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Higgs bundles and holomorphic forms

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Abstract: For a complex manifold X which has a holomorphic form ϖ of odd degree k, we endow $E^a = \bigoplus_{p\geqslant a} \Lambda^{(p,0)}(X)$ with a Higgs bundle structure θ given by $\theta(Z)(\phi) := \{i(Z)\varpi\} \wedge \phi$. The properties such as curvature and stability of these and other Higgs bundles are examined. We prove (Theorem 2, Section 2, for $k > 1$) E^a and additional classes of Higgs subbundles of E^a do not admit Higgs–Hermitian–Yang–Mills metric in any one of the cases: (i) $\deg(X) < 0$, (ii) $\deg(X) = 0$ and $a \leqslant n-k+1$, or (iii) $a \leqslant n-k+1$ and $k \geqslant \frac{1}{2}n + 1$. We give examples of (noncompact) Kähler manifolds with the above Higgs structure which admit Higgs–Hermitian–Yang–Mills metrics. We also examine vanishing theorems for (p,q)-forms with values in Higgs bundles.

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1. Introduction

The purpose of this paper is to give new examples of Higgs bundles which arise in a rather natural way, and to study their properties. Recall that a *Higgs bundle* [11] is a holomorphic vector bundle, $E \to X$ over a complex manifold X, together with a holomorphic section $\theta \in \vartheta\Gamma(\mathrm{Hom}(E) \otimes \Lambda^{1,0}(X))$ (the “Higgs” form), which satisfies the equation $\theta \wedge \theta = 0$. This equation means that if Z and W are holomorphic tangent vectors to X at a point, then $[\theta(Z), \theta(W)] = 0$ as an endomorphism of E at that point.

The examples consist of a complex manifold X of complex dimension n which is assumed to possess a nontrivial holomorphic k-form ϖ where k is odd. The bundle E is given by $E := \bigoplus_{p=0}^{n} \bigwedge^{(p,0)}(X)$, and the Higgs form θ is given by the prescription $\theta(Z)(\phi) := \{i(Z)\varpi\} \wedge \phi$, where ϕ is a section of E and Z is a holomorphic tangent vector. Defining E^a by $E^a := \bigoplus_{p=0}^{n} \bigwedge^{(p,0)}(X)(E = E^0)$, the E^a form a Higgs filtration of E (cf. 2.15). We now give some examples of complex manifolds possessing such forms.

(i) $X =$ any complex torus.

(ii) If X is the zero-locus in $\mathbf{P}^{n+1}$ of a homogeneous polynomial of large degree D, then $h^{n,0}(X) = \binom{D-1}{n+1}$ so if n is odd these are examples of the types of complex manifolds required.

(iii) Calabi–Yau manifolds—compact Kähler Ricci flat complex 3-manifolds with a nowhere vanishing holomorphic 3-form, i.e., trivial canonical bundle, and higher-dimensional analogs

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(cf. [2, pages 144–145]).

(iv) For any complex manifold X, its holomorphic cotangent bundle $\Lambda^{(1,0)}X$ admits a canonical holomorphic one-form $\phi \in \vartheta\Gamma\ \Lambda^{(1,0)}(\Lambda^{(1,0)}X)$ such that $\partial\phi$ is a (holomorphic) symplectic two-form. This ϕ can be given invariantly by the formula $\phi(Z_\alpha) = \alpha(\pi_{*\alpha}(Z_\alpha))$, $Z_\alpha \in T_\alpha^{(1,0)}(\Lambda^{(1,0)}X), \alpha \in \Lambda^{(1,0)}X$ with $\pi : \Lambda^{(1,0)}X \to X$ the projection (cf. [2, pages 85–86]). Replacing X with the complex manifold $\Lambda^{(1,0)}X$, one gets the corresponding holomorphic one-form $\Phi \in \vartheta\Gamma\ (\Lambda^{(1,0)}(\Lambda^{(1,0)}(\Lambda^{(1,0)}X)))$ and (symplectic) two-form $\partial\Phi \in \vartheta\Gamma\ (\Lambda^{(2,0)}(\Lambda^{(1,0)}(\Lambda^{(1,0)}X)))$ on $\Lambda^{(1,0)}(\Lambda^{(1,0)}X)$. Let $p : \Lambda^{(1,0)}(\Lambda^{(1,0)}X) \to \Lambda^{(1,0)}X$ be the projection. Then for any holomorphic functions a and b on $\Lambda^{(1,0)}(\Lambda^{(1,0)}X)$, one gets a holomorphic three-form $a\Phi \wedge p^*\partial\phi + b\partial\Phi \wedge p^*\phi \in \vartheta\Gamma\ (\Lambda^{(3,0)}(\Lambda^{(1,0)}(\Lambda^{(1,0)}X)))$. Computation of these 3-forms in local holomorphic coordinates (using coordinates on $\Lambda^{(1,0)}X$ given by "pulling up" a holomorphic chart on X and then "pulling up" these coordinates on $\Lambda^{(1,0)}X$ via p to $\Lambda^{(1,0)}(\Lambda^{(1,0)}X)$) shows that these forms are generally nonzero.

(v) If M is any of the above examples, then any complex manifold $\widetilde{M}$ from which there is a holomorphic submersion $p : \widetilde{M} \to M$ onto M, itself inherits nonzero holomorphic odd-degree forms from M by pull-back. For example, coverings or blowing up any of the above examples at any number of points and/or taking products of those examples will serve as such an $\widetilde{M}$.

We investigate the curvature, stability and other properties of these Higgs bundles (and also general Higgs Bundles) and prove the following :

Theorem. ([2, Sect. 2]) *Let X be a compact Kähler manifold with a nontrivial holomorphic k-form ϖ where $k > 1$ is odd. Let the Higgs structure of E be as above and let P be any Higgs subbundle of E of the form $P = \bigoplus_{s=1}^{z} \Lambda^{(p_s,0)}(X)$, $0 \leqslant p_1 < p_2 < \cdots < p_z \leqslant n$, $(z \geqslant 2)$. Then P does not admit any Higgs–Hermitian–Yang–Mills metric in any of the following cases*:

(i) $\deg(X) < 0$.

(ii) $\deg(X) = 0$ *and* $p_1 \leqslant n - k + 1$.

(iii) $k \geqslant \frac{1}{2}n + 1$, $p_1 \leqslant n - k + 1$, *and* ϖ *is a section of* P.

Note that the degree statement in (ii) is sharp because the Higgs form θ acts trivially on E^{n-k+2}. If X is compact Kähler with first Chern class $c_1(X) = 0$, then the Yau resolution of the Calabi conjecture [15], yields a Ricci-flat metric g on X. Extending g in the usual way to the complex exterior algebra of X gives a Hermitian metric on E^{n-k+2} which is Higgs–Hermitian–Yang–Mills in the "vacuous" sense that g is Hermitian–Yang–Mills and the Higgs form vanishes.

We also examine Bochner-type vanishing results ([1, Sect. 2]) and Kodaira–Nakano-type vanishing Theorems 3 and 4 in this setting.

The original study of Higgs bundles is due to Hitchin [4], where the case of rank 2 vector bundles over curves is considered. Hitchin studies the Yang–Mills equations with "interaction term" given by the Higgs field (cf. the discussion above (2.14)). Hitchin obtains a correspondence relating irreducible rank 2 flat vector bundles and degree zero stable Higgs bundles over Riemann surfaces. This correspondence has its genesis in the work of Narasimhan and Seshardi [6].

Higgs bundles also arise in the study of Variations of Hodge Structure. See, e.g., [7, Sect. 1 and 2], [3], [13, Ch. V, Sect. 6], [9, pp. 868–869], and [10, Sect. 1], for detailed information. Generalizing the idea that Hitchin had introduced, Simpson [9,10,11] defined the notion of Higgs bundles on higher-dimensional varieties, where the equation $\theta \wedge \theta = 0$ (automatically satisfied on a curve) is part of the definition. Simpson studied the moduli space of stable Higgs bundles with vanishing Chern classes in work which leads up to the following striking result (showing the “ubiquity” of VHS) among others:

If M is a smooth projective variety then any representation of $\pi_1(M)$ can be deformed to a representation arising from a complex variation of Hodge structure.

This result, among other things, restricts the types of groups which can arise as the fundamental group for any such M, cf. [1, Chapter 7].

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We now continue with the development of properties of Higgs bundles. Any Higgs bundle has a naturally defined operator $D'' : \Gamma E \to \Gamma(E \otimes \Lambda^1(X))$ defined by $D'' = \bar\partial + \theta$ where $\bar\partial$ is the complex structure on E. The three conditions: $\bar\partial$ is integrable ($\bar\partial^2 = 0$), θ is holomorphic and $\theta \wedge \theta = 0$ are simultaneously expressed in the single equation $(D'')^2 = 0$.

Let h be a Hermitian metric on E. The Hermitian connection of (E, h), ∇, can be uniquely written $\nabla = \partial_h + \bar\partial$. Define the Hermitian adjoint of θ, $\overline{\theta_h}$ by the formula

$$h(\overline{\theta_h}(Y)\, s, t) = h(s, \theta(\bar{Y})\, t), \tag{1.1}$$

where Y is a complex tangent vector and s and t are sections of E. Define D'_h by

$$D'_h = \partial_h + \overline{\theta_h} \tag{1.2}$$

and

$$D_h = D'_h + D''. \tag{1.3}$$

One checks that $(D'_h)^2 = 0$, that D_h is a connection on E and that the curvature of D_h is given by

$$F_h = (D_h)^2 = D'_h D'' + D'' D'_h. \tag{1.4}$$

Let $\Theta = \nabla^2$ be the curvature of h. Although Θ is a type (1,1) End(E)-valued form, in general F_h will have parts of type $(2, 0)$, $(1, 1)$ and $(0, 2)$. The relation between the components of F_h, Θ, θ will now be described. Let $\{e_\alpha\}_{\alpha=1}^r$ be a local holomorphic frame for E (r = rank of E), $h_{\alpha\bar\beta} = h(e_\alpha, e_\beta)$, and $(h^{\beta\bar\gamma})$ be the inverse matrix of $(h_{\alpha\bar\beta})$. Then $\nabla e_\alpha = \sum_{\beta=1}^r e_\beta \otimes C_\alpha^\beta$, where $C = h^{-1}\partial h$ and $\Theta = \bar\partial C$. Also, $\theta e_\alpha = \sum_{\beta=1}^r e_\beta \otimes \theta_\alpha^\beta$ where θ_α^β are the matrix representative $(1, 0)$-forms of θ relative to $\{e_\alpha\}$. In this setting we also have $\overline{\theta_h} e_\alpha = \sum_{\beta=1}^r e_\beta \otimes \overline{\theta_h}{}_\alpha^\beta$, where $\overline{\theta_h}{}_\alpha^\beta = \sum_{\gamma,\kappa} h^{\beta\bar\gamma}\overline{\theta_\gamma^\kappa} h_{\alpha\bar\kappa}$. If the frame $\{e_\alpha\}_{\alpha=1}^r$ is orthonormal at a point of evaluation, then $\overline{\theta_h}{}_\alpha^\beta = \overline{\theta_\beta^\alpha}$ at that point.

Now, writing $F_h e_\alpha = \sum_\beta \{e_\beta \otimes F_{h\alpha}^\beta\}$, and $(F_h e_\alpha)^{(a,b)} = \sum_\beta \{e_\beta \otimes (F_{h\alpha}^\beta)^{(a,b)}\}$ where $(a, b) = (2, 0)$, $(0, 2)$ or $(1, 1)$ one computes (cf. [9, page 879, fourth line from the top], and also Proposition 1 below)

$$F_h^{(2,0)} = \partial\theta + C \wedge \theta + \theta \wedge C, \tag{1.5}$$

$$F_h^{(0,2)} = \bar\partial\overline{\theta_h}, \tag{1.6}$$

$$F_h^{(1,1)} = \Theta + \theta \wedge \overline{\theta_h} + \overline{\theta_h} \wedge \theta. \tag{1.7}$$

In more detail

$$(F_h e_\alpha)^{(2,0)} = \sum_\beta \left\{ e_\beta \otimes \left\{ \partial\theta^\beta_\alpha + \sum_\gamma \left(C^\beta_\gamma \wedge \theta^\gamma_\alpha + \theta^\beta_\gamma \wedge C^\gamma_\alpha \right) \right\} \right\},$$

$$(F_h e_\alpha)^{(0,2)} = \sum_\beta \left\{ e_\beta \otimes \left\{ \bar\partial \overline{\theta_h}^\beta_\alpha \right\} \right\},$$

$$(F_h e_\alpha)^{(1,1)} = \sum_\beta \left\{ e_\beta \otimes \left\{ \Theta^\beta_\alpha + \sum_\gamma \left(\theta^\beta_\gamma \wedge \overline{\theta_h}^\gamma_\alpha + \overline{\theta_h}^\beta_\gamma \wedge \theta^\gamma_\alpha \right) \right\} \right\}.$$

In the course of proving (1.7) one must use the identity

$$\partial\overline{\theta_h} + C \wedge \overline{\theta_h} + \overline{\theta_h} \wedge C = 0$$

which in turn follows from the identities $C = h^{-1}\partial h$ and $\overline{\theta_h} = h^{-1}\bar\partial h$.

If M is any smooth manifold and $\mathbf{V} \to M$ is any real or complex vector bundle with a connection $\nabla\colon C^\infty\Gamma\mathbf{V} \to C^\infty\Gamma(\mathbf{V} \otimes \Lambda^1(V))$ there is a "natural" extension of ∇, $d^\nabla : \mathbf{V}^k \to \mathbf{V}^{k+1}$, where $\mathbf{V}^r := C^\infty\Gamma(\mathbf{V} \otimes \Lambda^r(V))$. This implies the following (cf. [9, page 879]):

Proposition 1. *$F_h^{(2,0)} = d^\nabla\theta$ and $F_h^{(0,2)} = d^\nabla\overline{\theta_h}$.*

At any point p we can always find a local holomorphic frame $\{e_\alpha\}_{\alpha=1}^r$ adapted to p, and also $\bar\partial\overline{\theta^\alpha_\beta} = \overline{\partial\theta^\alpha_\beta}$ so we conclude $d^\nabla\theta(p) = 0 \Leftrightarrow d^\nabla\overline{\theta_h}(p) = 0$. Now the above Proposition 1 implies

$$F_h^{(2,0)} = 0 \quad \Leftrightarrow \quad F_h^{(0,2)} = 0 \quad \Leftrightarrow \quad d^\nabla\theta = 0 \quad \Leftrightarrow \quad d^\nabla\overline{\theta_h} = 0. \tag{1.8}$$

We now examine the curvature terms appearing in (1.7). If Z, W are holomorphic tangent vectors at a point, then (1.7) implies

$$\begin{aligned} F_h(Z, \overline{W})s &= \Theta(Z, \overline{W})s + \theta(Z)\overline{\theta_h}(\overline{W})s - \overline{\theta_h}(\overline{W})\theta(Z)s \\ &= \Theta(Z, \overline{W})s + [\theta(Z), \overline{\theta_h}(\overline{W})]\, s, \end{aligned} \tag{1.9}$$

where s is any section of E. Relative to the local framing $\{e_\alpha\}_{\alpha=1}^r$ of E, (1.9) can be written

$$F_h(Z, \overline{W})e_\alpha = \sum_\beta \left\{ e_\beta \otimes \left\{ \Theta^\beta_\alpha(Z, \overline{W}) + \sum_\gamma \left(\theta^\beta_\gamma(Z)\, \overline{\theta_h}^\gamma_\alpha(\overline{W}) - \overline{\theta_h}^\beta_\gamma(\overline{W})\theta^\gamma_\alpha(Z) \right) \right\} \right\}. \tag{1.10}$$

One final identity we will use following from (1.9) is

$$h(F_h(Z, \overline{Z})s, s) = h(\Theta(Z, \overline{Z})s, s) + \|\overline{\theta_h}(\overline{Z})s\|_h^2 - \|\theta(Z)s\|_h^2. \tag{1.11}$$

One can see an earliest version of this formula in [7, Sect. 7, Lemma 7.18, pp. 271–272]. In the (VHS) context of that paper one would have $F_h = 0$.

If we now endow X with a Hermitian metric g, then use g to take the trace of the identity of (1.11) in the "Z" variables we get

$$h(i\Lambda F_h s, s) = h(i\Lambda\Theta s, s) + \sum_{i=1}^n \left\{ \|\overline{\theta_h}(\overline{Z}_i)s\|_h^2 - \|\theta(Z_i)s\|_h^2 \right\} \tag{1.12}$$

where the $\{Z_i\}_{i=1}^n$ forms an orthonormal basis for $T^{1,0}X$ at a point and also in (1.12) we have used the term $i\Lambda$ as a shorthand for "trace with respect to g over $T^{(1,0)}X$". This can be written, for example, $i\Lambda\Theta = \sum_{i=1}^n \Theta(Z_i, \overline{Z_i})$, where $\{Z_i\}_{i=1}^n$ is a g-orthonormal basis for $T^{(1,0)}X$ at a point, and in general $\Lambda = i\sum g^{i\bar{j}} i(Z_i)i(\overline{Z}_j)$. If g happens to be a Kähler metric then this agrees with the usual symbolism. If s is a holomorphic section of E, then we have the well-known identity ([5, Ch. III, Prop. 1.5, p. 50] and [14, p. 349, (3.30)])

$$h(i\Lambda\Theta s, s) = -i\Lambda\partial\bar{\partial}\,\|s\|_h^2 + \|\nabla s\|_h^2. \tag{1.13}$$

Now (1.12) and (1.13) lead, via the Bochner technique, to the following vanishing result (cf. [14, Theorem 5, pp. 347–349], [5, Ch. III, Theorem 1.9, p. 52] and [7, Lemma (7.18), pp. 271–272]):

Lemma 1. *Suppose X is compact, s is a holomorphic section of E satisfying $\theta s = 0$ and $i\Lambda F_h \leqslant 0$ (pointwise as an endomorphism of E). Then s is parallel $\nabla s = 0$ and satisfies $\overline{\theta_h} s = 0$ and $i\Lambda F_h(s) = 0$. If $i\Lambda F_h$ is a quasinegative operator* ([14], p. 323) *then $s = 0$.*

We will see in Section 3 how Lemma 1 extends to Kodaira–Nakano-type vanishing result for (p, q)-forms with values in a Higgs bundle.

2. Existence and nonexistence of special metrics

Let X be a complex manifold of complex dimension n. Let $E := \bigoplus_{p=0}^n \bigwedge^{(p,0)}(X)$ be the holomorphic vector bundle of forms of degree $(p, 0)$ for all p. Assume X has a holomorphic form ϖ (not everywhere zero) of type $(k, 0)$ where k is odd. We define a Higgs form θ on $E, \theta \in \vartheta\Gamma(\mathrm{Hom}(E) \otimes \Lambda^{1,0}(X))$, by the formula

$$\theta(Z)(\phi) := \{i(Z)\varpi\} \wedge \phi \tag{2.1}$$

where Z is a complex tangent vector to X, $i(Z)$ is interior multiplication by Z, and ϕ is any section of E. One can write θ without referring to a specific complex tangent vector locally by the formula

$$\theta(\phi) := \sum_{i=1}^n \left(\left\{i\left(\frac{\partial}{\partial z_i}\right)\varpi\right\} \wedge \phi\right) \otimes dz_i, \tag{2.2}$$

where $\{\partial/\partial z_i\}$ (dz_i) is a local framing for $T^{(1,0)}(X)$ $(\bigwedge^{(1,0)}(X))$. Formulas (2.1) and (2.2) imply that θ is actually a holomorphic section of $\mathrm{Hom}(E)\otimes\Lambda^{1,0}(X)$ and the condition $[\theta(Z), \theta(W)] = 0$ follows from the assumption that k is odd as follows:

$$[\theta(Z), \theta(W)](\phi) = \{i(Z)\varpi\} \wedge \{i(W)\varpi\} \wedge \phi - \{i(W)\varpi\} \wedge \{i(Z)\varpi\} \wedge \phi = 0$$

because $i(Z)\varpi$ is a form of even degree. This same idea shows that if ϖ is a sum of holomorphic forms of possibly different odd degrees, then (2.1) also defines a Higgs structure on E. If ϖ is a holomorphic k-form, where k is not necessarily assumed to be odd, then a "super Higgs" structure can be defined on E if we define a new bracket operation "$[\cdot\,,\cdot]^{\varpi}$" in $\mathrm{Hom}(E)$ by the prescription

$$[A,B]^{\varpi}(\phi) = \big(AB - (-1^{\deg(\varpi)})BA\big)(\phi).$$

We now examine some examples of this Higgs form for specific values of k in a purely linear algebraic setting. Let $(V_{\mathbb{R}}, J)$ be a real vector space with a complex structure J, $V = V_{\mathbb{R}} \otimes \mathbb{C} = V^{(1,0)} \oplus V^{(0,1)}$ be the complexification and decomposition into $\pm i$ eigenspaces of J. Let $\varpi \in \Lambda^{(k,0)}(V)$ (k odd) and define $\theta \in \mathrm{Hom}\big(\bigoplus_{p\geqslant 0} \Lambda^{(p,0)}(V)\big) \otimes \Lambda^{(1,0)}(V)$ by (2.1). If $k = 1$, then (2.1) yields $\theta(\phi) = \phi \otimes \varpi$, that is, $\theta(Z)(\phi) = \varpi(Z)\phi$. If $k = n$ is odd, then

$$\theta(\phi) = \begin{cases} \phi i(\bullet)\varpi & \text{if } \deg(\phi) = 0, \\ \varpi \otimes \phi & \text{if } \deg(\phi) = 1, \\ 0 & \text{if } \deg(\phi) \geqslant 2. \end{cases}$$

The middle expression means $\theta(Z)(\phi) = \phi(Z)\varpi$ and these formulas follow from $(i(Z)\varpi) \wedge \phi = i(Z)(\varpi \wedge \phi) + \varpi \wedge (i(Z)\phi)$, which is valid for any form ϕ. These examples show the kernel of θ is 0 if $\varpi \neq 0$ in the interesting cases where θ could act nontrivially. In general we have

Proposition 2. (i) *For* $\phi \in \bigoplus_{p\geqslant 0} \Lambda^{(p,0)}(V)$, $\theta(\phi) = 0 \Longleftrightarrow \varpi \wedge \phi = 0$ *and* $\varpi \wedge i(Z)\phi = 0$ $\forall\, Z \in V$.

(ii) *Let h be any Hermitian metric on $\bigoplus_{p\geqslant 0} \Lambda^{(p,0)}(V)$, and let $\overline{\theta_h}$ be the h-adjoint of θ, $h(\overline{\theta_h}(Y)\phi, \psi) = h(\phi, \theta(\bar{Y})\psi)$. Then $\overline{\theta_h}(\bar{Z})\phi = 0\ \forall Z \in V \Longleftrightarrow (\varepsilon(\varpi))^{*_h}\phi = 0$ and $(\varepsilon(\varpi))^{*_h} i(Z)^{*_h}\phi = 0\ \forall Z \in V$ where ^{*_h} means adjoint with respect to h.*

Proof. (i) We need only consider $Z \in V^{(1,0)}$. The formula $(i(Z)\varpi) \wedge \phi = i(Z)(\varpi \wedge \phi) + \varpi \wedge i(Z)\phi$ makes $\Longleftarrow$ clear. If $(i(Z)\varpi) \wedge \phi = 0\ \forall Z$, then $(\varepsilon(\eta) i(Z)\varpi) \wedge \phi = 0\ \forall \eta \in \Lambda^{(1,0)}(V)$. Therefore $0 = \sum_j (\varepsilon(Z_j^*) i(Z_j)\varpi) \wedge \phi$ where $\{Z_j\}$ ($\{Z_j^*\}$) is a basis (dual) for $V^{(1,0)}$ ($\Lambda^{(1,0)}(V)$) but this sum also equals $k\varpi \wedge \phi$ due to the identity $\sum_j \varepsilon(Z_j^*) i(Z_j)\varpi = k\varpi$ which is valid for any $(k,0)$ form (seemingly most easily proved by computing on basis elements $Z_{j_1}^* \wedge Z_{j_2}^* \wedge \cdots \wedge Z_{j_k}^*$). Thus $\theta(\phi) = 0$ implies $\varpi \wedge \phi = 0$. Therefore the assumption in $\Longrightarrow$ yields $0 = (i(Z)\varpi) \wedge \phi = i(Z)(\varpi \wedge \phi) + \varpi \wedge i(Z)\phi = \varpi \wedge i(Z)\phi\ \forall Z$.

(ii) $\overline{\theta_h}(\bar{Z})\phi = 0\ \forall Z \in V \Longleftrightarrow h(\phi, (i(Z)\varpi) \wedge \psi) = 0\ \forall Z$ and $\forall \psi$. Replacing ψ with $\varepsilon(Z^*)\psi$, this implies $h(\phi, \sum_j \varepsilon(Z_j^*)(i(Z_j)\varpi) \wedge \psi) = 0$, so $h(\phi, \varpi \wedge \psi) = 0$, i.e., $h(\varepsilon(\varpi)^{*_h}\phi, \psi) = 0$, and thus $\varepsilon(\varpi)^{*_h}\phi = 0$. The rest is as is in part (i). $\square$

Let us call a (positive definite) Hermitian metric h on $\bigoplus_{p,q\geqslant 0} \Lambda^{(p,q)}(V)$ *standard* if h is the unique extension to $\bigoplus_{p,q\geqslant 0} \Lambda^{(p,q)}(V)$ of a (real) metric on $V_{\mathbb{R}}$ for which J is orthogonal such that $\Lambda^{(p,q)}(V)$ is orthogonal to $\Lambda^{(p',q')}(V)$ if $(p,q) \neq (p',q')$ and $Z_{i_1}^* \wedge Z_{i_2}^* \wedge \cdots \wedge Z_{i_p}^* \wedge \overline{Z_{j_1}^*} \wedge \cdots \wedge \overline{Z_{j_q}^*}$, $1 \leqslant i_1 < \cdots < i_p \leqslant n$, $1 \leqslant j_1 < \cdots < j_q \leqslant n$ is an orthonormal basis for $\Lambda^{(p,q)}(V)$ if $\{Z_j\}_{j=1}^n$ is an orthonormal basis for $V^{(1,0)}$. If h is standard then one has the usual isomorphisms $\# : V^* \to V$ and $\flat : V \to V^*$ and then $\varepsilon(\varpi)^{*h} = i(\varpi^{\#})$ and $i(Z)^{*h} = \varepsilon(Z^{\flat})$. One proves the following statement:

$$\text{For a standard } h,\ \overline{\theta_h}(\phi) = 0 \Longleftrightarrow i(\varpi^{\#})\phi = 0 \quad\text{and}\quad \varpi \wedge \varepsilon(Z^{\flat})\phi = 0\ \ \forall Z \in V. \tag{2.3}$$

Let us call a (positive definite) Hermitian metric h on $\bigoplus_{p\geqslant 0}\Lambda^{(p,0)}(V)$ *natural* if h makes $\Lambda^{(p,0)}(V)$ orthogonal to $\Lambda^{(p',0)}(V)$ if $p\neq p'$. One verifies that

$$h \text{ natural on } \bigoplus_{p\geqslant 0}\Lambda^{(p,0)}(V) \quad\Longrightarrow\quad \overline{\theta_h}:\Lambda^{(a,0)}(V)\to\Lambda^{(a-k+1,0)}(V)\otimes\Lambda^{(1,0)}(V). \tag{2.4}$$

For use later in giving examples of Kähler manifolds which admit Higgs–Hermitian–Yang–Mills metrics (2.12) we now give a formula for the linear algebraic operator $T_h(s)$ defined by

$$T_h(s):=\sum_{n}^{i=1}\big[\theta(Z_i),\bar\theta_h(\overline{Z_i})\big]s,$$
$$h(T_h(s),s)=\big\{\|\overline{\theta_h}(\bar Z_i)s\|_h^2-\|\theta(Z_i)s\|_h^2\big\} \tag{2.5}$$

(cf. (2.10)), in the case where the Hermitian metric h on $\bigoplus_{p,q\geqslant 0}\Lambda^{(p,q)}(V)$ is standard, $s\in\bigoplus_{p\geqslant 0}\Lambda^{(p,0)}(V)$ and the θ operator is defined using an element $\varpi=aZ_1^*\wedge Z_2^*\wedge\cdots\wedge Z_n^*\in\Lambda^{(n,0)}(V)$ (n odd, >1) where $\{Z_j\}_{j=1}^n$ is an orthonormal basis for $V^{(1,0)}$. In this setting, one has the identity $i((Z_{i_1}^*\wedge Z_{i_2}^*\wedge\cdots\wedge Z_{i_p}^*)^\#)=i(Z_{i_p})i(Z_{i_{p-1}})\cdots i(Z_{i_1})$ for any p. If $\deg s\geqslant 2$ then $\theta(Z_i)s=\{i(Z_i)\varpi\}\wedge s=0$ and if $\deg s\leqslant n-2$ then $\overline{\theta_h}(\bar Z_i)s=i(\pm(aZ_1^*\wedge Z_2^*\wedge\cdots\wedge\widehat{Z_i^*}\wedge\cdots\wedge Z_n^*)^\#)s=0$. Thus $T_h(s)=0$ if $2\leqslant\deg s\leqslant n-2$. It is straightforward to check that $h(T_h(s),s)=f[i]\,\|\varpi\|_h^2\,\|s\|_h^2$, where $s\in\Lambda^{(i,0)}(V)$, with $f[i]=0$, if $2\leqslant i\leqslant n-2$, $f[0]=-n$, $f[1]=-1$, $f[n]=n$ and (because T_h must have trace 0, or by similar computations) $f[n-1]=1$. By polarizing, we get

$$\varpi\in\Lambda^{(n,0)}(V),\, s\in\Lambda^{(i,0)}(V)\quad\Rightarrow\quad T_h(s)=\|\varpi\|_h^2\,f[i]s, \tag{2.6}$$
$$f[i]=\begin{cases}-n & \text{if } i=0,\\ -1 & \text{if } i=1,\\ 0 & \text{if } 2\leqslant i\leqslant n-2,\\ 1 & \text{if } i=n-1,\\ n & \text{if } i=n.\end{cases}$$

We remark that one can prove the following identity: if (V,h) are as above, but now $\varpi=aZ_{i_1}^*\wedge Z_{i_2}^*\wedge\cdots\wedge Z_{i_k}^*\in\Lambda^{(k,0)}(V)$ is a simple $(k,0)$-form (k odd), then

$$T_h(s)=-\|\varpi\|_h^2\Big\{ks+\sum_{r=1}^{\min(k-2,\deg s)}(k-r)(-1)^r$$
$$\times\sum_{1\leqslant t_1<t_2<\cdots t_r\leqslant k}\varepsilon(Z_{i_{t_1}}^*\wedge Z_{i_{t_2}}^*\wedge\cdots\wedge Z_{i_{t_r}}^*)i((Z_{i_{t_1}}^*\wedge Z_{i_{t_2}}^*\wedge\cdots\wedge Z_{i_{t_r}}^*)^\#)s\Big\}.$$

Consequently if $s\in\Lambda^{(i,0)}(\operatorname{span}\{Z_{i_1}^*,Z_{i_2}^*,\ldots,Z_{i_k}^*\}^\perp)$, then $T_h(s)=-k\|\varpi\|_h^2 s$, while if $s\in\Lambda^{(i,0)}(\operatorname{span}\{Z_{i_1}^*,Z_{i_2}^*,\ldots,Z_{i_k}^*\})$, then one can show $T_h(s)=\|\varpi\|_h^2F[i]s$, with $F[0]=-k$, $F[1]=-1$, $F[k-1]=1$, $F[0]=k$, $F[i]=0$ if $2\leqslant i\leqslant k-2$.

Now consider again the differential geometric setting described in the beginning of Section 2. $E\to X$ is the holomorphic vector bundle $E=\bigoplus_{p\geqslant 0}\Lambda^{(p,0)}(X)$, $\varpi\in\vartheta\Gamma\Lambda^{(k,0)}(X)$, and θ the Higgs form defined by (2.1). Let h be any Hermitian metric on E and let g be any Hermitian metric on $TX \otimes \mathbb{C}$ (we do not assume any a priori relation between g and h). In this case formula (1.11) becomes

$$\begin{aligned} &h(F_h(Z,\overline{Z})s,s) \\ &\quad = h(\Theta(Z,\overline{Z})s,s) + \|(\varepsilon(i(Z)\varpi))^{*_h}s\|_h^2 - \|(i(Z)\varpi)\wedge s\|_h^2. \end{aligned} \tag{2.7}$$

Remark 1. 1. *If* $k = \deg\varpi = 1$, *then* $(i(Z)\varpi)\wedge s = (i(Z)\varpi)s$, $(\varepsilon(i(Z)\varpi))^{*_h}s = \overline{(i(Z)\varpi)}s$ (*even if* h *is not natural*) *and then* (2.7) *becomes* $h(F_h(Z,\overline{Z})s,s) = h(\Theta(Z,\overline{Z})s,s)$. *In fact, the operator corresponding to* $\theta\wedge\overline{\theta_h} + \overline{\theta_h}\wedge\theta$ (*cf.* (1.7)) *is zero. Partly this reason we will assume* $k \geqslant 3$ *unless specified otherwise. Another reason for assuming* $k \geqslant 3$ *is that we want to consider solutions to the equation* $(i(Z)\varpi)\wedge s = 0\ \forall Z$ (*locally defined*) *holomorphic tangent vector fields, and* $s \in \vartheta\Gamma E$. *If* ϖ *is a* 1*-form, then this would imply either* $\varpi = 0$ *or* $s = 0$.

2. *Note that* F_h *does not annihilate functions on* X, *i.e., sections of* $\Lambda^{(0,0)}(X)$, *unlike* Θ. *In particular, we conclude for the constant section* $1 \in \Gamma\Lambda^{(0,0)}(X)$, *that* $h(F_h(Z,\overline{Z})1,1) = \|(\varepsilon(i(Z)\varpi))^{*_h}1\|_h^2 - \|(i(Z)\varpi)\|_h^2$ *and if* h *is natural, then* $h(F_h(Z,\overline{Z})1,1) = -\|(i(Z)\varpi)\|_h^2$.

If $\deg s \geqslant n-k+2$, then $(i(Z)\varpi)\wedge s = 0$. If h is a natural metric then $\deg(\varepsilon(i(Z)\varpi))^{*_h}s = \deg s - k + 1$ and hence if $\deg s \leqslant k-2$ then $(\varepsilon(i(Z)\varpi))^{*_h}s = 0$. Now $k-2 \geqslant n-k+2 \to k \geqslant \frac{1}{2}n+2$, so for this range of k both of the last two terms on the right-hand side of (2.7) vanish. We summarize these observations below.

$$h(F_h(Z,\overline{Z})s,s) \geqslant h(\Theta(Z,\overline{Z})s,s) \qquad \forall s \in C^\infty\Gamma \bigoplus_{a\geqslant n-k+2}\Lambda^{(a,0)}(X), \tag{2.8}$$

$$\begin{aligned} &h \text{ natural} \quad \Rightarrow \quad h(F_h(Z,\overline{Z})s,s) \leqslant h(\Theta(Z,\overline{Z})s,s) \\ &\qquad\qquad\qquad \forall s \in C^\infty\Gamma \bigoplus_{a\leqslant k-2}\Lambda^{(a,0)}(X), \\ &h \text{ natural},\quad k \geqslant \tfrac{1}{2}n+2 \quad \Rightarrow \quad F_h(Z,\overline{Z})\,s = \Theta(Z,\overline{Z})s \\ &\qquad\qquad\qquad\qquad \forall s \in C^\infty\Gamma \bigoplus_{n-k+2\leqslant a\leqslant k-2}\Lambda^{(a,0)}(X). \end{aligned} \tag{2.9}$$

Theorem 1. *Assume* (X,g) *is a compact Hermitian manifold of complex dimension* n, $E \to X$ *is the Higgs bundle given by* (2.1) *and* h *is a Hermitian metric on* E. *If for all sections* t *of* E, $0 \geqslant h(i\Lambda F_h t,t)$, *pointwise, then*

$$\begin{aligned} s \in \vartheta\Gamma\bigoplus_{a\geqslant n-k+2}\Lambda^{(a,0)}(X) \quad &\Rightarrow \quad \nabla^h s \equiv 0, \quad (\varepsilon(i(Z)\varpi)^{*_h}s \equiv 0 \\ &\qquad \forall Z \text{ and } i\Lambda F_h(s) \equiv 0. \end{aligned}$$

If $0 \geqslant h(i\Lambda F_h t,t)$ *for all sections* t *of* E *and* $k \geqslant \frac{1}{2}n+1$, *then* $\varpi = 0$.

Proof. (1.12) in this setting can be written

$$h(i\Lambda F_h s,s) - h(i\Lambda\Theta s,s) = \sum\left\{\|(\varepsilon(i(Z_i)\varpi))^{*_h}s\|_h^2 - \|\{i(Z_i)\varpi\}\wedge s\|_h^2\right\}. \tag{2.10}$$

The argument of Lemma 1 implies the result in the first line of the theorem, because $\deg(s) \geqslant n-k+2$ implies $\{i(Z)\varpi\}\wedge s = 0\ \forall Z$. To prove the second statement, note that $k \geqslant \frac{1}{2}n + 1$ implies that $\deg(\varpi) = k \geqslant n - k + 2$ so we can use the first argument to conclude that (ϖ is h-parallel and) $(\varepsilon(i(Z)\varpi))^{*_h}\varpi = 0\ \forall Z$. From the second part of Proposition 2 we get $(\varepsilon(\varpi))^{*_h}\varpi = 0$. This yields $h((\varepsilon(\varpi))^{*_h}\varpi, t) = 0$ for all sections t of E, and taking $t = 1$ implies $\|\varpi\|_h^2 = 0$. $\square$

If $i\Lambda F_h$ is quasinegative and s is as in Theorem 1 then $s = 0$.

In the context of general Higgs bundles, the vanishing of the $F_h^{(2,0)}$ and $F_h^{(0,2)}$ is equivalent to the Higgs form being parallel (cf. Proposition 1). The next result examines the case of those Higgs bundles defined by 2.1, and with a special metric. The result will be used at the end of Section 3 in a vanishing theorem for (p, q) forms with values in E.

Proposition 3. *Let (X, g) be a Kähler manifold, and extend g to a standard metric on $\Lambda^*(X) \otimes \mathbb{C}$. In the above notation let the metric h on $E := \bigoplus_{p=0}^n \bigwedge^{(p,0)}(X)$ be $h = g$. Then $F_h^{(2,0)} = 0 \Longleftrightarrow \nabla\varpi = 0$ ($\Leftrightarrow F_h^{(0,2)} = 0$ by Proposition 1).*

Proof. From Proposition 1 it follows that on any Higgs bundle (E, θ) with a Hermitian metric h, $F_h^{(2,0)} = 0$ at a point $p \Leftrightarrow \partial\theta_s^t = 0$ at p where $\theta e_\alpha = \sum_\beta e_\beta \otimes \theta_\alpha^\beta$ for a local frame $\{e_\alpha\}_{\alpha=1}^r$ of E adapted to p. In the case we are considering, let $\{e_\alpha\}_{\alpha=1}^r$ ($r = 2^n$) be a local frame of $E = \bigoplus_{p=0}^n \Lambda^{(p,0)}(X)$ h-adapted to p. Then throughout the neighborhood where $\{e_\alpha\}_{\alpha=1}^r$ is defined we can write

$$\theta e_\alpha = \sum_{i=1}^n \big(\{i(\partial/\partial z_i)\varpi\} \wedge e_\alpha\big) \otimes dz_i = \sum_{\beta=1}^r \Big(e_\beta \otimes \sum_{i=1}^n A_{i,\alpha}^\beta dz_i\Big)$$

where

$$\{i(\partial/\partial z_i)\varpi\} \wedge e_\alpha = \sum_{\beta=1}^r A_{i,\alpha}^\beta e_\beta.$$

Therefore we have $\theta_\alpha^\beta = \sum_{i=1}^n A_{i,\alpha}^\beta dz_i$ and $\partial\theta_\alpha^\beta = \sum_{i=1}^n \partial A_{i,\alpha}^\beta \wedge dz_i$. One computes that $A_{i,\alpha}^\beta = \sum_{\gamma=1}^r h^{\beta\bar\gamma} h\big(\{i(\partial/\partial z_i)\varpi\} \wedge e_\alpha, e_\gamma\big)$, where $h_{a\bar b} = h(e_a, e_b)$ and $(h^{s\bar t}) = (h_{a\bar b})^{-1}$. Up to this point we have not used the assumption that h is a Kähler metric. We now exploit this assumption by writing $\partial = \sum_j \varepsilon(\partial/\partial z_j)\nabla_{\partial/\partial z_j}$ where $\{\partial/\partial z_j\}_{j=1}^n$ is a local holomorphic frame for $T^{(1,0)}(X)$ which is also h-adapted to p. Then, at p, the following equalities hold

$$\begin{aligned}
\partial\theta_\alpha^\beta &= \sum_{i,j} \frac{\partial}{\partial z_j} h\Big(\Big\{i\Big(\frac{\partial}{\partial z_i}\Big)\varpi\Big\} \wedge e_\alpha, e_\beta\Big) dz_j \wedge dz_i \\
&= \sum_{i,j} h\Big(\nabla_{\partial/\partial z_j}\Big(\Big\{i\Big(\frac{\partial}{\partial z_i}\Big)\varpi\Big\} \wedge e_\alpha\Big), e_\beta\Big) dz_j \wedge dz_i \\
&= \sum_{i,j} h\Big(\Big(\nabla_{\partial/\partial z_j}\Big\{i\Big(\frac{\partial}{\partial z_i}\Big)\varpi\Big\}\Big) \wedge e_\alpha, e_\beta\Big) dz_j \wedge dz_i \\
&= \sum_{i,j} h\Big(\Big(i\Big(\frac{\partial}{\partial z_i}\Big)\nabla_{\partial/\partial z_j}\varpi\Big) \wedge e_\alpha, e_\beta\Big) dz_j \wedge dz_i \\
&= \sum_{i<j} h\Big(\Big[i\Big(\frac{\partial}{\partial z_i}\Big)\nabla_{\partial/\partial z_j}\varpi - i\Big(\frac{\partial}{\partial z_j}\Big)\nabla_{\partial/\partial z_i}\varpi\Big] \wedge e_\alpha, e_\beta\Big) dz_j \wedge dz_i.
\end{aligned}$$

We conclude:

$$\partial\theta_\alpha^\beta(p) = 0 \Leftrightarrow h\Big(\Big[i\frac{\partial}{\partial z_i}\nabla_{\partial/\partial z_j}\varpi - i\frac{\partial}{\partial z_j}\nabla_{\partial/\partial z_i}\varpi\Big] \wedge e_\alpha, e_\beta\Big)(p) = 0 \qquad \forall i < j.$$

Thus $\nabla\varpi = 0 \Rightarrow \partial\theta_\alpha^\beta = 0$ and hence that $F_h^{(2,0)} = 0$. Conversely, $F_h^{(2,0)} = 0 \Rightarrow \partial\theta_\alpha^\beta(p) = 0 \Rightarrow h\big(\big[i(\partial/\partial z_i)\nabla_{\partial/\partial z_j}\varpi - i(\partial/\partial z_j)\nabla_{\partial/\partial z_i}\varpi\big]\wedge e_\alpha, e_\beta\big)(p) = 0$ for any adapted frame $\{e_\alpha\}_{\alpha=1}^r$ and we can conclude that $F_h^{(2,0)} = 0 \Rightarrow \big(i(\partial/\partial z_i)\nabla_{\partial/\partial z_j}\varpi - i(\partial/\partial z_j)\nabla_{\partial/\partial z_i}\varpi\big)(p) = 0\ \forall i, j$, e.g., by choosing $e_\alpha = 1 \in \Lambda^{(0,0)}(X) \subset E$. Finally we have

$$\begin{aligned}
i\Big(\frac{\partial}{\partial z_i}\Big)\nabla_{\partial/\partial z_j}\varpi &= i\Big(\frac{\partial}{\partial z_j}\Big)\nabla_{\partial/\partial z_i}\varpi \\
&\Rightarrow \sum_i \varepsilon(dz_i)i\Big(\frac{\partial}{\partial z_i}\Big)\nabla_{\partial/\partial z_j}\varpi = \sum_i \varepsilon(dz_i)i\Big(\frac{\partial}{\partial z_j}\Big)\nabla_{\partial/\partial z_i}\varpi \\
&\Rightarrow k\nabla_{\partial/\partial z_j}\varpi = \sum_i \varepsilon(dz_i)i\Big(\frac{\partial}{\partial z_j}\Big)\nabla_{\partial/\partial z_i}\varpi \\
&\Rightarrow k\sum_j \varepsilon(dz_j)\nabla_{\partial/\partial z_j}\varpi = \sum_{i,j}\varepsilon(dz_j)\varepsilon(dz_i)i\Big(\frac{\partial}{\partial z_j}\Big)\nabla_{\partial/\partial z_i}\varpi \\
&\Rightarrow k\partial\varpi = -\sum_{i,j}\varepsilon(dz_i)\varepsilon(dz_j)i\Big(\frac{\partial}{\partial z_j}\Big)\nabla_{\partial/\partial z_i}\varpi \\
&\Rightarrow k\partial\varpi = -k\partial\varpi \Rightarrow \partial\varpi = 0.
\end{aligned}$$

From the third line above we also get $k\nabla_{\partial/\partial z_j}\varpi = \nabla_{\partial/\partial z_j}\varpi - \sum_i i(\partial/\partial z_j)\varepsilon(dz_i)\nabla_{\partial/\partial z_i}\varpi = \nabla_{\partial/\partial z_j}\varpi - i(\partial/\partial z_j)\partial\varpi = \nabla_{\partial/\partial z_j}\varpi$ because $\partial\varpi = 0$. Finally we conclude $(k-1)\nabla_{\partial/\partial z_j}\varpi = 0$, and hence $\nabla\varpi = 0$. $\square$

In preparation for the examination of stability questions for the Higgs bundle E, we consider general Higgs subbundles of E. Suppose $P \subset E$ is a Higgs subbundle of E. This means that if s is a local section of P, then $(i(Z)\varpi) \wedge s$ is also a local section of P. If h is a Hermitian metric on P, then (2.7) and (2.10) apply to the Higgs bundle P with the Hermitian metric h. Additionally, the proof of Theorem 1 works as well in this setting, which we include as a

Remark 2. *Let $P \subset E$ be a Higgs subbundle and let h be a Hermitian metric on P, so all h Hermitian data applies to P. If $0 \geqslant i\Lambda F_h$, i.e., $i\Lambda F_h$ is a pointwise negative semidefinite operator, then any holomorphic section s of P which is a $(p,0)$-form with $p \geqslant n-k+2$ (or a sum of such forms) must be parallel for the Hermitian connection of h. If $i\Lambda F_h$ is quasinegative, then any such s must be* 0.

We now examine the question of stability for the Higgs bundles defined by (2.1). Assume (X, g) is a compact Kähler manifold. If $E \to X$ is any holomorphic vector bundle over X, then E is said to be *stable* (*semistable*) ([5], Ch. V, Sect. 5–7) if for every nontrivial coherent analytic subsheaf $\mathfrak{F}$ of the sheaf $\vartheta(E)$ of germs of holomorphic sections of E the following inequality holds

$$\mu(\mathfrak{F}) < (\leqslant)\ \mu(E). \tag{2.11}$$

If $F \subset E$ is any holomorphic subbundle of E, $\mu(F)$ (the "slope" of F) is defined to be $\mu(F) = \deg(F)/\mathrm{rank}(F) = \left(\int_X c_1(F) \wedge \omega^{n-1}\right)/\mathrm{rank}(F)$, ω being the Kähler form of g and $c_1(F)$ the first Chern class of F. If $\mathfrak{F}$ is the sheaf $\vartheta(F)$ of germs of holomorphic sections of F, $\mu(\mathfrak{F})$ means $\mu(F)$. A coherent subsheaf of $\vartheta(E)$ need not arise as the sheaf of germs of such a subbundle F, i.e., $\mathfrak{F}$ need not be locally free. Nevertheless, there is a well-defined rank for $\mathfrak{F}$, because $\mathfrak{F}$ is locally free outside a set of codimension at least 2. There is also a holomorphic line bundle associated to $\mathfrak{F}$, "$\det(\mathfrak{F})$" and one defines $c_1(\mathfrak{F})$ to be $c_1(\det(\mathfrak{F}))$ and $\mu(\mathfrak{F}) = \left(\int_X c_1(\mathfrak{F}) \wedge \omega^{n-1}\right)/\mathrm{rank}(\mathfrak{F})$. In case $\mathfrak{F}$ arises from a vector bundle F these definitions agree with the standard vector bundle ones.

A Hermitian metric h on the holomorphic vector bundle E over (X, g) is said to be an Einstein–Hermitian metric ([5, Chapter IV]) or a Hermitian–Yang–Mills (HYM) metric ([12]) if $i\Lambda\Theta = c\,\mathrm{Id}_E$, Θ being the Hermitian curvature of h, and where c is a constant determined by the rank and degree of E and the (class of) the Kähler form of g (cf. [5, Ch. IV, Sect. 2]). If E admits such a metric h, then E is semistable and splits into a direct sum of holomorphic, stable subbundles with the same slope ([5, Ch. V, Sect. 8, Theorem 8.3]). The converse theorem conjectured by Kobayashi was proved in [12]: A stable holomorphic vector bundle over a compact Kähler manifold admits a unique Hermitian–Yang–Mills metric.

In the category of Higgs bundles over compact Kähler manifolds, $(E, \theta) \to (X, g)$, E is said to be *Higgs stable* (*Higgs semistable*) ([9]) if for every nontrivial coherent analytic subsheaf $\mathfrak{F}$ satisfying $\theta : \mathfrak{F} \to \mathfrak{F} \otimes \vartheta(\Lambda^{(1,0)}(X))$ (i.e. a Higgs subsheaf) the inequality in (2.11) holds. A Hermitian metric h on the Higgs bundle $(E, \theta) \to (X, g)$ is said to be a (Higgs–)Hermitian–Yang–Mills (HHYM) metric ([11]) if $i\Lambda F_h = c\,\mathrm{Id}_E$, where F_h is defined in (1.4). Again it is true that if (E, θ) admits such a metric h, then (E, θ) is (Higgs) semistable and splits into a direct sum of holomorphic, (Higgs) stable subbundles with the same slope ("polystability" [11, Theorem 1, p. 19]), because the proof of [5, Ch. V, Sect. 8, Theorem 8.3] can be modified for the Higgs category, and the inequalities still go the right way. The converse of this theorem, for compact and certain classes of noncompact Kähler manifolds, is due to Simpson ([9], see also [11]) and plays an important part in the results described at the beginning of Section 1.

One would like to know when an HHYM h exists for the Higgs bundles defined by (2.1) for a X a compact Kähler manifold. The results we present below (2) indicate that such metrics may be quite rare for such X. In order to get some information about such metrics we give examples of HHYM metrics in noncompact cases where there are no topological or complex-analytic obstructions to their existence.

Let (X, g) be complex n-dimensional with Kähler metric g. Assume the following properties are satisfied:

(i) g is Kähler-Einstein.

(ii) Given any constant C, there is a smooth function $f : X \to \mathbb{C}$ such that $\bar{\Box}_g(f) = C$.

(iii) $\varpi \in \Lambda^{(n,0)}(X)$ is a holomorphic n-form with constant g-length.

Then the Higgs bundle $\left(E = \bigoplus_{p \geqslant 0} \Lambda^{(p,0)}(X), \theta\right)$ admits a HHYM metric g', $i\Lambda F_{g'} = c\,\mathrm{Id}_E$ with any number c (note that the Λ in $i\Lambda F_{g'}$ refers to interior multiplication by the g-dual to the g-Kähler form, we use g for all Riemannian data on X). In fact we will now show that such a g' can be obtained by taking the standard extension of g to E and changing it conformally on each $\Lambda^{(p,0)}(X)$ (with a conformal factor depending on p). $X = \mathbb{C}^n$ with the standard metric and a constant coefficient $(n,0)$-form ϖ is of course an example of such a manifold, and $f(z_1, z_2, \ldots, z_n) = (C/n)\sum_i |z_i|^2 +$ any harmonic function yields condition (ii). That condition excludes the possibility of X being compact.

For (X, g) satisfying (i), (ii) and (iii), say that $\mathrm{Ric}(g, T^{(1,0)}(X)) = -\lambda\,\mathrm{Id}_{T^{(1,0)}(X)}$, so for the induced action of the Ricci curvature on $\Lambda^{(1,0)}(X)$, $\mathrm{Ric}(g, \Lambda^{(1,0)}(X)) = \lambda\,\mathrm{Id}_{\Lambda^{(1,0)}(X)}$. Extend g as a standard Hermitian metric to E. Then the curvature Θ_g of the corresponding Hermitian connection on E then satisfies $i\Lambda\Theta_g = \binom{n-1}{p-1}\lambda\,\mathrm{Id}_{\Lambda^{(p,0)}(X)}$ on $\Lambda^{(p,0)}(X)$. The decomposition $E = \bigoplus_{p\geqslant 0}\Lambda^{(p,0)}(X)$ is g-orthogonal, hence each factor $\Lambda^{(p,0)}(X)$ is invariant under the Hermitian connection ∇^g, i.e., each factor is totally geodesic. Thus $i\Lambda\Theta_{(\Lambda^{(p,0)}(X), g|_{\Lambda^{(p,0)}(X)})} = i\Lambda\Theta_g\,|_{\Lambda^{(p,0)}(X)} = \binom{n-1}{p-1}\lambda\,\mathrm{Id}_{\Lambda^{(p,0)}(X)}$. Let g' be the Hermitian metric on E uniquely determined by the requirement that $E = \bigoplus_{p\geqslant 0}\Lambda^{(p,0)}(X)$ is g'-orthogonal and $g' = e^{f_p}g$ on $\Lambda^{(p,0)}(X)$, where f_p is a smooth function on X to be determined. The decomposition $E = \bigoplus_{p\geqslant 0}\Lambda^{(p,0)}(X)$ is g'-orthogonal and totally geodesic and we conclude that for g' we have $i\Lambda\Theta_{g'}|_{\Lambda^{(p,0)}(X)} = i\Lambda\Theta_{(\Lambda^{(p,0)}(X), g'|_{\Lambda^{(p,0)}(X)})} = (\bar{\Box}_g(f_p) + \binom{n-1}{p-1}\lambda)\,\mathrm{Id}_{\Lambda^{(p,0)}(X)}$ on $\Lambda^{(p,0)}(X)$.

For the Higgs structure θ defined by the form ϖ one checks that $\bar\theta_g = \bar\theta_{g'}$ because one has just changed the metric conformally on each of the orthogonal subspaces of E. Consequently we have $T_g = T_{g'}$ for the operator defined as in (2.5). Also because $E = \bigoplus_{p\geqslant 0}\Lambda^{(p,0)}(X)$ is both g and g' orthogonal, $T_g = T_{g'} : \Lambda^{(p,0)}(X) \to \Lambda^{(p,0)}(X)\ \forall p$. Because $i\Lambda F_g = i\Lambda\Theta_g + T_g$, combining all these observations yields

$$i\Lambda F_{g'}|_{\Lambda^{(p,0)}(X)} = \left(\bar{\Box}_g(f_p) + \binom{n-1}{p-1}\lambda\right)\mathrm{Id}_{\Lambda^{(p,0)}(X)} + T_g|_{\Lambda^{(p,0)}(X)}\,. \tag{2.12}$$

Now using the $(n,0)$-form ϖ, which we can assume has pointwise length 1, we see from (2.6) that $T_g\,|_{\Lambda^{(p,0)}(X)} = f[p]\,\mathrm{Id}_{\Lambda^{(p,0)}(X)}$ in the notation of (2.6). Because of assumption (ii) above, we can find, for each p, and for any constant C, a function f_p such that

$$\bar{\Box}_g(f_p) = C - \binom{n-1}{p-1}\lambda - f[p]. \tag{2.13}$$

Hence with such a choice of f_p for each p, the corresponding Hermitian metric g' on E is Higgs–Hermitian–Yang–Mills, with constant C. Note that in general g' (restricted to $\Lambda^{(1,0)}(X)$ and then defined on $T^{(1,0)}(X)$ by g'-duality), will not be a Kähler metric.

We now return to the question of the existence of HHYM metrics in the case where X a compact Kähler manifold. We do not have any examples of such metrics for the Higgs bundles defined by 2.2. In fact, the results we prove below on the nonexistence of such metrics came about as obstructions to such metrics in our investigations of this question. We need formulas for $c_1(F)$ for various subbundles of E. If X is any complex manifold of complex dimension n. Then there is the well-known formula

$$c_1(\Lambda^{(p,0)}(X)) = \binom{n-1}{p-1} c_1(\Lambda^{(1,0)}(X)). \tag{2.14}$$

If $p = 0$, we interpret $\binom{n-1}{p-1}$ to mean 0, so the formula is correct in this case, too. If we now assume X is a compact Kähler manifold, then it follows from (2.14) that $\deg(\Lambda^{(p,0)}(X)) = \binom{n-1}{p-1}\deg(\Lambda^{(1,0)}(X))$ and $\mu(\Lambda^{(p,0)}(X)) = p\mu(\Lambda^{(1,0)}(X))$.

Considering $E = \bigoplus_{p=0}^{n} \Lambda^{(p,0)}(X)$ as a Higgs bundle via (2.1), there are a large number of Higgs subbundles, hence Higgs subsheaves of ϑE. For example, $\bigoplus_{p \geqslant 0} \Lambda^{(2p,0)}(X) =: \Lambda^{\text{even}}$ and $\bigoplus_{p\geqslant 0} \Lambda^{(2p+1,0)}(X) =: \Lambda^{\text{odd}}$ are both Higgs subbundles of E. Also, using the fact that the first Chern class is additive over direct sums of bundles, one computes that $c_1(E) = 2^{n-1}c_1(\Lambda^{(1,0)})$. One computes $\mu(E) = \frac{1}{2}n\mu(\Lambda^{(1,0)})$, $c_1(\Lambda^{\text{even}}) = 2^{n-2}c_1(\Lambda^{(1,0)}) = c_1(\Lambda^{\text{odd}})$, and finally $\mu(\Lambda^{\text{even}}) = \frac{1}{2}n\mu(\Lambda^{(1,0)}) = \mu(\Lambda^{\text{odd}}) = \mu(E)$. One gets a Higgs "filtration" $\{E^a\}_{a=0}^{n}$ (i.e., a filtration by Higgs subbundles) of E as follows:

$$\begin{aligned} &0 \subset E^n \subset E^{n-1} \subset \cdots \subset E^1 \subset E^0 = E, \quad \text{where} \\ &E^a = \bigoplus_{p\geqslant a} \Lambda^{(p,0)}(X). \end{aligned} \tag{2.15}$$

This filtration also gives Higgs filtrations of $\Lambda^{\text{even (odd)}}$ by $\Lambda^{\text{even (odd)}} \cap E^a$. Writing $k = 2b+1$, each of the subbundles

$$\bigoplus_{p\geqslant 0} \Lambda^{(2bp+i,0)}(X), \qquad i = 1, 2, \ldots, 2b-1 \tag{2.16}$$

are Higgs subbundles of Λ^{even} or Λ^{odd}, and intersecting with the Higgs filtration gives more Higgs subbundles.

We now investigate the of the stability of some of these Higgs bundles. If $V \to X$ is a stable holomorphic vector bundle, then V cannot split holomorphically and nontrivially ($V = V_1 \oplus V_2$ holomorphic implies one $V_i = 0$), i.e., V is irreducible.

This irreducibility result also holds for Higgs bundles: if E is Higgs stable and $E = E_1 \oplus E_2$ with E_1 and E_2 Higgs, then one of these subbundles is 0 (and the proof follows [5, Lemma (7.3), Ch. 5., Sect. 7]). As a result, for the E defined by (2.1), if F is a Higgs subbundle of E for which there is a nontrivial splitting of the form $F = F \cap \Lambda^{\text{even}} \bigoplus F \cap \Lambda^{\text{odd}}$, then F cannot be Higgs stable (or "plain" stable). This type of splitting occurs in the bundles in the Higgs filtration (2.15). In particular, none of the E^a, $a = 0, 1, \ldots, n-1$ can be stable, although E^n, being a line bundle, is stable (cf. [5, Prop. (7.7), p. 170]).

It is natural to ask if any of the these Higgs subbundles could be semistable. If any of the components of the Higgs filtration were semistable (stable), say E^a, then $\mu(E^b) \leqslant (<) \mu(E^a)$ $\forall b > a$. The next result shows that if $\deg(\Lambda^{(1,0)}(X)) > 0$, then the bundles E^a, $E^a \cap \Lambda^{\text{even}}$, $E^a \cap \Lambda^{\text{odd}}$ and others cannot be semistable in the "ordinary" sense where no Higgs structure is assumed (again excluding the automatic case E^n, which is a line bundle and hence stable), and will be used to show that many of these bundles cannot admit HHYM metrics.

Proposition 4. *Let $d = \deg(\Lambda^{(1,0)}(X))$, let P be the holomorphic subbundle of $\bigoplus_{i=0}^{n} \Lambda^{(i,0)}(X)$ given by $P = \bigoplus_{s=1}^{z} \Lambda^{(p_s,0)}(X)$, $0 \leqslant p_1 < p_2 < \cdots < p_z \leqslant n$, and let Q be the holomorphic subbundle of P given by $Q = \bigoplus_{t=1}^{l} \Lambda^{(q_t,0)}(X)$, $p_1 \leqslant q_1 < q_2 < \cdots < q_l \leqslant p_z$, $\{q_1, q_2, \ldots, q_l\} \subset \{p_1, p_2, \ldots, p_z\}$. If Q is the "tail" of P, $q_i = p_{z-l+i}$, $i = 1, \ldots, l$, then $\mu(P) > \mu(Q) \Leftrightarrow d < 0$, $\mu(P) = \mu(Q) \Leftrightarrow d = 0$.*

Proof. We will show that $\mu(P) - \mu(Q) = d\ c(p_1, \ldots, p_z; q_1, \ldots, q_l)$ where $c(p_1, \ldots, p_z; q_1, \ldots, q_l)$ is a rational number which is strictly negative when $q_i = p_{z-l+i}$, $i = 1, \ldots, l$.

Write $\{p_1, p_2, \ldots, p_z\} = \{\{q_1, q_2, \ldots, q_l\}, \{r_1, r_2, \ldots, r_{z-l}\}\}$, $r_1 < r_2 < \cdots < r_{z-l}$. Using the formula (2.14) one computes

$$\mu(P) = \frac{\sum_{s=1}^{z} \binom{n-1}{p_s-1} d}{\sum_{s=1}^{z} \binom{n}{p_s}},$$

$$\begin{aligned}
\mu(P) - \mu(Q) &= \frac{d}{\mathrm{rk}(P)\,\mathrm{rk}(Q)} \left(\sum_{s=1}^{z} \binom{n-1}{p_s-1} \sum_{t=1}^{l} \binom{n}{q_t} - \sum_{t=1}^{l} \binom{n-1}{q_t-1} \sum_{s=1}^{z} \binom{n}{p_s} \right) \\
&= \frac{d}{\mathrm{rk}(P)\,\mathrm{rk}(Q)} \left(\sum_{b=1}^{z-l} \binom{n-1}{r_b-1} \sum_{t=1}^{l} \binom{n}{q_t} - \sum_{t=1}^{l} \binom{n-1}{q_t-1} \sum_{b=1}^{z-l} \binom{n}{r_b} \right) \qquad (2.17) \\
&= \frac{d}{\mathrm{rk}(P)\,\mathrm{rk}(Q)} \sum_{\substack{1 \leqslant t \leqslant l \\ 1 \leqslant b \leqslant z-l}} \binom{n-1}{r_b-1} \binom{n-1}{q_t-1} n \left\{ \frac{r_b - q_t}{r_b\, q_t} \right\}.
\end{aligned}$$

In the last line of (2.17) we have assumed $r_1,\ q_1 \geqslant 1$. If Q is the tail of P, then $r_b - q_t < 0$ for all b and t. In case one or both of r_1 or q_1 is 0, one has to write out some special cases of the expression in (2.17), but the basic result is the same: $\mu(P) - \mu(Q) = dc$ where c is a rational number, which is strictly negative if $r_b - q_t < 0$ for all b and t.

Theorem 2. *Let X be a compact Kähler manifold with a nontrivial holomorphic k-form ϖ where $k > 1$ is odd. Let the Higgs structure of E be as above, and let P be any Higgs subbundle of E of the form $P = \bigoplus_{s=1}^{z} \Lambda^{(p_s,0)}(X)$, $0 \leqslant p_1 < p_2 < \cdots < p_z \leqslant n$, $(z \geqslant 2)$. Then P does not admit any Higgs–Hermitian–Yang–Mills metric in any of the following cases*:

(i) $\deg(X) < 0$.

(ii) $\deg(X) = 0$ *and* $p_1 \leqslant n - k + 1$.

(iii) $k \geqslant \frac{1}{2}n + 1$, $p_1 \leqslant n - k + 1$, *and ϖ is a section of P.*

Proof. We first prove (i). Note that $\deg(\Lambda^{(s,0)}(X)) = \binom{n-1}{s-1} \deg(\Lambda^{(1,0)}(X))$, hence $\deg(P)$ is a positive multiple of $d = \deg(\Lambda^{(1,0)}(X)) = -\deg(X)$. Assume P admits a HHYM metric h, $i\Lambda F_h = c\,\mathrm{Id}_P$. Because P admits this HHYM metric, it is Higgs-semistable, so $\mu(P) \geqslant \mu(P')$, where P' is the Higgs subbundle of P given by $\bigoplus_{s=2}^{z} \Lambda^{(p_s,0)}(X)$. Now using P, and $Q = P'$ in Proposition 4, we get that $-\deg(X) = d \leqslant 0$, proving (i). We now prove (ii). Assume that P admits a HHYM metric h, $i\Lambda F_h = c\,\mathrm{Id}_P$. Representing $c_1(P)$ by $(i/2\pi)\,\mathrm{tr}_P F_h = (i/2\pi) \sum_{\alpha=1}^{\mathrm{rk} P} F^{\alpha}_{h\alpha}$ one computes $c_1(P) \wedge \omega^{n-1} = (1/2\pi n) \sum_{\alpha} i\Lambda F^{\alpha}_{h\alpha} \omega^n = (\mathrm{rk}(P)c/2\pi n)\omega^n$ ([5, Ch. 3, Sect. 1, (1.18)]) hence $\deg(P)$ is a positive multiple of c.

If $d = 0$, then we have $\mu(P) = 0$ and also $\mu(P') = 0$. If one adapts the proof [5, Prop. (8.2), Ch. V] to the Higgs setting one concludes the following, using the notation in [5]: if E is a Higgs bundle and $E' \subset E$ a Higgs subbundle, over a compact Kähler manifold (X, g), and if E admits a HHYM metric h, then $\mu(E') \leqslant \mu(E)$, with equality iff $E = E' \oplus (E')^{\perp_h}$ is a holomorphic splitting into Higgs subbundles (i.e., $(E')^{\perp_h}$ is a holomorphic, Higgs subbundle of E). In our setting ($d = 0$, so $\mu(P) = \mu(P') = 0$) this fact implies that $P = P' \oplus (P')^{\perp_h}$ is a holomorphic Higgs splitting, so $(P')^{\perp_h}$ is θ-invariant. However, $\theta P \subset P' \otimes \Lambda^{(1,0)}(X)$. Therefore $\theta\ (P')^{\perp_h} = 0$. Now let $s \in \Gamma\Lambda^{(p_1,0)}(X)$ and split $s = s' + s''$, $s' \in \Gamma P'$, $s'' \in \Gamma(P')^{\perp_h}$. Then $\theta s \in \Gamma\Lambda^{(p_1+k-1,0)}(X) \otimes \Lambda^{(1,0)}(X)$, while $\theta(s' + s'') = \theta s' \in \Gamma P' \otimes \Lambda^{(1,0)}(X)$, so the lowest possible degree "form" part of $\theta s'$ is $p_2 + k - 1 > p_1 + k - 1$. We conclude that $\theta s = 0$, for every $s \in \Gamma\Lambda^{(p_1,0)}(X)$. From Proposition 2, we conclude that $\varpi \wedge i(Z)s = 0$ for every $s \in \Gamma\Lambda^{(p_1,0)}(X)$ and every holomorphic tangent vector Z. Because $p_1 \leqslant n - k + 1$, this implies that $\varpi \equiv 0$ (one can see this pointwise by picking $s = dz_A$, where $A = \{1 \leqslant A_1 < \cdots < A_{p_1} \leqslant n\}$ can be any length p_1 multiindex, then $i(Z_{A_{p_1}})s = \pm dz_{A_1} \wedge dz_{A_2} \wedge dz_{A_3} \wedge \cdots \wedge dz_{A_{p_1-1}}$ so any of the simple $p_1 - 1$ forms $dz_{A_1} \wedge dz_{A_2} \wedge dz_{A_3} \wedge \cdots \wedge dz_{A_{p_1-1}}$ can be obtained as $i(Z)s$). We have reached a contradiction.

Now assume $k \geqslant \frac{1}{2}n + 1$, but not necessarily that $\deg(X) = 0$, and as above, assume P admits a HHYM metric h, $i\Lambda F_h = c\,\mathrm{Id}_P$. Because P admits this HHYM metric, it is semistable and using P, and $Q = P'$ in Proposition 4 we get, that $d \leqslant 0$ so $c \leqslant 0$. Now $c \leqslant 0$ implies that $i\Lambda F_h$ is a pointwise nonpositive operator. Since ϖ is a section of P, the formula (2.10) in the current setting, for the bundle P, with $s = \varpi$, (cf. Remark 2) becomes

$$c\|\varpi\|_h^2 = -i\Lambda\partial\bar\partial\|\varpi\|_h^2 + \|\nabla\varpi\|_h^2 + \sum\left\{\|\overline{\theta_h}(\overline{Z_i})\varpi\|_h^2\right\}$$

because $k \geqslant \frac{1}{2}n + 1$ implies $\theta\varpi = 0$. Integrating this equality over X implies $c \geqslant 0$, hence $c = 0$ and $\deg(X) = 0$. Now part (ii) gives a contradiction.

3. Kodaira-Nakano vanishing type results

In this section we examine analogs of the Kodaira and Nakano-type vanishing theorems for (p, q)-forms with values in a Higgs bundle over a Kähler manifold, cf. [5, Ch. 3] (see also [8, Ch. 1]) for the Kähler manifold operators and [11, Sect. 1] for the formulas on Higgs bundles.

Let (X, g) be a complex manifold of complex dimension n with a Kähler metric g. Let $(E, h) \to X$ be a rank r holomorphic vector bundle with a Hermitian metric h. As in [5], the Hermitian connection on E extends to an operator $d^\nabla : C^\infty\Gamma E \otimes \Lambda^i(X) \to C^\infty\Gamma E \otimes \Lambda^{i+1}(X)$ and there is the refinement of d^∇ into the two operators $d^\nabla = \partial_h + \bar\partial$,

$$\partial_h : C^\infty\Gamma E \otimes \Lambda^{(i,j)}(X) \to C^\infty\Gamma E \otimes \Lambda^{(i+1,j)}(X),$$
$$\bar\partial : C^\infty\Gamma E \otimes \Lambda^{(i,j)}(X) \to C^\infty\Gamma E \otimes \Lambda^{(i,j+1)}(X),$$

given relative to a local holomorphic frame $\{e_\alpha\}_{\alpha=1}^r$ of E by

$$\bar\partial(e_\alpha \otimes \phi) = e_\alpha \otimes \bar\partial\phi,$$
$$\partial_h(e_\alpha \otimes \phi) = \sum_{\beta=1}^{r} e_\beta \otimes C_\alpha^\beta \wedge \phi + e_\alpha \otimes \partial\phi$$

where $\nabla e_\alpha = \sum_{\beta=1}^r e_\beta \otimes C_\alpha^\beta$. The metric on X extends to $\Lambda^*(X) \otimes \mathbb{C}$ (we drop the $\mathbb{C}$ and write $\Lambda^*(X)$ for the complex exterior algebra of X) and the Hermitian metric on E combines to give a metric which we denote $\langle\!\langle\, ,\, \rangle\!\rangle$ on $E \otimes \Lambda^*(X)$ by the prescription

$$\langle\!\langle e \otimes \phi,\ f \otimes \psi \rangle\!\rangle = h(e,\ f) * (\phi \wedge *\overline{\psi}), \qquad e,\ f \in E\phi, \psi \in \Lambda^*(X)$$

where $*$ denotes the Hodge star operator on $\Lambda^*(X)$ determined by g (and extend $*$ to $E\otimes\Lambda^*(X)$ by $\mathrm{id}_E\otimes *$ which we also denote simply as $*$). We use the convention that the Hodge star operator is given on the complex exterior algebra by the method in [5, Ch. 3, Sect. 2]. Note that there is a sign typographical error in this reference in formula (2.6), which should read

$$\varepsilon(A, B) = (-1)^{np+n(n-1)/2}\sigma(AA')\cdot\sigma(BB').$$

The L^2 or formal adjoints of ∂^∇ and $\bar\partial$ with respect to $\langle\!\langle\, ,\, \rangle\!\rangle$ are given by (cf. [5, Ch. 3, Sect. 2])

$$\partial_h^* = -*\bar\partial * = i[\Lambda, \bar\partial],$$
$$\bar\partial^* = -*\partial_h * = -i[\Lambda, \partial_h]$$

where $\Lambda = i\sum_{i,j} g^{i\bar{j}} i(\partial/\partial z_i) i(\partial/\partial\overline{z_j})$ is the adjoint to exterior multiplication with the Kähler form ω. Let $\Box_{\bar\partial} = (\bar\partial + \bar\partial^*)^2$ and $\Box_{\partial_h} = (\partial_h + \partial_h^*)^2$. The Kodaira–Nakano formula (cf. [5, Ch. 3, Sect. 3, Proof (3.5), p. 69], or [8, Ch. 1, p. 16, (1.58)]) can be written

$$\Box_{\partial_h} - \Box_{\bar\partial} = i(\Lambda e(\Theta) - e(\Theta)\Lambda) \tag{3.1}$$

where as in [5] $e(\Theta) : E\otimes\Lambda^{(p,q)}(X) \to E\otimes\Lambda^{(p+1,q+1)}(X)$ is given by $e(\Theta)(e_\alpha\otimes\phi) = \sum_\beta\{e_\beta\otimes\Theta_\alpha^\beta\wedge\phi\}$. We will say that a section s of $E\otimes\Lambda^w(X) = \bigoplus_{i+j=w} E\otimes\Lambda^{(i,j)}(X)$, has Hodge type (p, q) if $s = \sum_{\alpha=1}^r e_\alpha\otimes\phi_\alpha$ where each ϕ_α is a section of $\Lambda^{(p,q)}(X)$. This terminology can become ambiguous if E is the Higgs bundle discussed in Section 2, since the E component of a section of $E\otimes\Lambda^{(p,q)}(X)$ will be a sum of forms with Hodge types. We will address this issue when it arises. Both $\Box_{\partial_h}$ and $\Box_{\bar\partial}$ preserve the Hodge (p, q) types.

Now suppose E also has the structure of a Higgs bundle with Higgs form θ. The operators $D'' = \bar\partial + \theta$, D_h' and D_h defined in (1.2) and (1.3) extend to $E\otimes\Lambda^*(X)$ as above ([11, Sect. 1]):

$$\begin{aligned}
D''(e_\alpha\otimes\phi) &= e_\alpha\otimes\bar\partial\phi + \sum_\beta e_\beta\otimes\theta_\alpha^\beta\wedge\phi,\\
D_h'(e_\alpha\otimes\phi) &= e_\alpha\otimes\partial\phi + \sum_\beta e_\beta\otimes(C_\alpha^\beta + \overline{\theta_h}_\alpha^\beta)\wedge\phi,\\
D_h'^* &= -*D''* = i[\Lambda, D''],\\
D''^* &= -*D_h'* = -i[\Lambda, D_h'],\\
D_h^* &= D_h'^* + D''^*.
\end{aligned}\tag{3.2}$$

One checks that as before that these extended operators D'', D_h' and their adjoints all square to zero. Note the adjoints of the Higgs forms are given by $\theta^* = *\bar\theta_h* = -i[\Lambda, \bar\theta_h]$, $\bar\theta_h^* = *\theta* = i[\Lambda, \theta]$. Define

$$\Box_{D''} = (D'' + D''^*)^2 = D''D''^* + D''^*D'', \tag{3.3}$$
$$\Box_{D_h'} = (D_h' + D_h'^*)^2 = D_h'D_h'^* + D_h'^*D_h', \tag{3.4}$$
$$\Box_{D_h} = D_hD_h^* + D_h^*D_h = \Box_{D''} + \Box_{D_h'}.$$

We now examine the relation between the Laplacians in (3.4) and (3.3), and the “ordinary” Laplacians corresponding to $\theta = 0$, $\Box_{\partial_h}$, $\Box_{\bar\partial}$ acting on $E\otimes\Lambda^t(X)$. The following formulas are given by computations using the definitions of D'', D'_h and their adjoints (3.2) (cf. the notation discussed at the beginning of the proof of Proposition 1):

$$\begin{aligned}\Box_{D''} &= \Box_{\bar\partial} + \theta\theta^* + \theta^*\theta + \bar\partial(\theta^*) + \bar\partial^*(\theta),\\ \Box_{D'_h} &= \Box_{\partial_h} + \bar\theta_h\bar\theta_h^* + \bar\theta_h^*\bar\theta_h + \partial_h(\bar\theta_h^*) + \partial_h^*(\bar\theta_h).\end{aligned}\tag{3.5}$$

In general the operators $\Box_{D''}$ and $\Box_{D'_h}$ will not preserve the Hodge type (p,q) of a section of $E\otimes\Lambda^{(p,q)}(X)$, although they do preserve the total degree $p+q$ because

$$\begin{aligned}\bar\partial(\theta^*),\ \partial_h^*(\bar\theta_h) &: C^\infty\Gamma E\otimes\Lambda^{(p,q)}(X)\to C^\infty\Gamma E\otimes\Lambda^{(p-1,q+1)}(X),\\ \bar\partial^*(\theta),\ \partial_h(\bar\theta_h^*) &: C^\infty\Gamma E\otimes\Lambda^{(p,q)}(X)\to C^\infty\Gamma E\otimes\Lambda^{(p+1,q-1)}(X),\end{aligned}\tag{3.6}$$

and defining

$$\begin{aligned}\bar\boxminus &:= \Box_{\bar\partial} + \theta\theta^* + \theta^*\theta,\\ \boxminus &:= \Box_{\partial_h} + \bar\theta_h\bar\theta_h^* + \bar\theta_h^*\bar\theta_h,\end{aligned}$$

each of $\boxminus$ and $\bar\boxminus$ preserve Hodge type (p,q) of a section of $E\otimes\Lambda^w(X)$, i.e., $\boxminus, \bar\boxminus : E\otimes\Lambda^{(p,q)}(X)\to E\otimes\Lambda^{(p,q)}$. The operators $\boxminus$ and $\bar\boxminus$ differ from the usual ($\theta=0$) Laplacians only by the zeroth order terms, which are nonnegative operators. If X is compact Kähler, then because $\bar\boxminus$ and $\boxminus$ preserve Hodge type (p,q) one has $\ker\bar\boxminus(\boxminus)\subset\ker\Box_{D''}(\Box_{D'_h})$ considering these operators acting on $C^\infty\Gamma E\otimes\Lambda^w(X)$. To wit, if $s=\sum_{p+q=w}s_{p,q}$ is the decomposition into Hodge (p,q) components, then $\bar\boxminus s=0\Leftrightarrow\bar\boxminus s_{p,q}=0\ \forall(p,q)\Leftrightarrow 0=\bar\partial s_{p,q}=\theta s_{p,q}=\bar\partial^* s_{p,q}=\theta^* s_{p,q}\ \forall(p,q)\Leftrightarrow 0=D''s_{p,q}=D''^*s_{p,q}\ \forall(p,q)\Rightarrow 0=D''s=D''^*s\Leftrightarrow\Box_{D''}s$.

An analog of the Kodaira–Nakano-type formula in this setting is

$$\Box_{D'_h}-\Box_{D''} = i(\Lambda e(F_h)-e(F_h)\Lambda)\tag{3.7}$$

where as in [5] $e(F_h)\colon E\otimes\Lambda^w(X)\to E\otimes\Lambda^{w+2}(X)$ is given $e(F_h)(e_\alpha\otimes\phi)=\sum_\beta\{e_\beta\otimes F^\beta_{h\alpha}\wedge\phi\}$. We note that

$$\begin{aligned}i(\Lambda e(F_h^{(1,1)})-e(F_h^{(1,1)})\Lambda) &: E\otimes\Lambda^{(p,q)}(X)\to E\otimes\Lambda^{(p,q)},\\ i(\Lambda e(F_h^{(2,0)})-e(F_h^{(2,0)})\Lambda) &: E\otimes\Lambda^{(p,q)}(X)\to E\otimes\Lambda^{(p+1,q-1)},\\ i(\Lambda e(F_h^{(0,2)})-e(F_h^{(0,2)})\Lambda) &: E\otimes\Lambda^{(p,q)}(X)\to E\otimes\Lambda^{(p-1,q+1)}.\end{aligned}\tag{3.8}$$

Because $\Box_{D''}$, $\Box_{D'_h}$ and $\Box_D$ are nonnegative operators on a compact Kähler manifold, formula (3.7) implies

Theorem 3. *If (X,g) is a compact Kähler manifold, $(E,\theta)\to X$ a Higgs bundle with a Hermitian metric h then with the notation above*

$$i\big(\Lambda e(F_h)-e(F_h)\Lambda\big)\leqslant 0\Rightarrow\ \ker\Box_{D''}\subseteq\ker\Box_{D'_h}\cap\ker\Box_D\tag{3.9}$$

where $i(\Lambda e(F_h)-e(F_h)\Lambda)$, $\Box_{D''}$, $\Box_{D'_h}$ and $\Box_D$ are considered as operators on $C^\infty\Gamma E\otimes\Lambda^w(X)$.

Proof. That $\ker\Box_{D''}\subseteq\ker\Box_{D'_h}$ follows from (3.7). Then, if $s\in\ker\Box_{D''}\cap\ker\Box_{D'_h}$, one gets $0=D''s=D''^*s=D'_hs=D'^*_hs$ and hence $0=Ds=D^*s$.

If the Higgs operator θ is parallel with respect to the operator d^{∇} defined by h as in Proposition 1, as a section of $\mathrm{Hom}(E) \otimes \Lambda^{(1,0)}(X)$, i.e., F_h has only $(1,1)$ form parts relative to a holomorphic frame, then the four operators in (3.6) all vanish (the proof is analogous to the computation in Proposition 1) and $\square_{D''} = \bar{\boxminus}$ and $\square_{D'_h} = \boxminus$ preserve Hodge type.

Let $s = \sum_{p+q=w} s_{p,q} \in C^{\infty}\Gamma E \otimes \Lambda^{w}(X)$ be as above, then the formulas above combine to give

$$\begin{aligned} \langle\!\langle(\square_{D'_h} - \square_{D''})s, s\rangle\!\rangle &= \langle\!\langle(i(\Lambda e(F_h) - e(F_h)\Lambda)s, s\rangle\!\rangle \\ &= \sum_{p+q=w} \big\{\langle\!\langle(\boxminus - \bar{\boxminus})s_{p,q}, s_{p,q}\rangle\!\rangle \\ &\quad + \langle\!\langle(\partial_h(\bar{\theta}^*_h) - \bar{\partial}^*(\theta)_h)s_{p,q}, s_{p+1,q-1}\rangle\!\rangle \\ &\quad + \langle\!\langle(\partial^*_h(\bar{\theta}_h) - \bar{\partial}(\theta^*))s_{p,q}, s_{p-1,q+1}\rangle\!\rangle\big\}. \end{aligned} \tag{3.10}$$

In formula (3.10) we see that if $s \in E \otimes \Lambda^{(p,q)}(X)$ then the term $\langle\!\langle(i(\Lambda e(F_h) - e(F_h)\Lambda)s, s\rangle\!\rangle$ depends only on the $(1,1)$ part of F_h, cf. (3.8). This observation yields the following vanishing result, which will be revisited in giving a type of Higgs bundle analog to the Kodaira–Nakano vanishing theorem.

Theorem 4. *Let (X, g) be a compact Kähler manifold, $(E, \theta) \to X$ a Higgs bundle with a Hermitian metric h. With the notation above, if the $(1,1)$ part of $i\big(\Lambda e(F_h) - e(F_h)\big)\Lambda) \leqslant 0$ pointwise as an operator on $E \otimes \Lambda^{(p,q)}(X)$ and if $s_{p,q} \in C^{\infty}\Gamma E \otimes \Lambda^{(p,q)}(X)$ satisfies $\square_{D''} s_{p,q} = 0$ then $\square_{D'_h} s_{p,q} = 0$ and $\square_{D_h} s_{p,q} = 0$. If the $(1,1)$-part of $i\big(\Lambda e(F_h) - e(F_h)\Lambda\big)$ is quasinegative on $E \otimes \Lambda^{(p,q)}(X)$, any then any such $s_{p,q}$ must be 0.*

If the Higgs form θ is parallel as a section of $\mathrm{Hom}(E) \otimes \Lambda^{(1,0)}(X)$, *i.e., F_h has only $(1,1)$ form parts with respect to a holomorphic frame, and if $i\big(\Lambda e(F_h) - e(F_h)\Lambda\big) \leqslant 0$ on $E \otimes \Lambda^{w}(X)$ then $s = \sum_{p+q=w} s_{p,q} \in E \otimes \Lambda^{w}(X)$, $\square_{D''} s = 0$ implies $\square_{D'_h} s_{p,q} = 0$ and $\square_{D_h} s_{p,q} = 0$ $\forall (p,q)$ (cf. Theorem 3).*

The Kodaira–Nakano vanishing theorems are generally stated as vanishing theorems for harmonic sections of (p,q)-forms with values in a holomorphic line bundle L, i.e., harmonic sections of $L \otimes \Lambda^{(p,q)}(X)$, with X compact Kähler or compact complex with $c_1(L) < 0$ ([5, Ch. 3, Sect. 3], and [8, Ch. 2, Th. 2.18]). However the proofs given work for $\square_{\bar{\partial}}$-harmonic sections of (p,q)-forms with values in a Hermitian holomorphic vector bundle, i.e., harmonic sections of $E \otimes \Lambda^{(p,q)}(X)$, if we assume that E admits a projectively flat Hermitian metric (every Hermitian metric on a line bundle is projectively flat). The main technical point is that conformally changing a projectively flat Hermitian metric yields another projectively flat Hermitian metric. This observation is used in the course of proving the next theorem.

Theorem 5. (Kodaira–Nakano) *Let $(E, \theta) \to (X, g)$ be a Higgs bundle over a compact complex manifold of complex dimension n with $c_1(E) < 0$ and assume E admits a Hermitian metric h for which the $(1,1)$ part of the Higgs curvature $F_h^{(1,1)}$satisfies the equation $F_h^{(1,1)} = \kappa\, \mathrm{Id}_E$, where $\kappa = \sum \kappa_{i\bar{j}} dz_i \wedge \overline{dz_j}$ is a $(1,1)$ form. If $s_{p,q} \in C^{\infty}\Gamma E \otimes \Lambda^{(p,q)}(X)$ and $\square_{D''} s_{p,q} = 0$, $p + q \leqslant n - 1$ $(= n)$, then $s_{p,q} = 0$ ($\square_{D'_h} s_{p,q} = 0$ and $\square_{D_h} s_{p,q} = 0$).*

Proof. Let a negative representative of $c_1(E)$ be given by $(i/2\pi) f = (i/2\pi)\sum_{i,j} f_{i\bar{j}} dz_i \wedge d\bar{z}_j$, a closed real (1, 1) form with $(f_{i\bar{j}})$ negative definite pointwise. Then $-i\sum_{i,j} f_{i\bar{j}} dz_i d\bar{z}_j$ is a Kähler metric g on X, and we use this metric as the Kähler metric on X. Suppose h is a Hermitian metric on E for which $F_h^{(1,1)} = \kappa\,\mathrm{Id}_E$ where $\kappa = \sum \kappa_{i\bar{j}} dz_i \wedge \overline{dz_j}$ is a (1, 1) form. From (1.7) we have $\mathrm{tr}_E F_h^{(1,1)} = \mathrm{tr}_E \Theta$ and thus we can represent $c_1(E)$ as $(i/2\pi)\,\mathrm{tr}_E F_h^{(1,1)} = (i/2\pi) r\kappa$. Now, any conformal change of h to a new Hermitian metric $h' = ah$ (a a smooth positive real-valued function on X) changes the Hermitian metric curvature from Θ to $\Theta' = \Theta - \partial\bar{\partial}\ln(a)\,\mathrm{Id}_E$ and does not change $\overline{\theta_h}$, i.e., $\overline{\theta_{h'}} = \overline{\theta_h}$. It therefore follows from (1.7) that for the new metric h', $F_{h'}^{(1,1)} = \kappa'\,\mathrm{Id}_E$ where $\kappa' = \kappa - \partial\bar{\partial}\ln(a)$.

For any real representative of $c_1(E)$, such as $(i/2\pi) f$, we can always change h (or any given Hermitian metric on a E) conformally to a new metric h' for which $\mathrm{tr}_E \Theta' = f$ ([5, Ch. 2, Sect. 2, p. 41, Prop. (2.23)]). Therefore we conclude: we can conformally change h to a metric h' for which $f = \mathrm{tr}_E \Theta' = \mathrm{tr}_E F_{h'}^{(1,1)} = r\kappa'$. Thus $F_{h'}^{(1,1)} = (1/r) f\,\mathrm{Id}_E$. Now with this Hermitian metric the formulas (3.7) and (3.8) yield, if $s \in C^\infty \Gamma E \otimes \Lambda^{(p,q)}(X)$,

$$\begin{aligned} h\big((\Box_{D_h'} - \Box_{D''})s, s\big) &= h\big(i(\Lambda e(F_h) - e(F_h)\Lambda)s, s\big) \\ &= h\big(i(\Lambda e(F_h^{(1,1)}) - e(F_h^{(1,1)})\Lambda)s, s\big) = h\big(-\tfrac{1}{r}(\Lambda L - L\Lambda)s, s\big) \\ &= -\tfrac{1}{r}(n - (p+q))\|s\|^2_{h'} \end{aligned}$$

(L is exterior multiplication by the Kähler form). Thus if $p + q \leqslant n - 1$ $(= n)$, $\Box_{D''} s = 0$ we conclude $s = 0$ ($\Box_{D_h'} s = 0$ and $\Box_{D_h} s = 0$). $\Box$

Remark 3. *The vanishing theorem of Gigante and Girbau,* ([5, *Ch.* 3, *Sect.* 3, *Theorem* 3.4] *and* [8, *Ch.* 3, *Theorem* 3.2]) *where the assumptions are*: X *is compact Kähler,* $c_1(E) \leqslant 0$ *and pointwise rank* k, *and the vanishing occurs in degrees* $p + q \leqslant k - 1$, *is also valid for a Hermitian holomorphic vector bundle with a projectively flat metric* h *and in the Higgs setting when* $F_h^{(1,1)} = \kappa\,\mathrm{Id}_E$. *The proof given in* [5, *Ch.* 3, *Sect.* 3, *Theorem* (3.4), *pp.* 69–73] *works in this setting. One has to extend the formula* (3.6), *page* 70 *as we now indicate. Let* $(E, \theta) \to (X, g)$ *be a Higgs bundle over a compact Kähler manifold and* E *admits a Hermitian metric* h *for which* $F_h^{(1,1)} = \kappa\,\mathrm{Id}_E$, *where* $\kappa = \sum \kappa_{i\bar{j}} dz_i \wedge \overline{dz_j}$ *is a* (1, 1) *form. Then* $\kappa\,\mathrm{Id}_E$ *has the same Hermitian symmetries as the Hermitian curvature,* Θ, *of* h, *so without loss of generality,* $\kappa = \sum \kappa_i dz_i \wedge \overline{dz_i}$ ($\{dz_i\}_{i=1}^n$ *orthonormal at the point of evaluation*). *Now for* $s \in C^\infty \Gamma E \otimes \Lambda^{(p,q)}(X)$ *we can write locally* $s = \sum e_a \otimes \varphi^a_{I\bar{J}} dz_I \wedge \overline{dz_J}$ *where the multiindices satisfy* $|I| = p, |J| = q$ *and at one particular point of evaluation the holomorphic frame* $\{e_a\}_{a=1}^r$ *is orthonormal. With this notation the* [5, *p.* 70, *formula* (3.6)] *translates into*

$$h(i(\Lambda e(F_h^{(1,1)}) - e(F_h^{(1,1)})\Lambda)s, s) = \sum_{\substack{a=1 \\ |I|=p, |J|=q}}^{r} \Bigg(- \sum_{i \in (I\cap J)} \kappa_i + \sum_{i \in (I\cup J)^c} \kappa_i \Bigg) |\varphi^a_{I\bar{J}}|^2.$$

The remainder of the proof goes through as in [5].

We now examine the consequences of the results in this section for the Higgs bundles defined by (2.1). With (E, θ) as in (2.1) and assuming (X, g) is a Kähler manifold, we get a second grading of the bundle

$$E \otimes \Lambda^{w}(X) = \bigoplus_{p+q=w} E \otimes \Lambda^{(p,q)}(X) = \bigoplus_{p+q=w, 1 \leqslant a \leqslant n} E_a^{(p,q)}$$

where $E_a^{(p,q)} = \Lambda^{(a,0)}(X) \otimes \Lambda^{(p,q)}(X)$. Then one checks

$$\theta(\bar{\theta}_h^*) : E_a^{(p,q)} \to E_{a+k-1}^{(p+1,q)}(E_{a+k-1}^{(p,q-1)}),$$

and if E is endowed with a natural metric h (2.4) then

$$\bar{\theta}_h(\theta^*) : E_a^{(p,q)} \to E_{a-k+1}^{(p,q+1)}(E_{a-k+1}^{(p-1,q)}).$$

We continue assuming E is endowed with a natural Hermitian metric h. Then the decomposition $E = \bigoplus_{a=1}^{n} \Lambda^{(a,0)}(X)$ is h orthogonal, hence is also preserved by the associated Hermitian connection and its curvature, Θ. It follows that $e(\Theta) : E_a^{(p,q)} \longrightarrow E_a^{(p+1,q+1)}$, that $i\big(\Lambda e(\Theta) - e(\Theta)\Lambda\big) : E_a^{(p,q)} \longrightarrow E_a^{(p,q)}$, and that $i\big(\Lambda e(F_h^{(1,1)}) - e(F_h^{(1,1)})\Lambda\big) = i\big(\Lambda e(\Theta) - e(\Theta)\Lambda\big) + \bar{\theta}_h\bar{\theta}_h^* + \bar{\theta}_h^*\bar{\theta}_h - \theta\theta^* - \theta^*\theta : E_a^{(p,q)} \longrightarrow E_a^{(p,q)}$ (cf. 3.8). Finally if we assume that the Higgs form θ is h parallel then $F_h = F_h^{(1,1)}$ (Proposition 294) and we have $i\big(\Lambda e(F_h) - e(F_h)\Lambda\big), \bar{\boxminus} = \Box_{D''}, \boxminus = \Box_{D_h'} : E_a^{(p,q)} \to E_a^{(p,q)}$. From this data we deduce the following interpretation of the last part of Theorem 4 in this setting.

Theorem 6. *Let (X, g) be a compact Kähler manifold of complex dimension n, let $(E, \theta) \to X$ be the Higgs bundle given by* (2.1). *Let h be a natural metric on E and assume that the Higgs form θ is parallel, $d^{\nabla}\theta = 0$. For a section s of $E \otimes \Lambda^{w}(X)$, write $s = \sum_{p+q=w, 1 \leqslant a \leqslant n} s_a^{(p,q)}$, where $s_a^{(p,q)} \in C^{\infty}\Gamma E_a^{(p,q)}$. If $i\,(\Lambda e(F_h) - e(F_h)\Lambda) \leqslant 0$ pointwise as an operator on $E \otimes \Lambda^{w}(X)$, then $\Box_{D''}s = 0$ implies $0 = \Box_{D''}s_a^{(p,q)} = \Box_{D_h'}s_a^{(p,q)} = \Box_{D_h}s_a^{(p,q)}$ for all (p, q) and all a.*

Using Proposition 3 we deduce the following

Corollary 1. *Let (X, g) be a compact Kähler manifold of complex dimension n, let $(E, \theta) \to X$ be the Higgs bundle given by* (2.1). *Assume that ϖ is g parallel and use the standard extension of g as the Hermitian metric on E. For a section s of $E \otimes \Lambda^{w}(X)$, write $s = \sum_{p+q=w, 1 \leqslant a \leqslant n} s_a^{(p,q)}$, where $s_a^{(p,q)} \in C^{\infty}\Gamma E_a^{(p,q)}$. If $i\big(\Lambda e(F_h) - e(F_h)\Lambda\big) \leqslant 0$ pointwise as an operator on $E \otimes \Lambda^{w}(X)$, then $\Box_{D''}s = 0$ implies $0 = \Box_{D''}s_a^{(p,q)} = \Box_{D_h'}s_a^{(p,q)} = \Box_{D_h}s_a^{(p,q)}$ for all (p, q) and all a.*

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