Available online at www.sciencedirect.com
ScienceDirect
J. Math. Anal. Appl. 325 (2007) 1327–1338
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Nontrivial solutions of singular two-point boundary value problems with sign-changing nonlinear terms

Guodong Han [a,*], Ying Wu [b]

[a] *Institute for Information and System Science, Xi'an Jiaotong University, Xi'an, Shaanxi 710049, PR China*
[b] *Department of Basic Courses, Xi'an University of Science & Technology, Xi'an, Shaanxi 710054, PR China*

Received 30 December 2005
Available online 27 March 2006
Submitted by D. O'Regan

Abstract

The singular two-point boundary value problem

$$-u''(t) = h(t) f\big(u(t)\big), \quad t \in (0,1); \qquad u(0) = u(1) = 0$$

is considered under some conditions concerning the first eigenvalue corresponding to the relevant linear problem, where h is allowed to be singular at both $t = 0$ and $t = 1$. Moreover, $f : (-\infty, +\infty) \to (-\infty, +\infty)$ is a sign-changing function and not necessarily bounded from below. By computing the topological degree of an completely continuous field, the existence results of nontrivial solutions are established.

Keywords: Singular boundary value problems; Nontrivial solution; Cone; Leray–Schauder degree

1. Introduction

In recent years, much attention has been given to singular second order two-point boundary value problem (BVP)

$$\begin{cases} -u''(t) = h(t) f(u(t)), & t \in (0,1), \\ u(0) = u(1) = 0 \end{cases} \tag{1.1}$$

* Corresponding author.
E-mail address: gdhan@mail.xjtu.edu.cn (G. Han).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2006.02.076

by a number of authors, see [1,2,4,8,10–12,17–20] and references therein. Most of them obtained the existence of positive solutions provided $f:[0,+\infty)\to[0,+\infty)$ is nonnegative, continuous and superlinear or sublinear by employing the cone expansion or compression fixed point theorem or super- and subsolution method. In a later paper [16], Sun and Zhang considered the general singular nonlinear Strum–Liouville problems

$$\begin{cases} -(Lu)(t)=h(t)f(u(t)), & t\in(0,1),\\ R_1(u)=\alpha_1u(0)+\beta_1u'(0)=0, & R_2(u)=\alpha_2u(1)+\beta_2u'(1)=0, \end{cases} \tag{1.2}$$

where $(Lu)=(p(t)u'(t))'+q(t)u(t)$ and h is allowed to be singular at 0 and 1. Besides, the main condition they imposed on f is that f is bounded from below, i.e.,

(H_0) There exists a constant $b\geqslant 0$ such that

$$f(u)\geqslant -b \quad \text{for all } u\in\mathbb{R}^1. \tag{1.3}$$

To our knowledge, there are not many references to deal with singular problems in the case that $f:(-\infty,+\infty)\to(-\infty,+\infty)$ is not necessarily nonnegative except [16]. When f is a sign-changing function, the fixed point index theory on a cone becomes invalid since the nonlinear operator generated by f does not map the positive cone into itself.

The present paper, for simplicity, is concern with the existence of nontrivial solutions of singular BVP (1.1). Replacing (H_0) by the following more general condition:

(H_1) There exist three constants $b>0$, $c>0$ and $\alpha\in(0,1)$ such that

$$f(u)\geqslant -b-c|u|^{\alpha} \quad \text{for all } u\in\mathbb{R}^1, \tag{1.4}$$

we still can obtained the existence of nontrivial solution of (1.1). Notice that condition (H_1) permits f to be a unbounded function. Our results can be generalized to singular nonlinear Strum–Liouville problems (1.2) without essential difficulties.

We also need the following hypotheses for our main results.

(H_2) $h\in C((0,1),[0,+\infty))$, $h(t)\not\equiv 0$ in $(0,1)$ and

$$\int_0^1 t(1-t)h(t)\,\mathrm{d}t<+\infty;$$

(H_3) $f:(-\infty,+\infty)\to(-\infty,+\infty)$ is continuous.

It is well known that the solution of singular BVP (1.1) in $C[0,1]\cap C^2(0,1)$ is equivalent to the solution of the following Hammerstein integral equation in $C[0,1]$:

$$u(t)=\int_0^1 G(t,s)h(s)f\bigl(u(s)\bigr)\,\mathrm{d}s, \quad t\in[0,1], \tag{1.5}$$

where $G:[0,1]\times[0,1]\to[0,1]$ is Green's function for $-u''(t)=0$ for all $t\in[0,1]$ subject to $u(0)=u(1)=0$, i.e.,

$$G(t,s)=\begin{cases} s(1-t), & 0\leqslant s\leqslant t\leqslant 1,\\ t(1-s), & 0\leqslant t\leqslant s\leqslant 1. \end{cases} \tag{1.6}$$

Define operators K, F and $A : C[0,1] \to C[0,1]$, respectively, by

$$(Ku)(t) = \int_0^1 G(t,s)h(s)u(s)\,\mathrm{d}s, \quad t \in [0,1],\ \forall u \in C[0,1],$$
$$(Fu)(t) = f\big(u(t)\big), \quad t \in [0,1],\ \forall u \in C[0,1],$$
$$(Au)(t) = (KFu)(t) = \int_0^1 G(t,s)h(s)f\big(u(s)\big)\,\mathrm{d}s. \tag{1.7}$$

Then the solution of singular BVP (1.1) is equivalent to the fixed point of A in $C[0,1]$. By (H_2), $K : C[0,1] \to C[0,1]$ is a completely continuous linear operator with the first eigenvalue $\lambda_1 > 0$ (see Lemma 3.1 in Section 3). Let φ_1 be the eigenfunction corresponding to λ_1, namely $\lambda_1 K\varphi_1 = \varphi_1$.

Theorem 1.1. *Assume that* (H_1)–(H_3) *hold. If*

$$\liminf_{u\to+\infty} \frac{f(u)}{u} > \lambda_1, \tag{1.8}$$
$$\limsup_{u\to 0} \left|\frac{f(u)}{u}\right| < \lambda_1, \tag{1.9}$$

then singular BVP (1.1) *has at least one nontrivial solution.*

Corollary 1.2. *Assume that* (H_0), (H_2), (H_3), (1.8) *and* (1.9) *hold. Then singular BVP* (1.1) *has at least one nontrivial solution.*

Theorem 1.3. *Assume that* (H_1)–(H_3) *and* (1.8) *hold. If*

$$\lim_{u\to 0} \frac{f(u)}{u} = \lambda, \tag{1.10}$$

where $\lambda \neq \lambda_n$, $\{\lambda_n\colon n = 1,2,\ldots\}$ *is the eigenvalue set of* K, *then the singular BVP* (1.1) *has at least one nontrivial solution.*

Corollary 1.4. *Assume that* (H_0), (H_2), (H_3), (1.8) *and* (1.10) *hold. Then singular BVP* (1.1) *has at least one nontrivial solution.*

Remark 1.5. In most papers, the nonlinear term f is a nonnegative function defined on $[0,+\infty)$ to guarantee the operator A generated by f is a cone mapping so that one can apply the fixed point index theory in cone. In [16, Theorem 1], f permits sign-changing but should be bounded from below. In our main results, f may be a sign-changing and unbounded function. If we impose suitable conditions (see [16, condition H_1]) on singular nonlinear Strum–Liouville problem (1.2), then the Green's function of

$$\begin{cases} -(Lu)(t) = 0, & 0 < t < 1, \\ R_1(u) = R_2(u) = 0 \end{cases}$$

has analogous properties as $G(t,s)$ defined in (1.6) (see [16, Lemma 1]). Hence, Theorem 2.1 can be applied to the singular nonlinear Strum–Liouville problem (1.2). So, similar results as

Theorems 1.1 and 1.3 can be established for singular nonlinear Strum–Liouville problem (1.2). Please refer to [16] for details.

Remark 1.6. For nonsingular BVP, there are analogous results as Theorem 1.1, Corollary 1.2, Theorem 1.3 and Corollary 1.4. For instance, we can easily prove the following result. Consider the following Dirichlet boundary value problem:

$$\begin{cases} -u''(t) = f(t, u(t)), \quad t \in [0, 1], \\ u(0) = u(1) = 0. \end{cases} \tag{1.11}$$

Theorem 1.7. *Suppose that the following conditions hold.*

(i) $f(t, u)$ *is continuous on* $[0, 1] \times \mathbb{R}^1$ *and there exist constants* $b > 0$, $c > 0$ *and* $\alpha \in (0, 1)$ *such that*

$$f(t, u) \geqslant -b - c|u|^{\alpha} \quad \text{for all } t \in [0, 1] \text{ and } u \in \mathbb{R}^1; \tag{1.12}$$

(ii)

$$\liminf_{u \to +\infty} \frac{f(t, u)}{u} > \pi^2 \quad \text{uniformly for } t \in [0, 1]; \tag{1.13}$$

$$\limsup_{u \to 0} \left| \frac{f(t, u)}{u} \right| < \pi^2 \quad \text{uniformly for } t \in [0, 1]. \tag{1.14}$$

Then BVP (1.11) *has at least one nontrivial solution.*

If we use Mountain pass lemma to deal with BVP (1.11), we should impose conditions on f such that the functional corresponding to BVP (1.11) satisfies PS condition. For example, we should suppose that there exist $\mu \in (0, 1/2)$ and $M > 0$ such that

$$F(t, u) \triangleq \int_0^u f(t, v)\, \mathrm{d}v \leqslant \mu u f(t, u) \quad \text{for all } |u| \geqslant M \text{ and } t \in [0, 1]. \tag{1.15}$$

In Theorem 1.7, we use condition (1.12) instead of condition (1.15).

Remark 1.8. Our main results can be applied to investigate the nontrivial radial solution of the following elliptic boundary value problem in an exterior domain:

$$\begin{cases} -\Delta u = a(\|x\|) g(u) \quad \text{for } \|x\| > 1,\ x \in \mathbb{R}^n,\ n \geqslant 3, \\ u = 0 \quad \text{for } \|x\| = 1, \\ u \to 0 \quad \text{as } \|x\| \to \infty. \end{cases} \tag{1.16}$$

In fact, by simple substitution of variable, (1.16) can be changed into a second order two-point singular boundary value problem as (1.1). So we can study Eq. (1.16) by our main results.

This paper is organized as follows. In Section 2, we compute the Leray–Schauder degree for a completely continuous field relating to singular BVP (1.1). Proofs of our main results and a simple example are given in Section 3. Please refer to [3,5,6] for the basic concepts and properties about the cone theory and the topological degree theory.

2. An abstract result of Leray–Schauder degree for a completely continuous field

The aim of this section is to compute the Leray–Schauder degree for a completely continuous field such as $I - A$, where A does not map the positive cone into itself.

Let E be a real Banach space, E^* the dual space of E, P a total cone in E, i.e., $E = \overline{P - P}$, and P^* the dual cone of P, namely $P^* = \{g \in E^*\colon\ g(u) \geqslant 0 \text{ for all } u \in P\}$. Let $K : E \to E$ be a completely continuous linear positive operator, r_1 the spectral radius of K and K^* the dual operator of K. On account of Krein–Rutman's theorem, if $r_1 \neq 0$, then there exist $\varphi_1 \in P\backslash\{\theta\}$ and $g_1 \in P^*\backslash\{\theta\}$, such that

$$K\varphi_1 = r_1\varphi_1, \qquad K^*g_1 = r_1 g_1. \tag{2.1}$$

Choose such an element $g_1 \in P^*\backslash\{\theta\}$, which enables the latter equation in (2.1) holds. Choose a number $\delta > 0$ and let

$$P(g_1, \delta) = \bigl\{u \in P\colon\ g_1(u) \geqslant \delta\|u\|\bigr\}, \tag{2.2}$$

then it is easy to see that $P(g_1, \delta)$ is a cone in E.

Theorem 2.1. *Suppose that the following conditions are satisfied.*

(C_1) *There exist $\varphi_1 \in P\backslash\{\theta\}$, $g_1 \in P^*\backslash\{\theta\}$ and $\delta > 0$ such that* (2.1) *holds and K maps P into $P(g_1, \delta)$;*
(C_2) *$T : E \to P$ is a continuous operator and there exist $\alpha \in (0, 1)$, $M > 0$ such that $\|Tu\| \leqslant M\|u\|^\alpha$ for all $u \in E$;*
(C_3) *$F : E \to E$ is a bounded continuous operator and there exists $u_0 \in E$ such that $Fu + u_0 + Tu \in P$ for all $u \in E$;*
(C_4) *There exist $v_0 \in E$ and $\eta > 0$ such that*

$$KFu \geqslant r_1^{-1}(1 + \eta)Ku - KTu - v_0 \quad \textit{for all } u \in E. \tag{2.3}$$

Let $A = KF$, then there exists $R > 0$ such that

$$\deg(I - A, B_R, \theta) = 0,$$

where $B_R = \{u \in E\colon\ \|u\| < R\}$ is the open ball of radius R in E.

Proof. Choose a number $R > 0$. Suppose there exist $u_1 \in \partial B_R$ and $\mu_1 \geqslant 0$ such that

$$u_1 = KFu_1 + \mu_1\varphi_1. \tag{2.4}$$

By (2.3) and (2.1), we have

$$\begin{aligned} g_1(u_1) &= g_1(KFu_1) + \mu_1 g_1(\varphi_1) \\ &\geqslant g_1(KFu_1) \\ &\geqslant r_1^{-1}(1 + \eta)g_1(Ku_1) - g_1(KTu_1) - g_1(v_0) \\ &= r_1^{-1}(1 + \eta)\bigl(K^*g_1\bigr)u_1 - \bigl(K^*g_1\bigr)(Tu_1) - g_1(v_0) \\ &= (1 + \eta)g_1(u_1) - r_1 g_1(Tu_1) - g_1(v_0). \end{aligned}$$

Thus

$$g_1(u_1) \leqslant \eta^{-1}r_1 g_1(Tu_1) + \eta^{-1}g_1(v_0). \tag{2.5}$$

Hence, it follows from (2.1), (2.5) and condition (C_2) that

$$\begin{aligned} g_1(u_1 + KTu_1 + Ku_0) &= g_1(u_1) + r_1 g_1(Tu_1) + r_1 g_1(u_0) \\ &\leqslant \big(1+\eta^{-1}\big) r_1 g_1(Tu_1) + \eta^{-1} g_1(v_0) + r_1 g_1(u_0) \\ &\leqslant \big(1+\eta^{-1}\big) r_1 M \|g_1\| \cdot \|u_1\|^\alpha + \eta^{-1} g_1(v_0) + r_1 g_1(u_0) \\ &= C_1 \|u_1\|^\alpha + C_2, \end{aligned} \tag{2.6}$$

where $C_1 = (1+\eta^{-1}) r_1 M \|g_1\| > 0$ and $C_2 = \eta^{-1} g_1(v_0) + r_1 g_1(u_0)$ are two constants. Since $Fu_1 + u_0 + Tu_1 \in P$ from condition (C_3) and $\mu_1 \varphi_1 = \mu_1 r_1^{-1} K \varphi_1 \in P(g_1, \delta)$ from condition (C_1), we have from condition (C_1) and (2.4)

$$u_1 + KTu_1 + Ku_0 = K(Fu_1 + Tu_1 + u_0) + \mu_1 \varphi_1 \in P(g_1, \delta).$$

So, from the definition of $P(g_1, \delta)$,

$$\begin{aligned} g_1(u_1 + KTu_1 + Ku_0) &\geqslant \delta \|u_1 + KTu_1 + Ku_0\| \\ &\geqslant \delta \|u_1\| - \delta \|KTu_1\| - \delta \|Ku_0\|. \end{aligned} \tag{2.7}$$

Thus, by (2.7) and (2.6), we have

$$\begin{aligned} R = \|u_1\| &\leqslant \delta^{-1} g_1(u_1 + KTu_1 + Ku_0) + \|KTu_1\| + \|Ku_0\| \\ &\leqslant \delta^{-1} C_1 \|u_1\|^\alpha + \delta^{-1} C_2 + M\|K\| \cdot \|u_1\|^\alpha + \|Ku_0\| \\ &= C_1' \|u_1\|^\alpha + C_2' \\ &= C_1' R^\alpha + C_2', \end{aligned} \tag{2.8}$$

where $C_1' = \delta^{-1} C_1 + M\|K\| > 0$ and $C_2' = \delta^{-1} C_2 + \|Ku_0\|$ are constants. Since $\alpha \in (0,1)$, (2.8) cannot hold when R is sufficiently large. Choose such a sufficiently large number $R > 0$, then for all $u \in \partial B_R$ and $\mu \geqslant 0$, we have $u \neq Au + \mu \varphi_1$. According to the property of omitting a direction for Leray–Schauder degree, we have

$$\deg(I - A, T_R, \theta) = 0.$$

The proof is done. □

Remark 2.2. If the operator T which satisfies the conditions of Theorem 2.1 is a null operator, then Theorem 2.1 turns into Theorem 1 in [15]. Hence, Theorem 2.1 is an improvement of Theorem 1 in [15]. On the basis of Theorem 2.1, in studying singular BVP (1.1), we can substitute condition (H_1) for condition (H_0). In [9], a result analogous to Theorem 2.1 was applied to investigate a kind of nonlinear Hammerstein integral equation.

3. Proofs of main theorems

In order to use Theorem 2.1, we choose $E = C[0,1]$ to be our real Banach space with the norm $\|u\| = \max_{t \in [0,1]} |u(t)|$ and $P = \{u \in C[0,1]\colon\ u(t) \geqslant 0 \text{ for all } t \in [0,1]\}$. Let K, F and A be the operators defined in (1.7). Then $F : E \to E$ is a nonlinear bounded continuous operator if (H_3) holds. It is obvious that $G(t,s) \leqslant G(s,s) = s(1-s)$ for all $t, s \in [0,1]$. Then we have the following lemma.

Lemma 3.1. *Assume* (H_2) *holds. Then*

(i) $K:E\to E$ *is a completely continuous positive linear operator.*
(ii) K *satisfies condition* (C_1) *of Theorem* 2.1.

Proof. (i) It follows from (H_2) that

$$\big|Ku(t)\big|\leqslant\int_0^1 G(t,s)h(s)\big|u(s)\big|\,\mathrm{d}s\leqslant\|u\|\cdot\int_0^1 G(s,s)h(s)\,\mathrm{d}s<+\infty.$$

Hence, by Lebesgue's dominated convergence theorem, it is easy to see that $K:E\to E$. Obviously, $K(P)\subset P$ from (H_2) and K is a linear operator, namely K is a positive linear operator. Next, we will show that K is completely continuous. For any natural number n ($n\geqslant 2$), let

$$h_n(t)=\begin{cases}\inf_{t<s\leqslant\frac{1}{n}}h(s), & 0\leqslant t\leqslant\frac{1}{n},\\ h(t), & \frac{1}{n}\leqslant t\leqslant\frac{n-1}{n},\\ \inf_{\frac{n-1}{n}\leqslant s<t}h(s), & \frac{1}{n}\leqslant t\leqslant 1.\end{cases}\tag{3.1}$$

Then $h_n:[0,1]\to[0,+\infty)$ is continuous and $h_n(t)\leqslant h(t)$ for all $t\in(0,1)$. Let

$$(K_nu)(t)=\int_0^1 G(t,s)h_n(s)u(s)\,\mathrm{d}s.\tag{3.2}$$

It is clearly that $K_n:E\to E$ is completely continuous. For any $r>0$ and $u\in B_r$, according to (3.1), (3.2) and the absolute continuity of integral, we have

$$\begin{aligned}\lim_{n\to\infty}\|K_nu-Ku\|&=\lim_{n\to\infty}\max_{t\in[0,1]}\left|\int_0^1 G(t,s)\big(h_n(s)-h(s)\big)u(s)\,\mathrm{d}s\right|\\&\leqslant\|u\|\lim_{n\to\infty}\int_0^1 G(s,s)\big(h(s)-h_n(s)\big)\,\mathrm{d}s\\&=\|u\|\lim_{n\to\infty}\int_{e(n)} G(s,s)\big(h(s)-h_n(s)\big)\,\mathrm{d}s\\&\leqslant\|u\|\lim_{n\to\infty}\int_{e(n)} G(s,s)\big(h(s)\big)\,\mathrm{d}s=0,\end{aligned}$$

where $e(n)=[0,1/n]\cup[(n-1)/n,1]$. Therefore, by [5, Chapter 1, Theorem 2.1], $K:E\to E$ is a completely continuous operator.

(ii) It is obvious that there is $t_1\in(0,1)$ such that $G(t_1,t_1)h(t_1)>0$. Thus there exists $[a_1,b_1]\subset(0,1)$ such that $t_1\in(a_1,b_1)$ and $G(t,s)h(s)>0$ for all $t,s\in[a_1,b_1]$. Choose $\psi\in P$ such that $\psi(t_1)>0$ and $\psi(t)=0$ for all $t\notin[a_1,b_1]$. Then for $t\in[a_1,b_1]$,

$$(K\psi)(t)=\int_0^1 G(t,s)h(s)\psi(s)\,\mathrm{d}s\geqslant\int_{a_1}^{b_1} G(t,s)h(s)\psi(s)\,\mathrm{d}s>0.$$

So there exists $c_1 > 0$ such that $c_1(K\psi)(t) \geqslant \psi(t)$ for $t \in [0,1]$. From [7, Chapter 5, Theorem 2.1], we know that the spectral radius $r_1 \neq 0$. Thus, corresponding to $\lambda_1 = r_1^{-1}$, the first eigenvalue of K, K has a positive eigenvector φ_1 and K^* has a positive eigenvector g_1, i.e.,

$$K\varphi_1 = r_1\varphi_1, \qquad K^*g_1 = r_1 g_1.$$

Since $G(0,s) = G(1,s) \equiv 0$ for $s \in [0,1]$, we have $\varphi_1(0) = \varphi_1(1) = 0$. This implies that $\varphi_1'(0) > 0$ and $\varphi_1'(1) < 0$ (see [14]). Define a function Φ on $[0,1]$ by

$$\Phi(s) = \begin{cases} \varphi_1'(0), & s = 0, \\ \frac{\varphi_1(s)}{s(1-s)}, & s \in (0,1), \\ -\varphi_1'(1), & s = 1. \end{cases} \tag{3.3}$$

Then it is easy to see that Φ is continuous on $[0,1]$ and $\Phi(s) > 0$ for all $s \in [0,1]$. So, there exist $\delta_1, \delta_2 > 0$ such that $\delta_1 \leqslant \Phi(s) \leqslant \delta_2$ for all $s \in [0,1]$. Thus

$$\delta_1 G(t,s) \leqslant \delta_1 s(1-s) \leqslant \varphi_1(s) \leqslant \delta_2 s(1-s) = \delta_2 G(s,s). \tag{3.4}$$

for all $t, s \in [0,1]$.

It is easy to see that g_1 can be explicitly given by

$$g_1(u) = \int_0^1 h(t)\varphi_1(t)u(t)\,\mathrm{d}t \quad \text{for all } u \in E. \tag{3.5}$$

In fact, firstly, by (3.4), $\int_0^1 h(t)\varphi_1(t)\,\mathrm{d}t \leqslant \delta_2 \int_0^1 t(1-t)h(t)\,\mathrm{d}t < +\infty$ and then g_1 is well defined. Secondly, notice that $G(t,s) = G(s,t)$ for all $t, s \in [0,1]$, we have

$$\begin{aligned} r_1 g_1(u) &= \int_0^1 h(t)\big(r_1\varphi_1(t)\big)u(t)\,\mathrm{d}t \\ &= \int_0^1 h(t)\left(\int_0^1 G(t,s)h(s)\varphi_1(s)\,\mathrm{d}s\right)\mathrm{d}t \\ &= \int_0^1 h(s)\varphi_1(s)\left(\int_0^1 G(s,t)h(t)u(t)\,\mathrm{d}t\right)\mathrm{d}s \\ &= \int_0^1 h(s)\varphi_1(s)\big[Ku(s)\big]\,\mathrm{d}s \\ &= g_1(Ku) = \big(K^*g_1\big)(u) \quad \text{for all } u \in E. \end{aligned} \tag{3.6}$$

So (3.5) holds. Take $\delta = r_1\delta_1 > 0$ in (2.2). In the following we prove that $K(P) \subset P(g_1, \delta)$. For any $u \in P$, by (3.4) and (3.6), we have

$$\begin{aligned} g_1(Ku) &= r_1 \int_0^1 h(s)\varphi_1(s)u(s)\,\mathrm{d}s \geqslant r_1\delta_1 \int_0^1 G(t,s)h(s)u(s)\,\mathrm{d}s \\ &= r_1\delta_1 (Ku)(t) \quad \text{for all } t \in [0,1]. \end{aligned}$$

Hence, $g(Ku) \geqslant \delta \|Ku\|$, i.e., $K(P) \subset P(g_1, \delta)$ and K satisfies condition (C_1) in Theorem 2.1. The proof is completed. □

Proof of Theorem 1.1. According to Lemma 3.1, K satisfies condition (C_1) in Theorem 2.1. Let $(Tu)(t) = c|u(t)|^\alpha$ for $u \in E$, then T satisfies condition (C_2) in Theorem 2.1. Take $u_0(t) \equiv b$, then it follows from (H_1) that

$$Fu + u_0 + Tu \in P \quad \text{for all } u \in E,$$

namely condition (C_3) in Theorem 2.1 holds. From (1.8), there exists $\varepsilon > 0$ such that

$$f(u) \geqslant \lambda_1(1+\varepsilon)u \tag{3.7}$$

as u is sufficiently large. Combining (H_1) and (3.7), it is easy to see that there exists $b_1 \geqslant 0$ such that

$$f(u) \geqslant \lambda_1(1+\varepsilon)u - b_1 - c|u|^\alpha \quad \text{for all } u \in \mathbb{R}^1. \tag{3.8}$$

Since K is a positive linear operator and (3.8),

$$(KFu)(t) \geqslant \lambda_1(1+\varepsilon)(Ku)(t) - Kb_1 - (KTu)(t) \quad \text{for all } t \in [0,1].$$

So condition (C_4) in Theorem 2.1 holds. According to Theorem 2.1, there exists a sufficiently large number $R > 0$ such that

$$\deg(I - A, B_R, \theta) = 0. \tag{3.9}$$

It follows from (1.9) that there exist $0 < \varepsilon < 1$ and $0 < r < R$ such that

$$\bigl|f\bigl(u(t)\bigr)\bigr| \leqslant (1-\varepsilon)\lambda_1\bigl|u(t)\bigr| \quad \text{for all } t \in [0,1], \tag{3.10}$$

for any $u \in E$ with $\|u\| \leqslant r$. If there exist $u_1 \in \partial B_r$ and $\mu_1 \in [0,1]$ such that $u_1 = \mu_1 A u_1$, then by (3.10) and (3.6),

$$\begin{aligned}
g_1\bigl(|u_1|\bigr) &= \mu_1 g_1\bigl(|Au_1|\bigr) \leqslant g_1\bigl(|KFu_1|\bigr) = g_1\left(\left|\int_0^1 G(t,s)h(s)f\bigl(u_1(s)\bigr)\,\mathrm{d}s\right|\right) \\
&\leqslant g_1\left(\int_0^1 G(t,s)h(s)\bigl|f\bigl(u_1(s)\bigr)\bigr|\,\mathrm{d}s\right) \leqslant (1-\varepsilon)\lambda_1 g_1\left(\int_0^1 G(t,s)h(s)\bigl|u_1(s)\bigr|\,\mathrm{d}s\right) \\
&= (1-\varepsilon)\lambda_1 g_1\bigl(K|u_1|\bigr) = (1-\varepsilon)\lambda_1 r_1 g_1\bigl(|u_1|\bigr) = (1-\varepsilon) g_1\bigl(|u_1|\bigr).
\end{aligned}$$

So

$$g_1\bigl(|u_1|\bigr) \leqslant 0.$$

On the other hand, $\varphi_1(t) > 0$ for all $t \in (0,1)$ by the maximum principle and $u_1(t)$ attains zero on isolated points by Sturm theorem. Hence

$$g_1\bigl(|u_1|\bigr) = \int_0^1 h(t)\varphi_1(t)\bigl|u_1(t)\bigr|\,\mathrm{d}t > 0.$$

This is a contradiction. Thus

$$u \neq \mu A u \quad \text{for all } u \in \partial B_r \text{ and } \mu \in [0,1].$$

Therefore, according to the homotopy invariance of Leray–Schauder degree, we have

$$\deg(I - A, B_r, \theta) = 1. \tag{3.11}$$

By (3.9), (3.11) and the additivity of Leray–Schauder degree, we have

$$\begin{aligned}\deg(I - A, B_R \backslash \overline{B}_r, \theta) &= \deg(I - A, B_R, \theta) - \deg(I - A, B_r, \theta)\\ &= 0 - 1 = -1.\end{aligned}$$

As a result, A has at least one fixed point on $B_R \backslash \overline{B}_r$, namely the singular BVP (1.1) has at least one nontrivial solution. □

Proof of Theorem 1.3. By (1.10) we know that

$$\bigl(A'_\theta u\bigr)(t) = \lambda \int_0^1 G(t,s)h(s)u(s)\,\mathrm{d}s = \lambda(Ku)(t). \tag{3.12}$$

Then 1 is not the eigenvalue of A'_θ. According to Leray–Schauder theorem [5, Chapter 2, Theorem 2.6], there exists $r \geqslant 0$ such that

$$\deg(I - A, B_r, \theta) = \deg\bigl(I - A'_\theta, B_r, \theta\bigr) = \pm 1. \tag{3.13}$$

Similar to the proof of Theorem 1.1, from (H_1) and (1.8), we know that there exists $R > r$ such that

$$\deg(I - A, B_R, \theta) = 0. \tag{3.14}$$

By (3.13), (3.14) and the additivity of Leray–Schauder degree, we have

$$\deg(I - A, B_R \backslash \overline{B}_r, \theta) = \deg(I - A, B_R, \theta) - \deg(I - A, B_r, \theta) = \mp 1.$$

As a result, A has at least one fixed point on $B_R \backslash \overline{B}_r$, namely the singular BVP (1.1) has at least one nontrivial solution. □

Since $(\mathrm{H}_0) \Rightarrow (\mathrm{H}_1)$, Corollaries 1.2 and 1.4 are obvious. Note that the first eigenvalue of K_1 define by

$$(K_1 u)(t) = \int_0^1 G(t,s)u(s)\,\mathrm{d}s \quad \text{for } u \in E$$

is π^2, Theorem 1.7 can be proved as Theorem 1.1. So, we skip it.

Next, we present a simple example to which Theorem 1.1 can be applied.

Example. Let $h(t) = t^{p-1}(1-t)^{q-1}$ with $p, q \in (0,1)$, then h is singular at both $t = 0$ and $t = 1$ and h satisfies (H_2). Let

$$f(u) = \begin{cases} 1 + \sum_{i=1}^n (-1)^i a_i - |u|^{1/2}, & u \in (-\infty, -1],\\ \sum_{i=1}^n a_i u^i, & u \in [-1, +\infty),\end{cases} \tag{3.15}$$

where $0 < a_1 < \lambda_1$ and $a_n > 0$. Then it is easy to see that all the conditions of Theorem 1.1 are satisfied and we infer that singular BVP (1.1) has at least a nontrivial solution. Theorem 1 in [16] is invalid for this example since f is not bounded from below.

To determine λ_1, the first eigenvalue of K, please refer to [13], in which D. O'Regan has given a complete discussion about the eigenvalues and eigenvectors of K. At the end of this paper, we give a rough estimate for λ_1. Without loss of generality, suppose φ_1, the eigenfunction corresponding to λ_1, satisfies $\|\varphi_1\| = \varphi_1(t_0) = 1$, then

$$1 = \varphi_1(t_0) = \lambda_1 \int_0^1 G(t_0, s)h(s)\varphi_1(s)\,\mathrm{d}s \leqslant \lambda_1 \int_0^1 G(t_0, s)h(s)\,\mathrm{d}s$$

$$\leqslant \lambda_1 \max_{t\in[0,1]} \int_0^1 G(t, s)h(s)\,\mathrm{d}s \leqslant \lambda_1 \int_0^1 G(s, s)h(s)\,\mathrm{d}s.$$

Hence, $\lambda_1 \geqslant (\int_0^1 s(1-s)h(s)\,\mathrm{d}s)^{-1}$. On the other hand, it is easy to see that there is $t_1 \in [1/4, 3/4]$ such that $\varphi_1(t_1) = \min_{t\in[1/4,3/4]} \varphi_1(t) > 0$ and $G(t_1, s) \geqslant (1/4)G(s, s)$ for all $s \in [0, 1]$. Thus

$$\varphi_1(t_1) = \lambda_1 \int_0^1 G(t_1, s)h(s)\varphi_1(s)\,\mathrm{d}s \geqslant \lambda_1 \int_{1/4}^{3/4} G(t_1, s)h(s)\varphi_1(s)\,\mathrm{d}s$$

$$\geqslant \lambda_1\varphi_1(t_1) \int_{1/4}^{3/4} G(t_1, s)h(s)\,\mathrm{d}s \geqslant \frac{1}{4}\lambda_1\varphi_1(t_1) \int_{1/4}^{3/4} G(s, s)h(s)\,\mathrm{d}s.$$

So, we have

$$m_1 \triangleq \left(\int_0^1 s(1-s)h(s)\,\mathrm{d}s\right)^{-1} \leqslant \lambda_1 \leqslant 4\left(\int_{1/4}^{3/4} s(1-s)h(s)\,\mathrm{d}s\right)^{-1} \triangleq M_1.$$

As a result, by Theorem 1.1, if

$$\liminf_{u\to+\infty} \frac{f(u)}{u} > M_1,$$

$$\limsup_{u\to 0} \left|\frac{f(u)}{u}\right| < m_1,$$

then singular BVP (1.1) has at least one nontrivial solution.

Acknowledgment

The authors thank the referee for his helpful remarks.

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