus, $g \mapsto z_g$ is an isomorphism from the additive group $G^\times = Z_{m_1} \times \cdots \times Z_{m_r}$ on the multiplicative group $G^* = \mathrm{Mor}(G, C_\times)$. Moreover, when speaking of the dual group, we shall refer either to G^* or to its representation $G^\times$ (the context makes things clear).

To distinguish between elements of G and elements of its dual $G^\times$, we shall use the letter u for the former and g for the latter. Similarly, we shall use v, w for the elements of a group H and h for those of its dual $H^\times$.

If $g = (g_1, \ldots, g_r)$, $u = (u_1, \ldots, u_r)$, we have $g'D_G u = g_1 u_1 p/m_1 + \cdots + g_r u_r p/m_r$, and the equality defining z_g can be rewritten

$$z_g(u) = \exp\left(2\pi i \sum \frac{g_j u_j}{m_j}\right)$$

(this is the form given by Ledermann [13, Theorem 2.4]).

COROLLARY. *There is a canonical isomorphism between* G *and its bidual* $(G^*)^*$, *which to* $u \in G$ *associate the mapping* $z \mapsto z(u)$ *from* G^* *into* $C_\times$.

We now give an illustration of Proposition 4, which shows that when G is a power of Z_2, the dual $G^\times$ is identical with the group introduced by Finney to represent the treatment effects.

EXAMPLE 6. Let us seek the dual $H^\times$ of the group $H = Z_2 \times Z_2 \times Z_2 = Z_2^3$ used as a reference design in Example 1 (Section 2). We can take $p = 2$, which gives $D_H = I_3$. The quantity $h'D_H v \in Z_2$ is then equal to 0 or 1 according to whether the triplets (h_1, h_2, h_3) and (v_1, v_2, v_3) have an even number or an odd number of 1 in common. Let us write each h in $H^\times$ as a formal product $A^{h_1}B^{h_2}C^{h_3}$, and let $O = A^0B^0C^0$, $A = A^1B^0C^0$, $AB = A^1B^1C^0$, and so on. It is then easy to check that z_h is precisely the column on the right of Table 1 labeled by the formal product associated to h. Thus the elements of the group $H^\times$ define the factorial effects in an obvious way. This result will be generalized in Section 4.4 (Proposition 9).

Several definitions and results concerning the duality for finite-dimensional vectorial spaces can be transposed to finite commutative groups. For instance the *dual*, $\theta^*: H^* \to G^*$, of a group morphism $\theta: G \to H$ is defined by:

$$\theta^*(z) = z \circ \theta \qquad \text{for every} \quad z \in H^*.$$

θ^* is a group morphism, and the classical properties

$$\mathrm{id}_G^* = \mathrm{id}_{G^*}, \qquad (\theta_1 \circ \theta_2)^* = \theta_2^* \circ \theta_1^*$$

are trivially verified. Moreover, θ^* is injective whenever θ is surjective, and θ^* surjective when θ is injective (this follows from the last result following Proposition 6). Then, if we identify each group and its bidual by the isomorphism of the preceding corollary, we have

$$(\theta^*)^* = \theta.$$

Matricial Representation of the Dual Morphism. Let $G = Z_{m_1} \times \cdots \times Z_{m_r}$ and $H = Z_{n_1} \times \cdots \times Z_{n_s}$ be two given representations of G and H as products of cyclic groups, and p be a common multiple of $m_1, \ldots, m_r, n_1, \ldots, n_s$. Let then D_G and D_H be defined as previously:

$$D_G = \mathrm{diag}\left(\frac{p}{m_1}, \ldots, \frac{p}{m_r}\right), \qquad D_H = \mathrm{diag}\left(\frac{p}{n_1}, \ldots, \frac{p}{n_s}\right).$$

If, using Proposition 4, we represent H^* by $H^\times = Z_{n_1} \times \cdots \times Z_{n_s}$, G^* by $G^\times = Z_{m_1} \times \cdots \times Z_{m_r}$, then the dual $\theta^*: H^* \to G^*$ of a morphism $\theta: G \to H$ can be represented by a morphism $\theta^\times: H^\times \to G^\times$, which will also be called the dual of θ and whose matrix $\theta^\times = (\theta_{ij}^\times)$ is defined by

$$(\theta^\times h)' D_G = h' D_H \theta \pmod{p} \quad \text{for every } h \text{ in } H^\times = Z_{n_1} \times \cdots \times Z_{n_s}$$

$$\Leftrightarrow \quad D_G \theta^\times = \theta' D_H \pmod{p}.$$

In terms of the components, this becomes

$$\frac{p}{m_i}\theta_{ij}^{\times} = \theta_{ji}\frac{p}{n_j} \pmod{p} \quad \Leftrightarrow \quad \theta_{ij}^{\times} = \frac{m_i}{p}\left(\theta_{ji}\frac{p}{n_j}\right) \pmod{m_i}$$

$$\Leftrightarrow \theta_{ij}^{\times} = \frac{m_i\theta_{ji}}{n_j} \pmod{m_i}.$$

Thus we have

PROPOSITION 5. *Let $\theta = (\theta_{ij})$ be the matrix of a morphism from $G = Z_{m_1} \times \cdots \times Z_{m_r}$ into $H = Z_{n_1} \times \cdots \times Z_{n_s}$. The matrix $\theta^{\times} = (\theta_{ij}^{\times})$ of the dual morphism is defined by the identity $D_G\theta^{\times} = \theta' D_H \pmod{p}$. Thus $\theta^{\times}$ can be calculated from θ by the formula $\theta^{\times} = D_G^{-1}(\theta' D_H)$, which is equivalent to the equalities $\theta_{ij}^{\times} = m_i\theta_{ji}/n_j \pmod{m_i}$.*

From the equality: $\theta^*(z_h) = z_{\theta^{\times}h}$, relating $\theta^{\times}$ to θ^*, it follows that the dual $\theta^{\times}$ can also be defined by

$$z_{\theta^{\times}h}(u) = z_h(\theta u)$$

Orthogonality. $g \in G^{\times}$ and $u \in G$ are said to be orthogonal iff $z_g(u) = 1$ or equivalently iff $g' D_G u = 0 \pmod{p}$. The set of elements in $G^{\times}$ orthogonal to all elements of a set $G_1 \subset G$ is a subgroup of $G^{\times}$, denoted $G_1^{\perp}$ and called the orthogonal of G_1. If G_1 is a subgroup of G, we have the following important relation between the orders $|G|$, $|G_1|$, and $|G_1^{\perp}|$ of the three groups G, G_1 and $G_1^{\perp}$, which immediately follows from Hall [9, Theorem 13.2.2]:

PROPOSITION 6. $|G_1^{\perp}| = |G|/|G_1|$.

The following results, where G_1, G_2 are subgroups of G and θ a group morphism, are also important when constructing designs based on group theory:

$$G_1^{\perp\perp} = G_1,$$

$$G_1 \subset G_2 \quad \Leftrightarrow \quad G_1^{\perp} \supset G_2^{\perp},$$

$$(G_1 \cap G_2)^{\perp} = G_1^{\perp} + G_2^{\perp}, \qquad (G_1 + G_2)^{\perp} = G_1^{\perp} \cap G_2^{\perp},$$

$$(\operatorname{Im}\theta)^{\perp} = \operatorname{Ker}\theta^{\times}, \qquad \operatorname{Im}\theta^{\times} = (\operatorname{Ker}\theta)^{\perp}.$$

4.3. The Association between Morphisms and Subgroups of the Dual

A factor A which is a morphism defined on G is equivalent to the quotient morphism $G \to G/\mathrm{Ker}\, A$, and thus can be deduced from it by simply relabelling the levels. Therefore, when orthogonal families of morphisms on G are sought, we can restrict our attention to the set $\mathcal{Q}$ of quotient morphisms. Alternatively, $\mathcal{Q}$ can be considered to be a set of equivalent classes of factors.

To each $A \in \mathcal{Q}$ we associate, as in Section 3.1, the subspace S_A of R^G. This clearly defines an injective mapping. Remember that when $B \leqslant A$, i.e. when A is nested in B, we have $S_B \subset S_A$, that $S_{A \wedge B} = S_A \cap S_B$ (Corollary 1 of Proposition 1) and finally that the dimension of S_A is equal to the order of $\mathrm{Im}\, A$, which is the number of levels of A on G.

Consider now the mapping $A \mapsto G_A = (\mathrm{Ker}\, A)^{\perp}$ from the set $\mathcal{Q}$ to the set of subgroups of $G^{\times}$. From the properties given at the end of Section 4.2, it follows at once that this mapping is a bijection and moreover that

$$B \leqslant A \quad \Leftrightarrow \quad G_B \subset G_A$$

$$G_{A \wedge B} = G_A \cap G_B,$$

$$G_{A \times B} = G_A + G_B,$$

$$|G_A| = |\mathrm{Im}\, A|.$$

The set of subgroups of $G^{\times}$ is a lattice for the order defined by inclusion, with infimum and supremum given by

$$\inf(G_A, G_B) = G_A \cap G_B, \qquad \sup(G_A, G_B) = G_A + G_B.$$

Consequently $\mathcal{Q}$ is also a lattice, whose infimum and supremum are given by

$$\inf(A, B) = A \wedge B, \qquad \sup(A, B) = A \times B,$$

and we can summarize the previous results in the following form:

PROPOSITION 7. *There is an isomorphism between the lattice $\mathcal{Q}$ of quotient morphisms on G and the lattice of subgroups of the dual group $G^{\times}$, given by $A \mapsto G_A = (\mathrm{Ker}\, A)^{\perp}$. Moreover, there is an injection from $\mathcal{Q}$ into the set, ordered by inclusion, of subspaces of R^G, given by $A \mapsto S_A$. This injection preserves the order and the infimum.*

The number of levels taken by a factor A and consequently the dimension of S_A is equal to the order of the subgroup G_A.

To simplify the notation, we shall sometimes write A instead of G_A, thus using the same letter for a morphism and the associated subgroup of $G^{\times}$. The quotient morphism associated to subgroup A is then the canonical projection $G \to G/(A^{\perp})$.

In practice, to get A from G_A, or conversely G_A from A, one uses the equality $G_A = \text{Im } A^{\times}$ and Proposition 5, as illustrated in the following example:

EXAMPLE 7. Let $G^{\times} = Z_2 \times Z_4 \times Z_3$. Let then G_A be the image of the morphism $A^{\times} : H^{\times} \to G^{\times}$ of the matrix

$$\begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 0 & 1 \end{bmatrix}.$$

Its columns are cyclic of order 2 and 6, and we can therefore take $H^{\times} = Z_2 \times Z_6$. To get the dual matrix of $A^{\times}$ which defines A, we perform the following operations:

multiplication of the columns by 2 and 6 respectively,
division of the rows by 2, 4, and 3 respectively,
replacement of the columns by the remainders of their integer division by 2 and 6 respectively,
transposition.

We thus get the following matrix for A:

$$\begin{bmatrix} 1 & 1 & 0 \\ 3 & 0 & 2 \end{bmatrix}.$$

Corollary 2 of Proposition 2 can now be rewritten as:

COROLLARY. *The family of morphisms associated to a set of subgroups of $G^{\times}$ is orthogonal iff this set is closed under intersection, i.e. contains the intersection of any two of its members.*

The same notation will generally be used for the orthogonal family of morphisms and the corresponding set of subgroups. If $\mathscr{E}$ is such a set, we shall denote by $\bar{A}$ (instead of $\bar{G}_A$) the set of elements in the subgroup A (i.e. G_A) of $\mathscr{E}$ which do not belong to any of the subgroups of the family strictly included in A. The number of elements in $\bar{A}$ is equal to the dimension of the space $\bar{S}_A$ introduced before Proposition 1, since they are calculated by the same recurrent method from the orders of subgroups of $\mathscr{E}$ equal to dimensions of corresponding subspaces.

The isomorphism between subgroups of $G^{\times}$ and morphisms on G can be used to construct designs, as in the following examples.

EXAMPLE 8. Let $G^\times = Z^2 \times Z^2$. Let A, B, C be the cyclic subgroups of order 2 spanned by $(0,1)',(1,0)',(1,1)'$. Adding to these three subgroups $O = \{0\}$ and $I = G^\times$, we get a family of distinct subgroups closed under intersection. The corresponding design is precisely the half fraction given in the example of Section 2.

EXAMPLE 9. Consider in $G^\times = Z_4 \times Z_8$ the four cyclic subgroups of respective orders 4, 8, 4, and 8 spanned by $(1,0)'$, $(0,1)'$, $(1,2)'$, and $(1,1)'$. The second and fourth ones have the subgroup spanned by $(0,4)'$ as intersection. If we add to these subgroups $\{0\}$ and $G^\times$, we obtain a family of subgroups closed under intersection.

The corresponding morphisms can be used to superimpose on a rectangular field with four columns (factor C) and eight rows (factor L) two treatment factors B and A having four and eight levels respectively. For instance, C, L, B, and A are the morphisms associated to subgroups respectively generated by $(1,0)'$, $(0,1)'$, $(1,2)'$, and $(1,1)'$. $A \wedge L$ is then associated to the subgroup generated by $(0,4)'$.

It is convenient to represent these subgroups and associated factors by the graph of Figure 1, where ascending paths indicate inclusions. For each subgroup H in it, we give the elements of $\overline{H}$, whose number gives the dimension of $\bar{S}_H$. This graph shows that the four factors C, L, B, and A are orthogonal except for one degree of freedom of A, confounded with differences between rows.

To explicitly obtain the levels of a factor, say A, on the unit (u_1, u_2) of $Z_4 \times Z_8$, we choose for $A^\times$ the morphism of the matrix $(1,1)'$, from Z_8 into $Z_4 \times Z_8$, whose image generates the associated subgroup G_A. The dual morphism has matrix $(2,1)$. Hence we have

$$A(u_1, u_2) = 2u_1 + u_2 \pmod 8.$$

EXAMPLE 10. As an alternative to the previous design, we can take as subgroups C, L, B, A the images in $G^\times = Z_2^5$ of the morphisms of the matrices

$$\overset{C^\times}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}}, \quad \overset{L^\times}{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}, \quad \overset{B^\times}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}}, \quad \overset{A^\times}{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}.$$

As in the preceding example, the four associated factors are orthogonal, except for 1 d.f. of A, confounded with differences between rows. However,

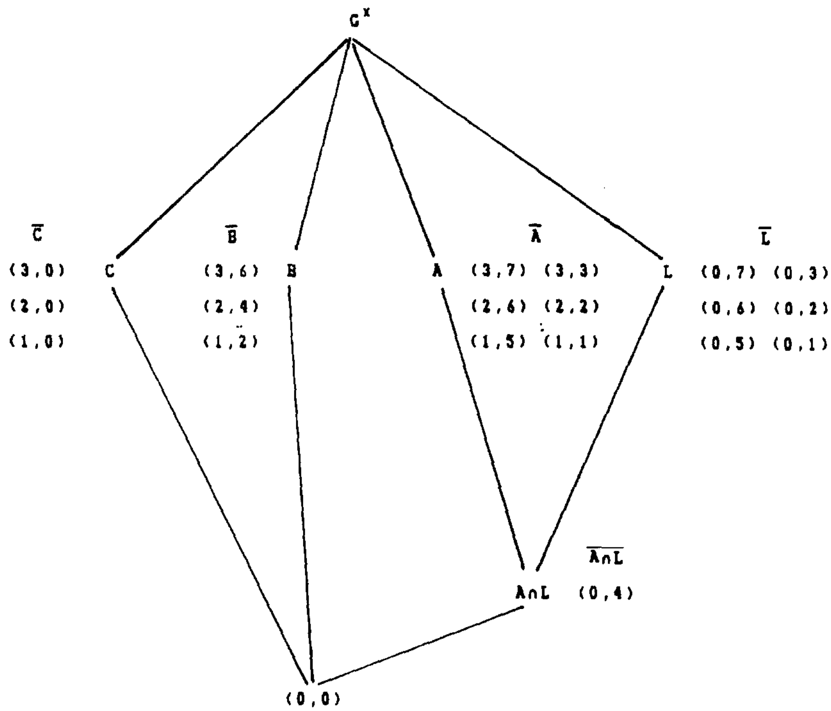

FIG. 1. A graph of some subgroups of $G^{\times} = Z_4 \times Z_8$.

this design is not equivalent to the previous one, in the sense that there is no bijection allowing to identify the two sets of units so that the corresponding factors are equivalent. This is clear, since the inequality $L \wedge A \leqslant C \times B$ holds in Example 9 but not in Example 10.

The remark following Proposition 3 can simplify the problem of searching for orthogonal families of group morphisms by using groups whose order is a power p^k of a prime number p (p-groups). The more useful p-groups for building designs are clearly the powers of the cyclic group Z_p, since they have a maximum number of subgroups. However, the use of other types of p-groups gives a greater flexibility, which has been used for instance by Bailey [3, 4] to build a catalog of resolution III designs.

4.4. *Basis Obtained from Irreducible Characters*

In the following, C^G is equipped with the usual hermitian product $\langle x, y \rangle = x'\bar{y}$, and R^G with the usual scalar product, induced by the hermitian product of C^G. Norms and orthogonality refer to these products.

We shall use the following well known result (Ledermann [13, Section 2.4]).

PROPOSITION 8. *The irreducible characters of G form an orthogonal basis of* C^G. *They all have the same norm* $\sqrt{n}$, *where n is the order of G. Moreover, the characters* z_g *and* z_{-g} *associated to opposite elements of* $G^{\times}$ *are conjugated vectors, i.e. have conjugate coordinates.*

This proposition can easily be deduced from Proposition 4.

We now define real vectors x_g of R^G as follows. If g is of order 2, we have $g = -g$; hence z_g is a real vector and we put

$$x_g = a_g z_g, \qquad \text{with} \quad a_g = +1 \text{ or } -1$$

(both choices are possible).

The elements of $G^{\times}$ of order strictly greater than 2 can be grouped by opposite pairs. For each such pair $g, -g$, we can define, from the orthogonal conjugate vectors z_g, $z_{-g} = \bar{z}_g$, two real orthogonal vectors x_g, x_{-g} spanning the same plane:

$$x_g = a_g z_g + \bar{a}_g \bar{z}_g,$$

$$x_{-g} = -i a_g z_g + i \bar{a}_g \bar{z}_g,$$

where a_g is any complex number of modulus $1/\sqrt{2}$, i.e. of the form $e^{i\xi}/\sqrt{2}$.

The family (x_g) can thus be defined in several ways, according to the choice made for the a_g. Yet, whatever the choice of the a_g is, we have the following important result:

PROPOSITION 9. *The vectors* $x_g/\sqrt{n}$, $g \in G^{\times}$, *constitute an orthonormal basis of* R^G. *If* $\mathcal{E}$ *is a set of subgroups of* $G^{\times}$ *closed under intersection, then the subfamily of vectors* $x_g/\sqrt{n}$ *associated to elements g of* $\bar{A}$, $A \in \mathcal{E}$, *constitute an orthonormal basis of* $\tilde{S}_A$.

A real basis x_g obtained by the process just described will be called a *real consistent basis of* R^G.

Proof. The x_g deduced from the z_g by a unitary transformation are orthogonal in C^G, hence in R^G. If $g \in A$, the map $u \to g'D_G u$ is constant on each coset of $A^{\perp} \subset g^{\perp}$. Since $x_g(u)$ depends on u only through $g'D_G u$, x_g belongs to S_A. If B is a subgroup of A not containing g, then x_g is orthogonal to each vector x_h for $h \in B$, and hence to S_B. The proposition follows. ∎

Thus, a real consistent basis provides a decomposition in individual orthogonal degrees of freedom (d.f.) which is compatible with any decomposition of d.f. induced by an orthogonal family of morphisms.

Let now $\theta : G \to H$ be a group morphism, w a given element of H, and ϕ the map from G to H defined by $\phi(u) = w + \theta(u)$. As in Section 2, ϕ induces a map ψ from C^H, the space of functions from H to C, into C^G: (ψ maps $z \in C^H$ on $z \circ \phi$). Let now z_h be the irreducible character of H associated to $h \in H^\times$. Since z_h is a group morphism, we have

$$\psi(z_h)(u) = z_h \circ \phi(u) = z_h(w + \theta(u)) = z_h(w) z_h(\theta(u)).$$

Using then vector and matrix notation for u, h, θ, we get, using Proposition 5,

$$z_h(\theta(u)) = \exp\left(2\pi i \frac{h' D_H \theta u}{p}\right) = \exp\left(2\pi i h' \theta^{\times\prime} D_G u\right) = z_{\theta^\times h}(u).$$

So we have the following proposition:

PROPOSITION 10. *Let $\phi(u) = w + \theta(u)$, where $\theta : G \to H$ is a group morphism and w an element of H. Let ψ be the linear map from C^H to C^G induced by ϕ. Then ψ maps each irreducible character z_h of H on the irreducible character $z_{\theta^\times h}$ of G, multiplied by the complex number $z_h(w)$ of modulus* 1:

$$\psi(z_h) = z_h(w) z_{\theta^\times h}.$$

Let now (x_h) be a real consistent basis of R^H. From Proposition 10 and the definition of vectors x_h, we get, with $b_h = a_h z_h(w)$,

when $h = -h$,

$$\psi(x_h) = b_h z_{\theta^\times h} \qquad (b_h = +1 \text{ or } -1 \text{ in this case});$$

when $h \neq -h$,

$$\psi(x_h) = b_h z_{\theta^\times h} + \bar{b}_h \bar{z}_{\theta^\times h},$$

$$\psi(x_{-h}) = -i b_h z_{\theta^\times h} + i \bar{b}_h \bar{z}_{\theta^\times h}.$$

We now define two real consistent bases of R^H and R^G: $(x_h)_{h \in H^\times}$ and $(x_g)_{g \in G^\times}$, so that ψ maps x_h on $x_{\theta^\times h}$. We first define the basis of R^G by choosing, for each pair g, $-g$ of opposite elements of $G^\times$, the complex a_g used in the definition of x_g, x_{-g}. The basis of R^H is then defined in a coherent way by choosing, if $\theta^\times h = g$, a coefficient a_h verifying

$$a_h z_h(w) = \begin{cases} a_g & \text{if} \quad h = -h, \\ a_g e^{i\pi/4}/\sqrt{2} & \text{if} \quad h \neq -h, \text{ but } g = -g, \\ a_g & \text{if} \quad h \neq -h \text{ and } g \neq -g. \end{cases}$$

The equality $\psi(x_h) = x_{\theta^\times h}$ can then be easily checked.

We sum up in the following proposition:

PROPOSITION 11. *With the notation of Proposition* 9, *we can get two real consistent bases of R^H and R^G such that ψ maps an element x_h of the basis of R^H on the element $x_{\theta^\times h}$ of the basis of R^G.*

This is the result announced in Section 2. The confounded effects thus correspond to those $h \in H^\times$ having the same image by $\theta^\times$. In short, we shall say that these "effect-indices" h are confounded together and with their common image $\theta^\times h$. We can now proceed to study the dual $\theta^\times$ in relation with the decomposition induced by the factors.

5. NATURE OF CONFOUNDING

EXAMPLE 11. In order to study confounding in the four-factor design of Example 9 (Section 4.2), let us take as reference design the product $H = Z_4 \times Z_8 \times Z_4 \times Z_8$, constituted by the 1024 combinations of the levels of factors C, L, B, and A. The map ϕ from $G = Z_4 \times Z_8$ to H is the product of these four factors. The rows of its matrix are the matrices (1,0), (0,1), (1,1), and (2,1) of the four factors, while the columns of the dual matrix $\psi = \phi^\times$ are the vectors spanning the associated subgroups of $G^\times$: (1,0)′, (0,1)′, (1,2)′, and (1,1)′. Thus

$$\phi = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}, \qquad \psi = \phi^\times = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 1 \end{bmatrix}.$$

It will now be convenient to change our notation: C, L, B, and A will be the projections from H on its successive coordinates. Consequently, the composition mappings $C \circ \phi, L \circ \phi, B \circ \phi, A \circ \phi$ will replace C, L, B, A to denote the factors induced on G.

The subgroups C, L, B, and A of $H^\times$ are spanned respectively by $(1,0,0,0)'$, $(0,1,0,0)'$, $(0,0,1,0)'$, and $(0,0,0,1)'$. Their images $\psi C, \psi L, \psi B, \psi A$, spanned by the successive columns of $\psi = \phi^\times$, are the subgroups associated to the morphisms $C \circ \phi, L \circ \phi, B \circ \phi, A \circ \phi$.

To determine which elements are confounded together, we must now find $N = \operatorname{Ker} \psi = \operatorname{Im} \phi^\perp$. This can be done by the method described by El Mossadeq et al. [7]. It gives as generators of N, whose order is 32, the elements $(1,0,1,6)$ and $(1,1,0,7)$. Once this kernel is known, the effect indices of $H^\times$ confounded with any $g = \psi h$ of $G^\times$ are easily obtained, since they form the coset $h + N$.

For instance, the effect indices confounded with the three effect indices $(1,0)' = \psi((1,0,0,0)')$, $(2,0)'$, and $(3,0)'$ of $\overline{C}$ are given in Table 2, whose first column gives N. To a product factor like $C \times L \times B$ is associated the sum of corresponding subgroups $C + L + B$, containing all elements of the form $(c, l, b, 0)$. If the model contains all products of some of the four factors, $\overline{C + L + B}$ is then the set of elements of the form $(c, l, b, 0)$ with c, l, b different from 0. These elements define the contrasts of the interaction $C \times L \times B$. Thus, the position of the nonzero coordinates of an element of $H^\times$ gives the factors of the associate effect or interaction.

From Table 2, we now see that each of the three elements $(1,0), (2,0), (3,0)$ of the subset $\overline{\psi C}$ of $G^\times$ is confounded with:

1 element associated to interaction $B \times A$,
2 elements associated to the interaction $L \times A$,
1 element associated to the interaction $L \times B$,
4 elements associated to the interaction $L \times B \times A$,
2 elements associated to the interaction $C \times B \times A$,
5 elements associated to the interaction $C \times L \times A$,
2 elements associated to the interaction $C \times L \times B$,
14 elements associated to the interaction $C \times L \times B \times A$.

It is not by chance that each element of the subset $\overline{\psi C}$ is confounded with the same number of elements associated to a given interaction. This is a particular case of a more general result that we shall now state.

Let $\phi: G \to H$ be a group morphism, $\psi = \phi^\times$ its dual. Let $\mathscr{F}$ denote an orthogonal family of morphisms on H as well as the corresponding family of subgroups of $H^\times$. Each morphism A of $\mathscr{F}$ induces a morphism $A \circ \phi$ on G, whose associated subgroup is the image ψA of A by ψ. From the set of such

TABLE 2

EFFECT INDICES CONFOUNDED[a]

Confounded with:															
(0,0)				(1,0)				(2,0)				(3,0)			
0	0	0	0	0	0	3	2	0	0	2	4	0	0	1	6
0	4	0	4	0	3	0	5	0	2	0	6	0	1	0	7
0	1	3	1	0	7	0	1	0	6	0	2	0	5	0	3
0	2	2	2	0	6	1	0	0	4	2	0	0	2	3	0
0	3	1	3	0	1	2	3	0	1	1	5	0	3	2	1
0	5	3	5	0	2	1	4	0	3	3	7	0	4	1	2
0	6	2	6	0	4	3	6	0	5	1	1	0	6	3	4
0	7	1	7	0	5	2	7	0	7	3	3	0	7	2	5
1	0	1	6	1	0	0	0	2	0	0	0	3	0	0	0
2	0	2	4	2	0	1	6	3	0	1	6	1	0	2	4
3	0	3	2	3	0	2	4	1	0	3	2	2	0	3	2
1	1	0	7	2	1	0	7	3	1	0	7	1	2	0	6
2	2	0	6	3	2	0	6	1	3	0	5	2	3	0	5
3	3	0	5	1	4	0	4	2	4	0	4	3	4	0	4
1	5	0	3	2	5	0	3	3	5	0	3	1	6	0	2
2	6	0	2	3	6	0	2	1	7	0	1	2	7	0	1
3	7	0	1	2	2	3	0	3	2	3	0	1	4	2	0
1	2	3	0	3	4	2	0	1	6	1	0	2	6	1	0
2	4	2	0	3	1	1	5	1	1	2	3	1	1	1	5
3	6	1	0	1	1	3	1	2	1	3	1	2	1	2	3
2	1	1	5	1	2	2	2	1	2	1	4	3	1	3	1
3	1	2	3	1	3	1	3	2	2	2	2	2	2	1	4
3	2	1	4	2	3	2	1	2	3	1	3	3	2	2	2
1	3	2	1	3	3	3	7	3	3	2	1	3	3	1	3
2	3	3	7	2	4	1	2	3	4	1	2	1	3	3	7
1	4	1	2	3	5	1	1	1	4	3	6	2	4	3	6
3	4	3	6	1	5	3	5	1	5	2	7	1	5	1	1
2	5	1	1	1	6	2	6	2	5	3	5	2	5	2	7
3	5	2	7	2	6	3	4	2	6	2	6	3	5	3	5
1	6	3	4	1	7	1	7	3	6	3	4	3	6	2	6
1	7	2	5	2	7	2	5	2	7	1	7	3	7	1	7
2	7	3	3	3	7	3	3	3	7	2	5	1	7	3	3

[a] The elements of each column are grouped according to the position of 0 in them.

morphisms we can get, by the method described at the end of Section 3.3, an orthogonal family $\mathscr{E}$ of morphisms, which will be called the orthogonal family of morphisms *induced by* $\mathscr{F}$ on G. $\mathscr{E}$ will also refer to the associate set of subgroups of $G^{\times}$, which is the closure for the intersection of the set of subgroups ψA, $A \in \mathscr{F}$. With this notation, we have:

THEOREM 1. *If $A \in \mathscr{E}$ and $B \in \mathscr{F}$, the number of elements of $\psi^{-1}(g) \cap B$ for $g \in \bar{A}$ does not depend on g, and neither does the number of elements of $\psi^{-1}(g) \cap \bar{B}$ which can be calculated by recurrence.*

The number of elements of $\psi^{-1}(g) \cap \bar{B}$ for $g \in \bar{A}$ will be called the *degree of confounding of $\bar{B}$ with $\bar{A}$*, and we shall say that *$\bar{B}$ is confounded with $\bar{A}$* if this degree is at least 1 or equivalently, if $(\psi^{-1}\bar{A}) \cap \bar{B}$ is not empty.

Proof. If B belongs to $\mathscr{F}$, then ψB belongs to $\mathscr{E}$. Its intersection with $\bar{A}$ is therefore empty, or equal to $\bar{A}$. If it is empty, the number of elements of $\psi^{-1}(g) \cap B$ for $g \in \bar{A}$ is zero, and thus independent of g. If it is equal to $\bar{A}$, the number of elements of $\psi^{-1}(g) \cap B$ is equal to the order of the kernel of the map ψ_B, the restriction of ψ to B; hence it is also independent of g. Now, since B is the disjoint union of subsets $\bar{C}$, where C is a subgroup of $\mathscr{F}$ included in B, the number of elements of $\psi^{-1}(g) \cap \bar{B}$ can be deduced recurrently from the number of elements of sets $\psi^{-1}(g) \cap C$. Therefore, it is also independent of g. ∎

EXAMPLE 12. Let us go back to Example 9 (Section 4.2). Let $\mathscr{F}$ be the set of morphisms on the reference design $H = Z_4 \times Z_8 \times Z_4 \times Z_8$ constituted by the constant, the four factors C, L, B, and A, and all the products of any of them. As a set of subgroups of $H^{\times}$, $\mathscr{F}$ contains, besides C, L, B, and A, the sums of any of these four subgroups and the subgroup $\{0\}$ associated to the constant. As noted previously, the images by ψ of C, L, B, A are the subgroups of $G^{\times}$ spanned by $(1,0)'$, $(0,1)'$, $(1,2)'$, and $(1,1)'$ respectively. The order $|\psi A + \psi L|$ of the image by ψ of the sum $A + L$ is obtained by using the parallelogram rule, which gives

$$\frac{|\psi A + \psi L|}{|\psi A|} = \frac{|\psi L|}{|\psi A \cap \psi L|}.$$

Hence, $|\psi A + \psi L| = 8 \times 8/2 = 32$, and the image of $A + L$ by ψ is the whole group $G^{\times}$. Similarly, it can be shown that the images of the sums of two, three, or four of the subgroups C, L, B, A are equal to $G^{\times}$, except that of $C + B$, which has only 16 elements, given in Table 3.

TABLE 3
ELEMENTS OF THE SUM $\psi C + \psi B$

ψC	$\psi B = 0\,0$	1 2	2 4	3 6
0 0	0 0	1 2	2 4	3 6
1 0	1 0	2 2	3 4	0 6
2 0	2 0	3 2	0 4	1 6
3 0	3 0	0 2	1 4	2 6

Besides the images $\{0\}$, ψC, ψL, ψB, ψA, $\psi C + \psi B$, and $G^{\times}$ of subgroups of $\mathscr{F}$, we must add to $\mathscr{E}$, in order to get a set closed under intersection, the subgroup $\psi A \cap \psi L$ generated by $(0,4)$, and the intersections of $\psi C + \psi B$ with ψA and ψL, generated respectively by $(2,2)$ and $(0,2)$.

Table 4 gives the degrees of confusion of each subset of the partition induced by $\mathscr{F}$ on $H^{\times}$ with each subset of the partition induced by $\mathscr{E}$ on $G^{\times}$. Table 4 is obtained by recurrence, starting on the left and proceeding to the right. By instance, suppose we want to obtain the degree of confounding of $\overline{F}$ with $\overline{E}$, where $F = L + B + A$ and $E = (\psi B + \psi C) \cap \psi A$. Denoting by $|D|$ the number of elements of a set D, we have if $g \in \overline{E}$

$$\psi F = G^{\times} \supset E;$$

hence $g \in \psi F$, and

$$|\psi^{-1}(g) \cap F| = |\psi^{-1}(0) \cap F| = \frac{|F|}{|\psi F|} = \frac{8\times 4\times 8}{4\times 8} = 8.$$

Moreover, F is the disjoint union of the sets: $\{0\}$, $\overline{L}$, $\overline{B}$, $\overline{A}$, $\overline{L+B}$, $\overline{L+A}$, $\overline{B+A}$, and $\overline{F}$. Therefore, $\psi^{-1}(g) \cap F$ is the disjoint union of intersections of these sets with $\psi^{-1}(g)$. The sum of the numbers of elements of these last intersections is 8. These numbers have been previously calculated and appear on the line corresponding to $\overline{E}$ in Table 4, except for $|\psi^{-1}(g) \cap \overline{F}|$, which can therefore be obtained by difference:

$$\begin{aligned} |\psi^{-1}(g) \cap \overline{F}| &= |\psi^{-1}(g) \cap F| - \left(|\psi^{-1}(g) \cap \{0\}| + |\psi^{-1}(g) \cap \overline{L}| + \cdots\right) \\ &= 8 - 3 = 5. \end{aligned}$$

The author is grateful to Mr. Collombier and to the referees for helpful suggestions.

TABLE 4

DEGREES OF CONFOUNDING IN EXAMPLE 12

		1	3	7	3	7	21	9	21	21	49	21	147	63	147	63	441
		O	$\overline{C}$	$\overline{L}$	$\overline{B}$	$\overline{A}$	$\overline{C+L}$	$\overline{C+B}$	$\overline{C+A}$	$\overline{L+B}$	$\overline{L+A}$	$\overline{B+A}$	$\overline{L+B+A}$	$\overline{C+B+A}$	$\overline{C+L+A}$	$\overline{C+L+B}$	$\overline{H^{\times}}$
1	$\overline{\{0\}}$	1									1		6	3	6	3	12
1	$\overline{\psi A \cap \psi L}$			1		1		1					6	2	6	2	13
2	$\overline{(\psi B+\psi C)\cap\psi A}$					1	1	1		1	1		5	2	5	1	14
2	$\overline{(\psi B+\psi C)\cap\psi L}$			1				1	1		1	1	5	1	5	2	14
3	$\overline{\psi C}$		1							1	2	1	4	2	5	2	14
4	$\overline{\psi L}$			1					1		1	1	5	2	5	3	13
3	$\overline{\psi B}$				1		1		1		2		5	2	4	2	14
4	$\overline{\psi A}$					1	1			1	1		5	3	5	2	13
4	$\overline{\psi B+\psi C}$						1	1	1	1	2	1	4	1	4	1	15
8	$\overline{G^{\times}}$						1		1	1	2	1	4	2	4	2	14

REFERENCES

1 R. A. Bailey, Patterns of confounding in factorial designs, *Biometrika* 64:597–603 (1977).

2 R. A. Bailey, Dual abelian groups in the design of experiments, in *Algebraic Structures and Applications* (P. Schultz et al., Eds.), Lecture Notes in Pure and Applied Mathematics, 74, Dekker, New York, 1981, pp. 45–54.

3 R. A. Bailey, Fractional factorial design with resolution III. Unpublished manuscript, Open University, 1983, (revised as [4]).

4 R. A. Bailey, Factorial design and Abelian groups. (Paper based on [3]), *Linear Algebra Appl.*, 70:349–368 (1985).

5 R. C. Bose, Mathematical theory of the symmetrical factorial design, *Sankhyā* 8:107–166 (1947).

6 A. M. Dean and J. A. John, Single replicate factorial experiments in generalized cyclic designs: II. Asymmetrical arrangements, *J. Roy. Statist. Soc. Ser. B* 37, 72–76 (1975).

7 A. El Mossadeq, A. Kobilinsky, and D. Collombier, Construction d'orthogonaux dans les groupes abeliens finis et confusion d'effets dans les plans factoriels, *Linear Algebra Appl.*, 70:303–320 (1985).

8 D. J. Finney, The fractional replication of factorial arrangements, *Ann. Eugen.* 12:291–301 (1945).

9 M. Hall, *The Theory of Groups*, Macmillan, 1959, 434 pp.

10 B. Huppert, *Endliche Gruppen I*, Springer, Berlin, 1967, Section V.6.

11 J. A. John, Generalized cyclic designs in factorial experiments, *Biometrika* 60:55–63 (1973).

12 S. Lang, *Algebra*, Addison-Wesley, 1965, 508 pp.

13 W. Ledermann, *Introduction to Group Characters*, Cambridge U.P., 174 pp.

14 H. D. Patterson, Generation of factorial designs, *J. Roy. Statist. Soc. Ser. B* 38:175–179 (1976).

15 T. Tjur, Analysis of variance models in orthogonal designs (with discussion), *Internat. Statist. Rev.* 52:33–81 (1984).

16 D. White and R. A. Hultquist, Construction of confounding plans for mixed factorial designs, *Ann. Math. Statist.* 36:1256–1271 (1965).

Received 18 April 1984; revised 6 June 1985