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Matrix factorizations and elliptic fibrations

Harun Omer

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Abstract

I use matrix factorizations to describe branes at simple singularities of elliptic fibrations. Each node of the corresponding Dynkin diagrams of the ADE-type singularities is associated with one indecomposable matrix factorization which can be deformed into one or more factorizations of lower rank. Branes with internal fluxes arise naturally as bound states of the indecomposable factorizations. Describing branes in such a way avoids the need to resolve singularities. This paper looks at gauge group breaking from E_8 fibers down to $SU(5)$ fibers due to the relevance of such fibrations for local F-theory GUT models. A purpose of this paper is to understand how the deformations of the singularity are understood in terms of its matrix factorizations. By systematically factorizing the elliptic fiber equation, this paper discusses geometries which are relevant for building semi-realistic local models. In the process it becomes evident that breaking patterns which are identical at the level of the Kodaira type of the fibers can be inequivalent at the level of matrix factorizations. Therefore the matrix factorization picture supplements information which the conventional less detailed descriptions lack.
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1. Introduction

Historically, the idea of a category-theoretical description of string theory even predates the discovery of D-branes and first appeared in the context of homological mirror symmetry, where Kontsevich conjectured that the Fukaya category of a Calabi–Yau is equivalent to the derived category of coherent sheaves of the mirror Calabi–Yau [1]. Later it was suggested that D-branes can be described as sheaves [2] and brane/anti-brane systems were described as derived categories [3]. It was in the context of topological boundary Landau–Ginzburg models that D-branes

E-mail address: harunomer@gmail.com.

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have been explicitly described as and written out as matrix factorizations, using a result of Eisenbud [4]. A selection of papers with particular emphasis on review articles to introduce the subject is found in the references [5–14]. The category-theoretical description is powerful and its application extends beyond Landau–Ginzburg (LG) models. A strength of the description is its ability to deal with branes at singularities naturally without the need to resolve the geometry. This comes particularly handy in F-theory where 7-branes are located at geometric singularities arising from elliptic fibrations [15–28]. In particular it has been shown that models with phenomenologically viable features can be built only around a single singularity. This singularity can be thought of as the location of a 7-brane with GUT gauge group, whose singularity type is further enhanced as it intersects other branes at the location of the singularity. In such local F-theory models it is sufficient to focus on the vicinity of the singularity in order to derive physical properties of the theory. Numerous papers have been published in this direction in the last few years.

It is to date an open problem to fully characterize the (p, q)-branes arising in F-theory, the strong coupling limit of type IIB theory. F-theory treatments of 7-branes are usually limited to describing the Kodaira fiber type and the behavior under monodromies. This paper does not fully remedy the situation, however the formulation in terms of matrix factorizations does encode additional information as it is quite generic. The category of matrix factorizations is isomorphic to the derived category of coherent sheaves. Mathematically, a sheaf tracks local data on the open sets of a topological space. In physics a sheaf is often thought of as a brane with a gauge bundle. Sheaves together with morphisms form a category. Morphisms are defined to contain an identity element and satisfy the associativity property $h \circ (g \circ f) = (h \circ g) \circ f$, which reflects the recombination property of open strings. Consequently matrix factorizations capture rather generic topological properties of branes and strings, avoiding specialized or restrictive assumptions. To the extent that in F-theory any of these properties are violated or are insufficient, the present approach also finds its limits. Nevertheless, the description encodes more information than just the fiber type. Collinucci and Savelli state "We will argue that the complete way to specify an F-theory compactification is to define a geometry, and a corresponding choice of matrix factorization" [29]. In their work [30,29], they sought to rederive a theoretical foundation for applying matrix factorizations to F-theory. The authors use simple toy models of matrix factorizations. One goal of this paper is to make available complete sets of indecomposable factorizations for any given fibration. Neither in the physics nor in the mathematics literature exist systematic parameter-dependent factorizations of the fiber equations. Individual factorizations for singularities such as E_8 and E_7 can look rather similar, but that holds only after a suitable similarity transformation and no efforts have so far been made to write all factorizations in a consistent manner. Furthermore, as will be seen, there can be more than one way to deform a set of factorizations for a given ADE group into a set of factorization of a lower rank group. This should be thought of as reflecting the way how the deeper geometric structure of the singularity is deformed.

For a formal discussion to the connection to F-theory to the extent that it is known, see the work of Collinucci and Savelli. Readers more familiar with or more interested on the Landau–Ginzburg (LG) models can continue to think in terms of them. LG models are valued in a weighted projective target space defined by some equation $W = 0$ where W is the superpotential of the LG model. LG models with singularities, namely the ADE minimal models, have been the subject of extensive research and LG realizations of matrix factorizations with a torus as target space have also been discussed at length, for instance in [31–34]. The torus can be parameter-dependent and degenerate at certain regions of the parameter space, without affecting the description as matrix factorization. In principle, the branes defined by the factorizations of W

are not limited to an LG interpretation. According to the theorem of Eisenbud mentioned earlier, they describe certain types of sheaves. The difficulty has been to embed them into theories other than LG models. From the LG perspective, the LG models of the factorizations appearing in this paper could be regarded as a type of weak-coupling limit of an F-theory compactification. The elliptic fibration can be written in various ways, the simplest of which is the Weierstrass equation $y^2 = x^3 + f(z)x + g(z)$ with f and g appropriate sections. This Weierstrass equation (or any alternative description of an elliptic fibration) is the equation of a torus in weighted projective space, so that one can construct a matrix factorization for this torus. We can define,

$$W(x, y, z) = -y^2 + x^3 + f(z)x + g(z),$$

where $W(x, y, z) = 0$ defines our parameter-dependent torus. The same can be done with other equations of elliptic curves. The matrix factorizations of $W(x, y, z)$ will describe the branes. Matrix factorizations can be parameter-dependent and require no special treatment at points where the parameters take such values that the torus degenerates.

A goal of this paper is to move beyond toy models and work with the types of branes which actually appear in phenomenologically viable models. Typically such models start with one GUT brane with gauge group $SU(5)$ or $SO(10)$ which intersects other branes. At the brane intersections, the rank of the gauge group enhances, giving rise to larger symmetry groups. At multiple intersections, enhancements up to E_8 are possible. The GUT group can be further broken down by internal fluxes. Particularly relevant for local F-theory models are the gauge groups $SU(5)$, $SU(6)$, $SO(10)$, $SO(12)$, E_6, E_7 and E_8. All of these symmetry groups have a geometric description as simple singularities. Simple singularities are classified by the following equations:

$$f(x, y, z) = \begin{cases} -y^2 + x^2 + z^{n+1}, & A_n \text{ with } n \geq 1, \\ -y^2 + x^2 z + z^{n-1}, & D_n \text{ with } n \geq 4, \\ -y^2 + x^3 + z^4, & E_6, \\ -y^2 + x^3 + xz^3, & E_7, \\ -y^2 + x^3 + z^5, & E_8\,. \end{cases} \tag{1}$$

In the main part of this paper the starting point will be the maximal gauge group E_8 which is then gradually broken down to smaller subgroups. As the reader will see, there are generally several inequivalent ways to deform the factorizations, reflecting the additional information content of the description.

2. Elliptic fibrations

2.1. Equations for elliptic curves

Elliptic curves can be described by different equations in weighted projective space, such as by a cubic, a quartic or a sextic equation [35]:

$$\begin{aligned} &x^3 + y^3 + z^3 - a\,xyz \in \mathbb{P}_2^{1,1,1} \\ &x^4 + y^4 + z^2 - a\,xyz \in \mathbb{P}_2^{1,1,2} \\ &x^6 + y^3 + z^2 - a\,xyz \in \mathbb{P}_2^{1,2,3} \end{aligned} \tag{2}$$

An elliptic curve is a nonsingular curve of genus 1 with a rational point. Although every elliptic curve is topologically equivalent to a torus, different elliptic curves will in general not be isomorphic as Riemann surfaces. Isomorphic curves over a field K have the same j-invariant. Conversely, two curves with the same j-invariant are isomorphic over the closure $\bar{K}$. As a consequence of these equivalence theorems, it would be sufficient to consider only one type of equation to describe an elliptic curve. The caveat is that it is the singular fibers of the elliptic fibrations which play a key role in physics, notably F-theory. Elliptic fibrations which are equivalent as long as they are smooth generically give rise to different types of singularities where the fiber degenerates. It is therefore not sufficient to consider only the standard sextic Weierstrass equation as is done in the vast majority of all research papers in the field. To more efficiently work with the quartic equation it would be helpful to have a birational transformation between the quartic equation and the standard Weierstrass equation. The so-called Tate form is one way to write an elliptic fibration:

$$y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6. \tag{3}$$

Each term in the equation can be thought of as being in a graded ring where x has weight 2, y weight 3 and a_i weight i. By completing the square on the left hand side the xy and y terms can be eliminated after appropriate variable substitution and one obtains,

$$y^2 = x^3 + b_2 x^2 + b_4 x + b_6, \tag{4}$$

with,

$$b_2 = \frac{1}{4}a_1^2 + a_2$$
$$b_4 = \frac{1}{2}a_1 a_3 + a_4$$
$$b_6 = \frac{1}{4}a_3^2 + a_6$$

The advantage of this form for our purposes is its similarity with the equations of simple singularities. By completing the cube in x the x^2-term is eliminated and one obtains the standard Weierstrass equation,

$$y^2 = x^3 + fx + g. \tag{5}$$

These equations in affine form use a local coordinate chart where a point at infinity (the point $z = 0$) is left out. This selection of a point defines a global section of the fibration. Alternatively to the Weierstrass equation, an elliptic curve can also be described by a quartic equation. The quartic equation with general coefficients reads in homogeneous coordinates:

$$v^2 = c_0 u^4 + c_1 u^3 z + c_2 u^2 z^2 + c_3 u z^3 + c_4 z^4. \tag{6}$$

By shifting and rescaling coordinates, we can reduce the number of coefficients by two:

$$v^2 = u^4 + c_2 u^2 z^2 + c_3 u z^3 + c_4 z^4. \tag{7}$$

In analogy to the Weierstrass form, we used this to set the coefficient of the highest order term in u to unity and of the second highest-order term to zero. In affine coordinates the general quartic equation simplifies to,

$$v^2 = c_0 u^4 + c_1 u^3 + c_2 u^2 + c_3 u + c_4. \tag{8}$$

2.2. A birational transformation between the quartic and the Tate form

An elliptic curve is an algebraic curve of genus 1 with a rational point on the curve. One point the curve is the point at infinity with projective coordinates $(0:1:0)$. This rational point with coordinates $(u, v) = (p, q)$ can with the help of a coordinate shift u to $u + p$ be brought into the form $(u, v) = (0, q)$. Since the point must solve Eq. (8) we find $c_4 = q^2$. To show birational equivalence to the Weierstrass form, we distinguish between the cases $q = 0$ and $q \neq 0$.

The quartic in Eq. (8) with $c_4 = q^2 \neq 0$ is birationally equivalent to the cubic $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$ under the transformation [36],

$$x = \frac{2q(v+q) + c_3 u}{u^2}$$
$$y = \frac{2q\left[2q(v+q) + c_3 u + (c_2 - \frac{c_3^2}{4q})u^2\right]}{u^3} \tag{9}$$

and the identification,

$$a_1 = \frac{c_3}{q}, \quad a_2 = c_2 - \frac{c_3^2}{4q^2}, \quad a_3 = 2c_1q, \quad a_4 = -4c_0q^2, \quad a_6 = a_4a_2. \tag{10}$$

The equivalence between (a_1, a_2, a_3, a_4) and (c_0, c_1, c_2, c_3) is one-to-one. In addition we have the constraint

$$a_6 = a_4a_2 \tag{11}$$

on the coefficients of the Weierstrass form as a result of the constraint placed on the quartic curve. The point $(u, v) = (0, q)$ on the curve corresponds to the point at infinity $(x, y) = \infty$ in projective space and the point $(u, v) = (0, -q)$ corresponds to $(x, y) = (-a_2, a_1a_2 - a_3)$. The inverse transform is given by,

$$u = \frac{2q\left[x + c_2 - \frac{c_3^2}{4q^2}\right]}{y}$$
$$v = -q + \frac{-c_3 u + xu^2}{2q}. \tag{12}$$

The above statements are proven by direct calculation, where the equation is multiplied by $2y/u^3$ before the substitution. The argument does not cover the case when the rational point on the curve is at infinity $(u, v) = \infty$. That case implies either $c_0 = 0$ in which case the quartic reduces to a generic cubic, or we have $c_0 = q^2 \neq 0$, which allows us to apply the transformation $u \to 1/u$ and $v \to v/u^2$ to again obtain an equation of the form in Eq (8).

When $q = 0$ the transformation can be written as,

$$u = \frac{c_3}{x}$$
$$v = \frac{c_3\left[2y + a_1x + a_3\right]}{2x^2} \tag{13}$$

where,

$$c_0 = \frac{\frac{a_3^2}{4} + a_6}{c_3^2}, \; c_1 = \frac{a_1 a_3 + 2a_4}{2c_3}, \; c_2 = \frac{a_1^2}{4} + a_2 \tag{14}$$

The quotients in the variable identification are always defined since $q = 0$ requires $c_3 \neq 0$, otherwise the curve would be singular at the origin $(0, 0)$.

It has been emphasized that the equivalence between birationally equivalent elliptic curves only holds for smooth curves. With the help of Tate's algorithm, the singularities of elliptic curves described by the Tate form has been classified according to the vanishing orders of a_i in the Tate form. Eq. (14) is not a valid equation to map the vanishing orders of singularities since the rational transformation between elliptic curves does not necessarily preserve the type of singularity. In [37] the Tate algorithm has already been applied to the quartic curve in and a table of singularities was created. In that reference, the general equation is written as,

$$c_0 u^4 + c_1 u^3 z + c_2 u^2 z^2 + c_3 u z^3 + c_4 z^4 = a v^2 + b_0 z^2 v + b_1 v u z + b_2 u^2 v \tag{15}$$

The authors argue that a rank one Mordell–Weil group implies $a = 1$, $c_4 = 0$ and $b_0 \neq 0$. By applying these constraints and shifting and scaling the v coordinate, b_1 and b_2 can be absorbed into new coefficients c_i,

$$v^2 = c_0 u^4 + c_1 u^3 z + c_2 u^2 z^2 + c_3 u z^3 + \frac{1}{4} b_0^2 z^4 \tag{16}$$

Going to affine coordinate by setting $z = 1$ we have reproduced Eq. (8) with $c_4 = q^2$ and $q = \frac{b_0}{2} \neq 0$. One needs to be careful in realizing that Mordell–Weil rank 1 does not necessarily imply that eq. (11) is always satisfied, only that the elliptic curve can always be brought into a form such that this equation holds. A birational transformation also exists when c_4 is not a perfect square, but it is much more complicated and I am not writing it down here.

2.3. Gauge group breaking of the sextic curve and quartic curves

In principle one could start with the equation for an elliptic fibration, write out all sections a_i as polynomial expansions and attempt to factorize the equation into a product of two matrices with polynomial entries. To reduce the complexity of the endeavor, we restrict to the main gauge groups of physical interest. The symmetry type determines the vanishing orders of a_i of the Tate form [38]:

$$\begin{aligned}
SU(5)&: \quad -y^2 + x^3 - a_{1,0}xy + a_{2,1}x^2 z - a_{3,2}yz^2 + a_{4,3}xz^3 + a_{6,5}z^5 \\
SO(10)&: \quad -y^2 + x^3 - a_{1,1}xyz + a_{2,1}x^2 z - a_{3,2}yz^2 + a_{4,3}xz^3 + a_{6,5}z^5 \\
E_6&: \quad -y^2 + x^3 - a_{1,1}xyz + a_{2,2}x^2 z^2 - a_{3,2}yz^2 + a_{4,3}xz^3 + a_{6,5}z^5 \\
E_7&: \quad -y^2 + x^3 - a_{1,1}xyz + a_{2,2}x^2 z^2 - a_{3,3}yz^3 + a_{4,3}xz^3 + a_{6,5}z^5 \\
E_8&: \quad -y^2 + x^3 - a_{1,1}xyz + a_{2,2}x^2 z^2 - a_{3,3}yz^3 + a_{4,4}xz^4 + a_{6,5}z^5
\end{aligned}$$

Here the sections a_i have been expanded into $a_i = \sum_j a_{i,j} z^j$. We will want to deal with all the symmetry groups from $SU(5)$ up to E_8 at and therefore wish to preserve all coefficients $a_{i,j}$ which are non-vanishing for any of the groups of interest:

$$\begin{aligned}
W(x, y, z) := & -y^2 + x^3 - a_{1,0}xy - a_{1,1}xyz + a_{2,1}x^2 z + a_{2,2}x^2 z^2 \\
& - a_{3,2}yz^2 - a_{3,3}yz^3 + a_{4,3}xz^3 + a_{4,4}xz^4 + a_{6,5}z^5.
\end{aligned}$$

We bring this into the b-form of eq. (4) with the transformation,

$$y \mapsto y - \frac{1}{2}\left(a_{1,0} + a_{1,1}z\right)x - \frac{1}{2}\left(a_{3,2}z^2 + a_{3,3}z^3\right)$$

and obtain,

$$\boxed{W := -y^2 + f_1x^3 + f_2x^2z + f_3^2x^2 + 2f_3g_3xz^2 + g_1z^5 + g_2xz^3 + g_3^2z^4} \tag{17}$$

where,

$$\begin{aligned} f_1 &= 1 & g_1 &= \frac{1}{2}a_{3,2}a_{3,3} + a_{6,5} \\ f_2 &= \frac{1}{2}a_{1,0}a_{1,1} + a_{2,1} & g_2 &= \frac{1}{2}a_{1,1}a_{3,2} + \frac{1}{2}a_{1,0}a_{3,3} + a_{4,3} \\ f_3 &= \frac{1}{2}a_{1,0} & g_3 &= \frac{1}{2}a_{3,2} \end{aligned}$$

In the expression for W, three higher order terms have been suppressed:

$$W_{sup} = \left(\frac{1}{4}a_{1,1}^2 + a_{2,2}\right)x^2z^2 + \left(\frac{1}{2}a_{1,1}a_{3,3} + a_{4,4}\right)xz^4 + \frac{1}{4}a_{3,3}^2z^6$$

These terms do not affect the singularity type and can therefore be ignored. After all, Eq. (17) is the more fundamental equation with respect to the singularity type and the derivation from the Kodaira classification only serves to relate it to elliptic fibrations.

Instead of beginning with the Tate form, we could perform the analogous transformation with the full quartic equation in (15). In that case one obtains [37]:

$$\begin{aligned} &SU(5): && -y^2 - b_{0,0}x^2y - b_{1,0}xy \ - b_{2,2}yz^2 + c_{0,0}z^5 + c_{1,3}xz^3 + c_{2,1}x^2z \ + c_{3,0}x^3 \\ &SO(10): && -y^2 - b_{0,0}x^2y - b_{1,1}xyz - b_{2,2}yz^2 + c_{0,0}z^5 + c_{1,3}xz^3 + c_{2,1}x^2z \ + c_{3,0}x^3 \\ &E_6: && -y^2 - b_{0,0}x^2y - b_{1,1}xyz - b_{2,2}yz^2 + c_{0,0}z^5 + c_{1,3}xz^3 + c_{2,2}x^2z^2 + c_{3,0}x^3 \\ &E_7: && -y^2 - b_{0,0}x^2y - b_{1,1}xyz - b_{2,3}yz^3 + c_{0,0}z^5 + c_{1,3}xz^3 + c_{2,2}x^2z^2 + c_{3,0}x^3 \\ &E_8: && -y^2 - b_{0,0}x^2y - b_{1,1}xyz - b_{2,3}yz^3 + c_{0,0}z^5 + c_{1,4}xz^4 + c_{2,2}x^2z^2 + c_{3,0}x^3 \end{aligned}$$

The sections b_i and c_i have been expanded in the obvious manner. The equation with all non-vanishing coefficients preserved reads,

$$\begin{aligned} W(x,y,z) := -y^2 - b_{0,0}x^2y - b_{1,0}xy - b_{1,1}xyz - b_{2,2}yz^2 - b_{2,3}yz^3 \\ + c_{0,5}z^5 + c_{1,3}xz^3 + c_{1,4}xz^4 + c_{2,1}x^2z + c_{2,2}x^2z^2 + c_{3,0}x^3 \end{aligned}$$

After completing the square the resulting equations again has the structure of eq. (17). This time the coefficients mapped as follows:

$$\begin{aligned} f_1 &= \tfrac{1}{2}b_{0,0}b_{1,0} + c_{3,0} & g_1 &= \tfrac{1}{2}b_{2,3}b_{2,2} + c_{0,5} \\ f_2 &= \tfrac{1}{2}b_{1,0}b_{1,1} + c_{2,1} & g_2 &= \tfrac{1}{2}b_{1,1}b_{2,2} + \tfrac{1}{2}b_{1,0}b_{2,3} + c_{1,3} \\ f_3 &= \tfrac{1}{2}b_{1,0} & g_3 &= \tfrac{1}{2}b_{2,2} \end{aligned}$$

Again irrelevant higher order terms have been suppressed:

$$\begin{aligned} W_{sup} &= \tfrac{1}{4}b_{0,0}^2x^4 + \tfrac{1}{2}b_{0,0}b_{1,1}x^3z + \tfrac{1}{2}b_{0,0}b_{2,3}x^2z^3 + (\tfrac{1}{4}b_{1,1}^2 + \tfrac{1}{2}b_{0,0}b_{2,2} + c_{2,2})x^2z^2 \\ &\quad + (\tfrac{1}{2}b_{1,1}b_{2,3} + c_{1,4})xz^4 + \tfrac{1}{4}b_{2,3}^2z^6 \end{aligned}$$

Eq. (17) will be the starting point of the matrix-factorization based analysis.

3. Review of matrix factorizations

Given a polynomial W of some coordinate ring in an affine space, a matrix factorization of W is a square matrix Q with polynomial entries so that,

$$Q^2 = W\mathbb{1}. \tag{18}$$

The matrix is in $\mathbb{Z}_2$-graded space and with the grading operator in suitable form, the matrix can be written as,

$$Q = \begin{pmatrix} 0 & E \\ J & 0 \end{pmatrix}, \tag{19}$$

so that,

$$EJ = JE = W\mathbb{1}. \tag{20}$$

Both Q or the pair (E, J) may be referred to as matrix factorization. The Q-notation is rooted in the topological string and the relationship to the boundary Q_{BRST} operator. The simplest factorization is the trivial 1×1 factorization,

$$(1)(W) = (W)(1) = W\mathbb{1}. \tag{21}$$

This trivial factorization describes an 'empty' brane and does not contain physical information. Two matrix factorizations Q and Q' are equivalent, $Q \simeq Q'$, if they can be related by a similarity transformation by invertible matrices with polynomial entries,

$$Q' = UQU^{-1}. \tag{22}$$

The transformation can also be written as,

$$E' = U_1EU_2^{-1} \qquad J' = U_2JU_1^{-1}, \tag{23}$$

with,

$$U = \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix}. \tag{24}$$

Given two matrix factorizations of W we can define the direct sum which is again a matrix factorization of W. We have,

$$(E_1, J_1) \oplus (E_2, J_2) \equiv \left(\begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}, \begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix} \right). \tag{25}$$

With the help of a similarity transformation, some matrix factorizations can be decomposed into a direct sum of factorizations. Given a brane, its anti-brane is easily found by applying the shift functor T which swaps E and J,

$$T : (E, J) \mapsto (\bar{E}, \bar{J}) = (J, E). \tag{26}$$

The matrix factorizations $Q_{1,2}$ define a graded differential that acts as follows:

$$d\Psi_{12} := Q_1\Psi_{12} - (-1)^{|\Psi_{12}|}\Psi_{12}Q_2. \tag{27}$$

The open string states lie in the cohomology, which is defined as usual as the quotient of the kernel of d by the image of d. The even states are block diagonal on the $\mathbb{Z}_2$ graded space and are interpreted as bosonic states. The odd states are block off-diagonal and are interpreted as fermions. Specifically, for the fermions we have,

$$d\psi_{12} = \begin{pmatrix} 0 & E_1 \\ J_1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \psi_{12}^0 \\ \psi_{12}^1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \psi_{12}^0 \\ \psi_{12}^1 & 0 \end{pmatrix}\begin{pmatrix} 0 & E_2 \\ J_2 & 0 \end{pmatrix}, \tag{28}$$

and for the bosons,

$$d\phi_{12} = \begin{pmatrix} 0 & E_1 \\ J_1 & 0 \end{pmatrix}\begin{pmatrix} \phi_{12}^0 & 0 \\ 0 & \phi_{12}^1 \end{pmatrix} - \begin{pmatrix} \phi_{12}^0 & 0 \\ 0 & \phi_{12}^1 \end{pmatrix}\begin{pmatrix} 0 & E_2 \\ J_2 & 0 \end{pmatrix}. \tag{29}$$

These states can be used to build new factorizations through tachyon condensation. In tachyon condensation can be described by a short exact sequence of modules,

$$0 \to Q_1 \to Q_c \to Q_2 \to 0.$$

The bound states resulting from tachyon condensation correspond to branes with a flux turned on. In [39] it has been shown how non-trivial branes in an orbifold limit of the K3 can be described by matrix factorizations. Fractional branes, discrete Wilson lines as well as unusual orientifold actions all arise from the framework almost automatically.

Factorizations which can neither be obtained from other factorization by tachyon condensation nor are equivalent to a direct sum of branes are called indecomposable matrix factorization. For a complete description of all branes on some space described by $W = 0$ it is reasonable to begin by finding all indecomposable matrix factorizations of W. For a more detailed and rigorous introduction I refer to the literature cited in the introduction.

4. Simple singularities

Simple singularities are defined by the following equations:

$$f(x,y,z) = \begin{cases} -y^2 + x^2 + z^{n+1}, & A_n \text{ with } n \geq 1, \\ -y^2 + x^2 z + z^{n-1}, & D_n \text{ with } n \geq 4, \\ -y^2 + x^3 + z^4, & E_6, \\ -y^2 + x^3 + xz^3, & E_7, \\ -y^2 + x^3 + z^5, & E_8\ . \end{cases}$$

The indecomposable matrix factorizations for these ADE singularities are known in the mathematics literature and are given for example in [40,41]. In the following I list all indecomposable factorizations in a gauge more suitable for the purposes of this paper. These sets of factorizations should be regarded as the elementary building blocks which through tachyon condensation can fuse into bound states. The bound states obtained in such a fashion correspond to branes with a non-trivial flux turned on. The indecomposable factorizations are also relevant for global models since in the vicinity of a simple singularity in a global F-theory model, the factorization of the global surface will locally take the same form as these factorizations. In practice, we will usually work with birational extensions of singularities. For instance the singularity $y^2 = x^3 + z^4 + z^5$ is an extension of the E_6 singularity $y^2 = x^3 + z^4$. Results from homological algebra prove that the morphism between the modules are identical for both singularities. From the factorizations, the morphisms between them can be found. For the simple singularities this is a solved problem and can be performed with existing computer algebra systems such as 'Singular' [42]. Given a set of matrix factorizations, a quiver diagram can be drawn. Essentially for each distinct irreducible morphism between two factorizations one draws an arrow between them to obtain the so-called Auslander–Reiten quiver. For a proper treatment of them see for example [43]. In practice, one sets up short exact sequences and reads off the quiver diagram from them. For the ADE-singularities the quivers are essentially the Dynkin diagrams. To each node in the Dynkin diagram of the corresponding ADE Lie Group corresponds one of the indecomposable factorizations. The rank of the factorization is identical to the degree of the irreducible representation of the Lie Algebra. For example for the E_6 singularity we have six indecomposable factorizations and the short exact sequences,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M_1 & \longrightarrow & M_2 \oplus W & \longrightarrow & M_1 & \longrightarrow & 0 \\
0 & \longrightarrow & M_2 & \longrightarrow & M_1 \oplus M_3 \oplus M_4 & \longrightarrow & M_2 & \longrightarrow & 0 \\
0 & \longrightarrow & M_3 & \longrightarrow & M_2 \oplus M_5 & \longrightarrow & M_3 & \longrightarrow & 0 \\
0 & \longrightarrow & M_4 & \longrightarrow & M_2 \oplus M_6 & \longrightarrow & M_4 & \longrightarrow & 0 \\
0 & \longrightarrow & M_5 & \longrightarrow & M_3 & \longrightarrow & M_5 & \longrightarrow & 0 \\
0 & \longrightarrow & M_6 & \longrightarrow & M_4 & \longrightarrow & M_6 & \longrightarrow & 0
\end{array}$$

where W stands for the trivial factorization. From these exact sequences we can immediately set up the quiver diagram,

$$\begin{array}{ccccccccc}
 & & & & [W] & & & & \\
 & & & & \uparrow\downarrow & & & & \\
 & & & & [M^1] & & & & \\
 & & & & \uparrow\downarrow & & & & \\
[M^5] & \rightleftarrows & [M^3] & \rightleftarrows & [M^2] & \rightleftarrows & [M^4] & \rightleftarrows & [M^6]
\end{array} .$$

The trivial factorization $(1, W)$ always corresponds to the extended node of the Dynkin diagram. In case of a singularity which birationally dominates a simple singularity, the quiver diagram will be identical to the quiver diagram of the dominated singularity except that the extended node with the trivial factorization is removed along with all arrows into and out of it. Greuel and Knörrer proved that a ring R is of finite representation type if and only if R birationally dominates a simple singularity [44]. This is a rather strong statement. Finite representation types means that the Auslander–Reiten quiver is of finite size, which we must require for a physically sensible model. This restricts us to simple singularities only.

4.1. A_n factorizations

Let,

$$S_j := \begin{pmatrix} -x & z^j \\ z^{n+1-j} & x \end{pmatrix}$$

Then the factorizations of the A_n series singularity are given by,

$$M^j : (S_j - y\mathbb{1}, S_j + y\mathbb{1}) \qquad j = 1, ..., n.$$

The quiver diagram can be shown to be,

$$[M^1] \rightleftarrows [M^2] \rightleftarrows \cdots\cdots \rightleftarrows [M^{n-1}] \rightleftarrows [M^n]$$

The factorizations could also be defined for $j = 0$ and $j = n+1$, but for these values they reduce to a direct sum of trivial factorizations after a similarity transformation. The shift functor flips the quiver diagram,

$$T(M^j) \simeq M^{n+2-j}.$$

4.2. D_n factorizations

Let,

$$S_1 := \begin{pmatrix} 0 & x^2 + z^{n-2} \\ z & 0 \end{pmatrix},$$

$$S_j := \begin{pmatrix} 0 & 0 & xz & z^{n-1-\frac{j}{2}} \\ 0 & 0 & z^{\frac{j}{2}} & -x \\ x & z^{n-1-\frac{j}{2}} & 0 & 0 \\ z^{\frac{j}{2}} & -xz & 0 & 0 \end{pmatrix} \quad \text{for } j \text{ even and } 2 \le j \le n-2,$$

$$S_j := \begin{pmatrix} 0 & 0 & xz & z^{n-1+\frac{1-j}{2}} \\ 0 & 0 & z^{\frac{j+1}{2}} & -xz \\ x & z^{n-2+\frac{1-j}{2}} & 0 & 0 \\ z^{\frac{j-1}{2}} & -x & 0 & 0 \end{pmatrix} \quad \text{for } j \text{ odd and } 2 \le j \le n-2,$$

$$S_{n-1} := \begin{pmatrix} 0 & xz - iz^{\frac{n}{2}} \\ x - iz^{\frac{n}{2}-1} & 0 \end{pmatrix} \text{ when } n \text{ is even},$$

$$S_{n-1} := \begin{pmatrix} -z^{\frac{n-1}{2}} & xz \\ x & z^{\frac{n-1}{2}} \end{pmatrix} \text{ when } n \text{ is odd},$$

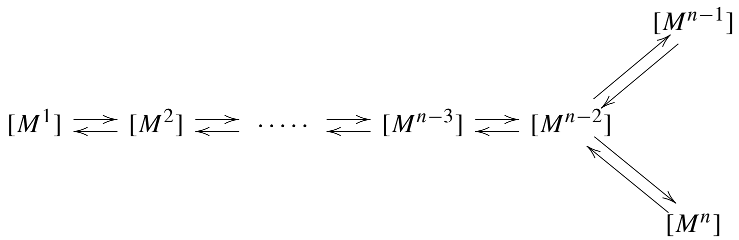
$$S_n := \begin{pmatrix} 0 & xz + iz^{\frac{n}{2}} \\ x - iz^{\frac{n}{2}-1} & 0 \end{pmatrix} \text{ when } n \text{ is even},$$

$$S_n := \begin{pmatrix} z^{\frac{n-1}{2}} & xz \\ x & -z^{\frac{n-1}{2}} \end{pmatrix} \text{ when } n \text{ is odd}.$$

Then the factorizations of the D_n series singularities are given by,

$$M^j : (S_j - y\mathbb{1}, S_j + y\mathbb{1}) \qquad j = 1, \ldots, n.$$

The quiver diagram takes the form,



All D_n factorizations are self-dual under T except for M^n and M^{n-1} which are duals of each other:

$$\begin{aligned} T(M^j) &\simeq M^j \text{ for } j = 1, ..., n-2 \\ T(M^{n-1}) &\simeq M^n \\ T(M^n) &\simeq M^{n-1} \end{aligned}$$

4.3. E_6 factorizations

Let,

$$S_1 := \begin{pmatrix} 0 & 0 & x^2 & z^3 \\ 0 & 0 & z & -x \\ x & z^3 & 0 & 0 \\ z & -x^2 & 0 & 0 \end{pmatrix},$$

$$S_2 := \begin{pmatrix} 0 & 0 & 0 & x^2 & -xz & z^2 \\ 0 & 0 & 0 & z^3 & x^2 & -xz \\ 0 & 0 & 0 & -xz^2 & z^3 & x^2 \\ x & z & 0 & 0 & 0 & 0 \\ 0 & x & z & 0 & 0 & 0 \\ z^2 & 0 & x & 0 & 0 & 0 \end{pmatrix},$$

$$S_3 := \begin{pmatrix} z^2 & ix^2 & 0 & -xz \\ -ix & -z^2 & -z & 0 \\ 0 & 0 & z^2 & ix^2 \\ 0 & 0 & -ix & -z^2 \end{pmatrix},$$

$$S_5 := \begin{pmatrix} z^2 & x^2 \\ x & -z^2 \end{pmatrix}.$$

Then the six indecomposable factorizations of the E_6 singularity are given by,

$$\begin{aligned} &M^1 : (S_1 - y\mathbb{1}, S_1 + y\mathbb{1}) \\ &M^2 : (S_2 - y\mathbb{1}, S_2 + y\mathbb{1}) \\ &M^3 : (S_3 - y\mathbb{1}, S_3 + y\mathbb{1}) \\ &M^4 : (S_3 + y\mathbb{1}, S_3 - y\mathbb{1}) \\ &M^5 : (S_5 - y\mathbb{1}, S_5 + y\mathbb{1}) \\ &M^6 : (S_5 + y\mathbb{1}, S_5 - y\mathbb{1}) \end{aligned}$$

The quiver diagram is,

$$\begin{array}{ccccccccc} & & & & [M^1] & & & & \\ & & & & \uparrow\downarrow & & & & \\ [M^5] & \rightleftarrows & [M^3] & \rightleftarrows & [M^2] & \rightleftarrows & [M^4] & \rightleftarrows & [M^6] \end{array}.$$

The actions of the functor T is given by,

$$
\begin{aligned}
T(M^1) &= M^1\\
T(M^2) &= M^2\\
T(M^3) &= M^4\\
T(M^4) &= M^3\\
T(M^5) &= M^6\\
T(M^6) &= M^5
\end{aligned}
$$

4.4. *E_7 factorizations*

Let,

$$
S_1 := \begin{pmatrix} 0 & 0 & x^2 & xz^2 \\ 0 & 0 & z & -x \\ x & xz^2 & 0 & 0 \\ z & -x^2 & 0 & 0 \end{pmatrix},
$$

$$
S_2 := \begin{pmatrix} 0 & 0 & 0 & x^2 & xz^2 & -x^2z \\ 0 & 0 & 0 & -xz & x^2 & xz^2 \\ 0 & 0 & 0 & z^2 & -xz & x^2 \\ x & 0 & xz & 0 & 0 & 0 \\ z & x & 0 & 0 & 0 & 0 \\ 0 & z & x & 0 & 0 & 0 \end{pmatrix},
$$

$$
S_3 := \begin{pmatrix} 0 & 0 & 0 & 0 & x & xz & 0 & z \\ 0 & 0 & 0 & 0 & z^2 & -x^2 & -xz & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & x^2 & z^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & xz & -x \\ x^2 & xz & 0 & xz & 0 & 0 & 0 & 0 \\ z^2 & -x & -z & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x & z^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & xz & -x^2 & 0 & 0 & 0 & 0 \end{pmatrix},
$$

$$
S_4 := \begin{pmatrix} 0 & 0 & xz & x^2 \\ 0 & 0 & x^2 & -xz^2 \\ z^2 & x & 0 & 0 \\ x & -z & 0 & 0 \end{pmatrix},
$$

$$
S_5 := \begin{pmatrix} 0 & 0 & 0 & x^2 & z^2 & -xz \\ 0 & 0 & 0 & -xz & x & z^2 \\ 0 & 0 & 0 & xz^2 & -xz & x^2 \\ x & 0 & z & 0 & 0 & 0 \\ xz & x^2 & 0 & 0 & 0 & 0 \\ 0 & xz & x & 0 & 0 & 0 \end{pmatrix},
$$

$$
S_6 := \begin{pmatrix} 0 & 0 & x^2 & xz \\ 0 & 0 & z^2 & -x \\ x & xz & 0 & 0 \\ z^2 & -x^2 & 0 & 0 \end{pmatrix},
$$

$$
S_7 := \begin{pmatrix} 0 & z^3 + x^2 \\ x & 0 \end{pmatrix}.
$$

Then the factorizations of the E_7 singularity are given by,

$$M^j : (S_j - y\mathbb{1}, S_j + y\mathbb{1}) \qquad j = 1, ..., 7.$$

The quiver diagram is,

$$\begin{array}{ccccccccccc} & & & & & & [M^4] & & & & \\ & & & & & & \upharpoonleft\downharpoonright & & & & \\ [M^7] & \rightleftarrows & [M^6] & \rightleftarrows & [M^5] & \rightleftarrows & [M^3] & \rightleftarrows & [M^2] & \rightleftarrows & [M^1] \end{array} .$$

All the E_7 factorizations are self-dual under T.

4.5. E_8 factorizations

Let,

$$S_1 := \begin{pmatrix} 0 & 0 & x^2 & z^4 \\ 0 & 0 & z & -x \\ x & z^4 & 0 & 0 \\ z & -x^2 & 0 & 0 \end{pmatrix},$$

$$S_2 := \begin{pmatrix} 0 & 0 & 0 & x^2 & z^4 & -xz^3 \\ 0 & 0 & 0 & -xz & x^2 & z^4 \\ 0 & 0 & 0 & z^2 & -xz & x^2 \\ x & 0 & z^3 & 0 & 0 & 0 \\ z & x & 0 & 0 & 0 & 0 \\ 0 & z & x & 0 & 0 & 0 \end{pmatrix},$$

$$S_3 := \begin{pmatrix} 0 & 0 & 0 & 0 & x & z^3 & 0 & z \\ 0 & 0 & 0 & 0 & z^2 & -x^2 & -xz & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & x^2 & z^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & z^3 & -x \\ x^2 & z^3 & 0 & xz & 0 & 0 & 0 & 0 \\ z^2 & -x & -z & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x & z^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & z^3 & -x^2 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$S_4 := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & x^2 & z^2 & z^3 & 0 & -xz \\ 0 & 0 & 0 & 0 & 0 & z^3 & -x & 0 & -z^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & x^2 & -xz & z^3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & z^4 & x^2 & -xz^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -xz^2 & z^3 & x^2 \\ x & z^2 & 0 & 0 & z & 0 & 0 & 0 & 0 & 0 \\ z^3 & -x^2 & 0 & -z^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x & z & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x & z^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & z^2 & 0 & x & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$S_5 := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & x^2 & -xz & z^3 & 0 & x & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & z^4 & x^2 & -xz^2 & 0 & 0 & xz \\ 0 & 0 & 0 & 0 & 0 & 0 & -xz^2 & z^3 & x^2 & xz & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x & z & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x & z^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & z^2 & 0 & x \\ x & z & 0 & 0 & -x & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & x & z^2 & 0 & 0 & -xz & 0 & 0 & 0 & 0 & 0 & 0 \\ z^2 & 0 & x & -xz & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x^2 & -xz & z^3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & z^4 & x^2 & -xz^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -xz^2 & z^3 & x^2 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$S_6 := \begin{pmatrix} 0 & 0 & 0 & x^2 & -xz & z^3 \\ 0 & 0 & 0 & z^4 & x^2 & -xz^2 \\ 0 & 0 & 0 & -xz^2 & z^3 & x^2 \\ x & z & 0 & 0 & 0 & 0 \\ 0 & x & z^2 & 0 & 0 & 0 \\ z^2 & 0 & x & 0 & 0 & 0 \end{pmatrix},$$

$$S_7 := \begin{pmatrix} 0 & 0 & 0 & 0 & x & z^2 & 0 & -z \\ 0 & 0 & 0 & 0 & z^3 & -x^2 & xz^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & x^2 & z^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & z^3 & -x \\ x^2 & z^2 & 0 & -xz & 0 & 0 & 0 & 0 \\ z^3 & -x & z^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x & z^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & z^3 & -x^2 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$S_8 := \begin{pmatrix} 0 & 0 & x^2 & z^3 \\ 0 & 0 & z^2 & -x \\ x & z^3 & 0 & 0 \\ z^2 & -x^2 & 0 & 0 \end{pmatrix}.$$

Then the factorizations of the E_8 singularity are given by,

$$M^j : (S_j - y\mathbb{1}, S_j + y\mathbb{1}) \qquad j = 1, ..., 8.$$

The quiver diagram is,

$$\begin{array}{ccccccccccccc} & & & & & & & & [M^6] & & & & \\ & & & & & & & & \uparrow\downarrow & & & & \\ [M^1] & \rightleftarrows & [M^2] & \rightleftarrows & [M^3] & \rightleftarrows & [M^4] & \rightleftarrows & [M^5] & \rightleftarrows & [M^7] & \rightleftarrows & [M^8]\,. \end{array}$$

All the E_7 factorizations are self-dual under T.

5. Deformations of matrix factorizations

The matrix factorizations for the A_n surface singularity were listed above as,

$$E_j = \begin{pmatrix} -x-y & z^j \\ z^{n+1-j} & x-y \end{pmatrix} \qquad J_j = \begin{pmatrix} -x+y & z^j \\ z^{n+1-j} & x+y \end{pmatrix} \qquad j = 1, ..., n.$$

To be concrete, I set $n = 5$ and select the factorization with $j = 2$ so that,

$$E_2 \cdot J_2 = \begin{pmatrix} -x-y & z^2 \\ z^4 & x-y \end{pmatrix} \cdot \begin{pmatrix} -x+y & z^2 \\ z^4 & x+y \end{pmatrix} = (-y^2 + x^2 + z^6)\mathbb{1}. \tag{30}$$

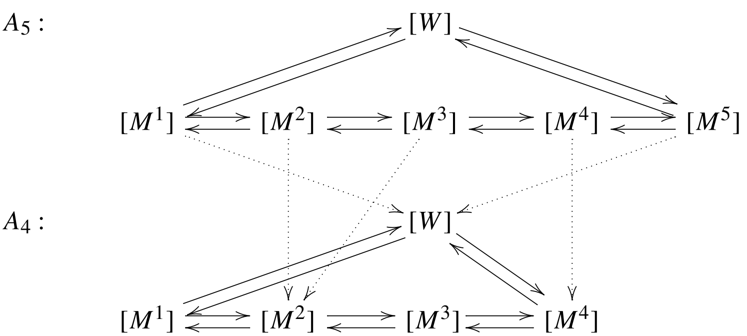
This factorization can be continuously deformed into an A_4-factorization in two different ways. The first option is,

$$\begin{pmatrix} -x-y & fz^2+gz \\ z^4 & x-y \end{pmatrix} \cdot \begin{pmatrix} -x+y & fz^2+gz \\ z^4 & x+y \end{pmatrix} = (-y^2 + x^2 + g_0 z^6 + g_1 z^5)\mathbb{1}, \tag{31}$$

where $g_i \in \mathbb{C}$ are deformation parameters. The values $(g_0, g_1) = (1, 0)$ restore the pure A_5 singularity and $(g_0, g_1) = (0, 1)$ the A_4 singularity. The second way is,

$$\begin{pmatrix} -x-y & z^2 \\ fz^4+gz^3 & x-y \end{pmatrix} \cdot \begin{pmatrix} -x+y & z^2 \\ fz^4+gz^3 & x+y \end{pmatrix} = (-y^2 + x^2 + g_0 z^6 + g_1 z^5)\mathbb{1}. \tag{32}$$

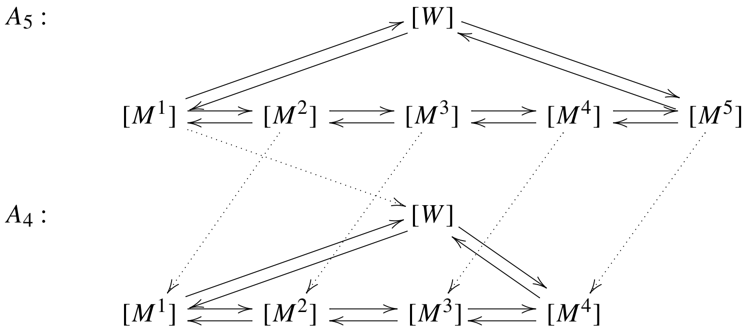
In a conventional treatment of F-theory the only information about a local 7-brane is the term on the right-hand side of the equation. At the level of matrix factorizations we have a much richer structure: Every node of the Dynkin diagram has a separate description, the open string spectrum between these components of branes can be found by computing the morphisms and, as we have just seen, there can be more than one way to deform a brane. It remains an open question to classify all deformations of the ADE-singularities. It is also an open question which deformations may be ruled out by physical principles. For instance, given a brane located at an A_n singularity, one would expect that the factorization should be a direct sum of all n factorizations (or bound states derived thereof). When deforming these n branes in an arbitrary way to A_{n-1}, it is not guaranteed that each of the $n-1$ different A_{n-1} factorizations will be obtained. One could for example deform the factorizations in the following manner:



In the above diagram the dotted arrows points from the original factorization to the deformed factorization and $[W]$ denotes the trivial factorization which sits at the extended node of the Dynkin diagram. In this example, no brane is deformed into either M^1 or M^3. To avoid this situation, we can look at the entire set of all n branes and deform them in the same manner. Then we have either,

$$E_j = \begin{pmatrix} -x-y & g_0 z^j + g_1 z^{j-1} \\ z^{n+1-j} & x-y \end{pmatrix} \qquad J_j = \begin{pmatrix} -x+y & g_0 z^j + g_1 z^{j-1} \\ z^{n+1-j} & x+y \end{pmatrix}$$
$$j = 1, ..., n,$$

and for $n = 5$ get,

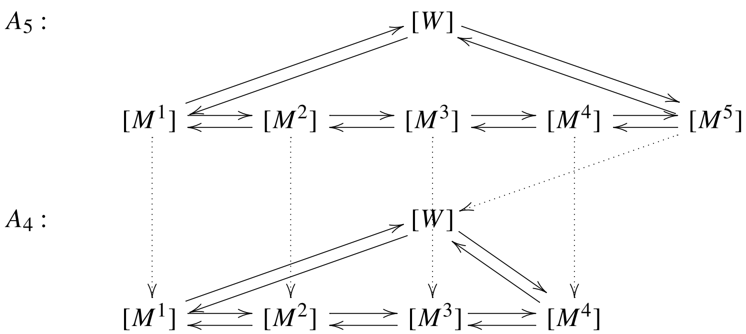


Alternatively we deform,

$$E_j = \begin{pmatrix} -x - y & z^j \\ g_0 z^{n+1-j} + g_1 z^{n-j} & x - y \end{pmatrix} \qquad J_j = \begin{pmatrix} -x + y & z^j \\ g_0 z^{n+1-j} + g_1 z^{n-j} & x + y \end{pmatrix}$$

$$j = 1, ..., n,$$

and obtain,



The latter option is the more conventional choice since the process of deformation here simply removes the n-th node of the Dynkin diagram. The former choice is equivalent to the latter after the exchange of branes with anti-branes (which maps $T(M^j) \mapsto M^{n+1-j}$). In the simple case of the A_n singularity we were able to make an argument of deforming all branes consistently and relied on the fact that all branes have the same factorization structure. With other types of singularities things are not as straightforward and it remains unclear which of the possible deformations are preferred.

6. Deformation from E_8 to D_5

As stated in the introduction, one purpose of this paper is to move beyond branes for simple toy models and demonstrate that branes which appear in phenomenologically viable models can be described. In this section 1 will take all indecomposable matrix factorizations and deform them in the sequence $E_8 \to E_7 \to E_6 \to D_5$. The A_n factorizations are rather simple and the further deformation to A_4 is not worked out here. Given the singularity of Eq. (17),

$$W(x, y, z) = -y^2 + f_1 x^3 + f_2 x^2 z + f_3^2 x^2 + g_1 z^5 + g_2 x z^3 + g_3^2 z^4,$$

we can reproduce the $E_{8,7,6}$ and D_5 singularities by setting the appropriate coefficients of f_i and g_i to zero. In principle the f_3-term is not necessary for the breaking pattern, but it is useful to preserve it for the straightforward extension to the A_n singularities. For a more compact notation of the factorizations we define for later use,

$$\begin{aligned} F &:= f_1 x + f_2 z + f_3^2 \\ G &:= g_1 z^2 + g_2 x + g_3^2 z \end{aligned}$$

6.1. Brane M^1

A deformation of the brane M^1 of E_8 is given by,

$$\tilde{M}^1 : (\tilde{S}_1 - y\mathbb{1}, \tilde{S}_1 + y\mathbb{1})$$

where,

$$\tilde{S}_1 := \begin{pmatrix} 0 & 0 & Fx & Gz^2 \\ 0 & 0 & z & -x \\ x & Gz^2 & 0 & 0 \\ z & -Fx & 0 & 0 \end{pmatrix}.$$

By direct computation it is easy to verify that the factorization condition is satisfied,

$$(\tilde{S}_1 - y\mathbb{1}) \cdot (\tilde{S}_1 + y\mathbb{1}) = (-y^2 + f_1 x^3 + f_2 x^2 z + f_3^2 x^2 + g_1 z^5 + g_2 x z^3 + g_3^2 z^4)\mathbb{1}.$$

The deformation sequence for this brane is rather simple:

$$\begin{array}{ll} E_8 : & [M^1] \\ & \downarrow \\ E_7 : & [M^1] \\ & \downarrow \\ E_6 : & [M^1] \\ & \downarrow \\ D_5 : & [M^1] \end{array}$$

This can be seen by setting the appropriate coefficients f_i and g_i to zero respectively unity and comparing with the factorization list of the ADE-singularities.

6.2. Brane M^2

A deformation of the brane M^2 of E_8 is given by,

$$\tilde{M}_2 : (\tilde{S}_2 - y\mathbb{1}, \tilde{S}_2 + y\mathbb{1})$$

where,

$$\tilde{S}_2 := \begin{pmatrix} 0 & 0 & 0 & Fx & Gz^2 & -Gxz \\ 0 & 0 & 0 & -Fz & Fx & Gz^2 \\ 0 & 0 & 0 & z^2 & -xz & x^2 \\ x & 0 & Gz & 0 & 0 & 0 \\ z & x & 0 & 0 & 0 & 0 \\ 0 & z & F & 0 & 0 & 0 \end{pmatrix}$$

The deformation sequence of this brane is:

$$\begin{array}{ll} E_8: & [M^2] \\ & \downarrow \\ E_7: & [M^2] \\ & \downarrow \\ E_6: & [M^2] \\ & \downarrow \\ D_5: & [M^1] \oplus [M^3] \end{array}$$

Again it is manifest from the list of factorizations how the E_8 factorizations deforms down to E_6, but the last step is non-trivial. For $f_1 = f_3 = g_1 = g_2 = 0$ we obtain a deformation to D_5 which reads,

$$\tilde{M}^2(D_5) : (\tilde{S}_2(D_5) - y\mathbb{1}, \tilde{S}_2(D_5) + y\mathbb{1})$$

where,

$$\tilde{S}_2(D_5) := \begin{pmatrix} 0 & 0 & 0 & f_2xz & g_3^2z^3 & -g_3^2xz^2 \\ 0 & 0 & 0 & -f_2z^2 & f_2xz & g_3^2z^3 \\ 0 & 0 & 0 & z^2 & -xz & x^2 \\ x & 0 & g_3^2z^2 & 0 & 0 & 0 \\ z & x & 0 & 0 & 0 & 0 \\ 0 & z & f_2z & 0 & 0 & 0 \end{pmatrix}$$

After an appropriate gauge transformation, this factorization decomposes into a direct sum of smaller matrices. To realize the gauge transformation we define the matrix,

$$U := \begin{pmatrix} 0 & 1 & f_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & f_2 & 0 & -g_3^2z \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

We can assume f_2 and g_3 to be locally non-zero, therefore U is invertible as required for a well-defined similarity transformation. Then,

$$U \cdot (\tilde{S}_2(D_5) \pm y\mathbb{1}) \cdot U^{-1} = \begin{pmatrix} \pm y & g_3^2 z^3 + f_2 x^2 & 0 & 0 & 0 & 0 \\ z & \pm y & 0 & 0 & 0 & 0 \\ 0 & 0 & \pm y & 0 & xz & g_3^2 z^3 \\ 0 & 0 & 0 & \pm y & z^2 & -f_2 xz \\ 0 & 0 & f_2 x & g_3^2 z^2 & \pm y & 0 \\ 0 & 0 & z & -x & 0 & \pm y \end{pmatrix}$$

For $f_2 = g_3 = 1$ the two block matrices on the right-hand side are identified with $M^1(D_5)$ and $M^3(D_5)$ in the factorization list of the simple singularities.

6.3. Brane M^3

A deformation of the brane M^3 of E_8 is given by,

$$\tilde{M}^3 : (\tilde{S}_3 - y\mathbb{1}, \tilde{S}_3 + y\mathbb{1})$$

where,

$$\tilde{S}_3 := \begin{pmatrix} 0 & 0 & 0 & 0 & x & Gz & 0 & z \\ 0 & 0 & 0 & 0 & z^2 & -Fx & -Fz & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & Fx & z^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & Gz & -x \\ Fx & Gz & 0 & Fz & 0 & 0 & 0 & 0 \\ z^2 & -x & -z & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x & z^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & Gz & -Fx & 0 & 0 & 0 & 0 \end{pmatrix}$$

The deformation sequence is:

$$\begin{array}{ll} E_8 : & [M^3] \\ & \quad\downarrow \\ E_7 : & [M^3] \\ & \quad\downarrow \\ E_6 : & [M^3] \oplus [M^4] \\ & \quad\downarrow \\ D_5 : & [M^3] \oplus [M^3] \end{array}$$

The decomposition at the E_6 level is proven by the gauge transformation,

$$U_1 \cdot \begin{pmatrix} -y & 0 & 0 & 0 & x & g_3^2 z^2 & 0 & z \\ 0 & -y & 0 & 0 & z^2 & -f_1 x^2 & -f_1 xz & 0 \\ 0 & 0 & -y & 0 & 0 & 0 & f_1 x^2 & z^2 \\ 0 & 0 & 0 & -y & 0 & 0 & g_3^2 z^2 & -x \\ f_1 x^2 & g_3^2 z^2 & 0 & f_1 xz & -y & 0 & 0 & 0 \\ z^2 & -x & -z & 0 & 0 & -y & 0 & 0 \\ 0 & 0 & x & z^2 & 0 & 0 & -y & 0 \\ 0 & 0 & g_3^2 z^2 & -f_1 x^2 & 0 & 0 & 0 & -y \end{pmatrix} \cdot U_2 =$$

$$\begin{pmatrix} iy - ig_3 z^2 & f_1 x^2 & 0 & g_3 xz & 0 & 0 & 0 & 0 \\ -x & iy + ig_3 z^2 & g_3 z & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & iy - ig_3 z^2 & x^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -f_1 x & iy + ig_3 z^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & iy + ig_3 z^2 & -f_1 x^2 & 0 & -g_3 xz \\ 0 & 0 & 0 & 0 & x & iy - ig_3 z^2 & -g_3 z & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & iy + ig_3 z^2 & -x^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & f_1 x & iy - ig_3 z^2 \end{pmatrix}$$

with,

$$U_1 = \begin{pmatrix} 0 & g_3 & 0 & 0 & 1 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 & 0 & ig_3 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & -\frac{1}{g_3} \\ 0 & 0 & 0 & -\frac{if_1}{g_3} & 0 & 0 & if_1 & 0 \\ 0 & -ig_3 & 0 & 0 & i & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & -g_3 & 0 & 0 \\ 0 & 0 & i & 0 & 0 & 0 & 0 & -\frac{i}{g_3} \\ 0 & 0 & 0 & -\frac{f_1}{g_3} & 0 & 0 & -f_1 & 0 \end{pmatrix}$$

$$U_2 = \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{i}{2} & 0 & 0 \\ -\frac{i}{2g_3} & 0 & 0 & 0 & \frac{1}{2g_3} & 0 & 0 & 0 \\ 0 & 0 & \frac{i}{2} & 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{g_3}{2f_1} & 0 & 0 & 0 & \frac{ig_3}{2f_1} \\ -\frac{i}{2} & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & -\frac{1}{2g_3} & 0 & 0 & 0 & \frac{i}{2g_3} & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2f_1} & 0 & 0 & 0 & \frac{i}{2f_1} \\ 0 & 0 & \frac{ig_3}{2} & 0 & 0 & 0 & \frac{g_3}{2} & 0 \end{pmatrix}$$

At the D_5-level a similar transformation exists which is not explicitly written down here. Remember that $M^3(E_6)$ and $M^4(E_6)$ are brane/anti-brane pairs,

$$T(M_3(E_6)) = M_4(E_6). \tag{33}$$

On the other hand, $M_3(D_5)$ is its own anti-brane, i.e. on the level of factorizations it is self-dual,

$$T(M_3(D_5)) = M_3(D_5) \tag{34}$$

Just from these relations it is clear that the deformation from $M_3(E_6)$ to $M_3(D_5)$ implies that $M_4(E_6)$ deforms also into $M_3(D_5)$. Therefore one brane at the E_8 level has been deformed into a direct sum of two identical branes. However, we expect the physics of a system with a direct sum of identical branes not to differ from the physics of a system with only single copy of the factorization.

6.4. Brane M^4

A deformation of the brane M^4 of E_8 is given by,

$$\tilde{M}^4 : (\tilde{S}_4 - y\mathbb{1}, \tilde{S}_4 + y\mathbb{1})$$

where,

$$\tilde{S}_4 := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & Fx & z^2 & z^3 & 0 & -Fz \\ 0 & 0 & 0 & 0 & 0 & gz & -x & 0 & -z^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & x^2 & -xz & gz \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & gz^2 & Fx & -Fg \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -xz^2 & z^3 & Fx \\ x & z^2 & 0 & 0 & z & 0 & 0 & 0 & 0 & 0 \\ gz & -Fx & 0 & -z^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & F & z & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x & g & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & z^2 & 0 & x & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The deformation sequence is:

$$\begin{array}{ll} E_8 : & [M^4] \\ & \downarrow \\ E_7 : & [M^4] \oplus [M^5] \\ & \downarrow \\ E_6 : & [M^2] \oplus [M^5] \oplus [M^6] \\ & \downarrow \\ D_5 : & [M^1] \oplus [M^2] \oplus [M^4] \oplus [M^5] \end{array}$$

6.5. Brane M^5

A deformation of the brane M^5 of E_8 is given by,

$$\tilde{M}^5 : (\tilde{S}_5 - y\mathbb{1}, \tilde{S}_5 + y\mathbb{1})$$

where,

$$
\tilde{S}_5 := \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & Fx & -Fz & Gz & 0 & F & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & Gz^2 & Fx & -Gx & 0 & 0 & xz \\
0 & 0 & 0 & 0 & 0 & 0 & -xz^2 & z^3 & x^2 & xz & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x & z & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & F & z^2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & G & 0 & x \\
x & z & 0 & 0 & -x & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & x & G & 0 & 0 & -xz & 0 & 0 & 0 & 0 & 0 & 0 \\
z^2 & 0 & F & -Fz & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & Fx & -xz & z^3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & Gz^2 & x^2 & -xz^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -FG & Gz & Fx & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

The deformation sequence is:

$$
\begin{array}{ll}
E_8: & [M^5] \\
 & \quad\downarrow \\
E_7: & [M^3] \oplus [M^7] \oplus [M^7] \\
 & \quad\downarrow \\
E_6: & [M^2] \oplus [M^2] \\
 & \quad\downarrow \\
D_5: & [M^1] \oplus [M^1] \oplus [M^3] \oplus [M^3]
\end{array}
$$

In the deformation sequences one could most of the times specify which of the factorizations deform to which lower rank factorizations, for instance each $M^2(E_6)$ deforms into one copy of $M^1(D_5) \oplus M^3(D_5)$, but this does not hold every time. For example at the E_7-level of this brane, a sum of branes is required to deform to the E_6 factorization.

6.6. Brane M^6

A deformation of the brane M^6 of E_8 is given by,

$$\tilde{M}^6 : (\tilde{S}_6 - y\mathbb{1}, \tilde{S}_6 + y\mathbb{1})$$

where,

$$
\tilde{S}_6 := \begin{pmatrix}
0 & 0 & 0 & Fx & -Fz & Gz \\
0 & 0 & 0 & Gz^2 & Fx & -Gx \\
0 & 0 & 0 & -xz^2 & z^3 & x^2 \\
x & z & 0 & 0 & 0 & 0 \\
0 & x & G & 0 & 0 & 0 \\
z^2 & 0 & F & 0 & 0 & 0
\end{pmatrix}
$$

The deformation sequence is:

$$
\begin{array}{ll}
E_8: & [M^6] \\
 & \quad\downarrow \\
E_7: & [M^4] \oplus [M^7] \\
 & \quad\downarrow \\
E_6: & [M^2] \\
 & \quad\downarrow \\
D_5: & [M^1] \oplus [M^3]
\end{array}
$$

Note that the E_8 factorization first falls apart into a direct sum of two E_7 factorizations but at further deformation to E_6 the two components recombine into a single one. The fact that deformations can involve and sometimes has to involve a direct sum of factorizations rather than being restricted to single factorization only makes a systematic treatment of all possible deformations much more difficult.

6.7. Brane M^7

A deformation of the brane M^7 of E_8 is given by,

$$\tilde{M}^7 : (\tilde{S}_7 - y\mathbb{1}, \tilde{S}_7 + y\mathbb{1})$$

where,

$$
\tilde{S}_7 := \begin{pmatrix}
0 & 0 & 0 & 0 & x & z^2 & 0 & -z \\
0 & 0 & 0 & 0 & Gz & -Fx & Gx & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & x^2 & z^2 \\
0 & 0 & 0 & 0 & 0 & 0 & Gz & -F \\
Fx & z^2 & 0 & -xz & 0 & 0 & 0 & 0 \\
Gz & -x & G & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & F & z^2 & 0 & 0 & 0 & 0 \\
0 & 0 & Gz & -x^2 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

The deformation sequence is:

$$
\begin{array}{ll}
E_8: & [M^7] \\
 & \quad\downarrow \\
E_7: & [M^5] \oplus [M^7] \\
 & \quad\downarrow \\
E_6: & [M^3] \oplus [M^4] \\
 & \quad\downarrow \\
D_5: & [M^1] \oplus [M^1] \oplus [M^4] \oplus [M^5]
\end{array}
$$

6.8. Brane M^8

A deformation of the brane M^8 of E_8 is given by,

$$\tilde{M}^8 : (\tilde{S}_8 - y\mathbb{1}, \tilde{S}_8 + y\mathbb{1})$$

where,

$$\tilde{S}_8 := \begin{pmatrix} 0 & 0 & Fx & Gz \\ 0 & 0 & z^2 & -x \\ x & Gz & 0 & 0 \\ z^2 & -Fx & 0 & 0 \end{pmatrix}$$

The deformation sequence is:

$$\begin{array}{ll} E_8 : & [M^8] \\ & \downarrow \\ E_7 : & [M^6] \\ & \downarrow \\ E_6 : & [M^5] \oplus [M^6] \\ & \downarrow \\ D_5 : & [M^4] \oplus [M^5] \end{array}$$

Given the small matrix dimensions of this example, it is worth looking at it explicitly.

$$E_8 : \begin{pmatrix} \pm y & 0 & f_1x^2 & g_1z^3 \\ 0 & \pm y & z^2 & -x \\ x & g_1z^3 & \pm y & 0 \\ z^2 & -f_1x^2 & 0 & \pm y \end{pmatrix}$$

$$E_7 : \begin{pmatrix} \pm y & 0 & f_1x^2 & g_2xz \\ 0 & \pm y & z^2 & -x \\ x & g_2xz & \pm y & 0 \\ z^2 & -f_1x^2 & 0 & \pm y \end{pmatrix}$$

$$E_6 : U_a \begin{pmatrix} \pm y & 0 & f_1x^2 & g_3^2z^2 \\ 0 & \pm y & z^2 & -x \\ x & g_3^2z^2 & \pm y & 0 \\ z^2 & -f_1x^2 & 0 & \pm y \end{pmatrix} U_a^{-1}$$

$$= \begin{pmatrix} \pm y + g_3z^2 & f_1x^2 & 0 & 0 \\ x & \pm y - g_3z^2 & 0 & 0 \\ 0 & 0 & \pm y - g_3z^2 & f_1x^2 \\ 0 & 0 & x & \pm y + g_3z^2 \end{pmatrix}$$

$$D_5 : U_a \begin{pmatrix} \pm y & 0 & f_2xz & g_3^2z^2 \\ 0 & \pm y & z^2 & -x \\ x & g_3^2z^2 & \pm y & 0 \\ z^2 & -f_2xz & 0 & \pm y \end{pmatrix} U_a^{-1}$$

$$= \begin{pmatrix} \pm y + g_3 z^2 & f_2 x z & 0 & 0 \\ x & \pm y - g_3 z^2 & 0 & 0 \\ 0 & 0 & \pm y - g_3 z^2 & f_2 x z \\ 0 & 0 & x & \pm y + g_3 z^2 \end{pmatrix}$$

$$A_4 : U_b \begin{pmatrix} \pm y & 0 & f_3^2 x & g_1 z^3 \\ 0 & \pm y & z^2 & -x \\ x & g_1 z^3 & \pm y & 0 \\ z^2 & -f_3^2 x & 0 & \pm y \end{pmatrix} U_b^{-1}$$

$$= \begin{pmatrix} \pm y - f_3 x & z^2 & 0 & 0 \\ g_1 z^3 & \pm y + f_3 x & 0 & 0 \\ 0 & 0 & \pm y - f_3 x & g_1 z^3 \\ 0 & 0 & z^2 & \pm y + f_3 x \end{pmatrix}$$

The two transformation matrices which were used to turn the matrices into a sum of indecomposable factorizations are given by,

$$U_a = \begin{pmatrix} 1 & 0 & 0 & g_3 \\ 0 & -g_3 & 1 & 0 \\ 1 & 0 & 0 & -g_3 \\ 0 & g_3 & 1 & 0 \end{pmatrix} \qquad U_b = \begin{pmatrix} 0 & f_3 & 0 & 1 \\ 1 & 0 & f_3 & 0 \\ 1 & 0 & -f_3 & 0 \\ 0 & -f_3 & 0 & 1 \end{pmatrix}.$$

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