Journal of Number Theory **87**, 253–269 (2001)
doi:10.1006/jnth.2000.2616, available online at http://www.idealibrary.com on IDEAL®

An Adelic Causality Problem Related to Abelian L-Functions

Jean-François Burnol

Université de Nice—Sophia Antipolis, Laboratoire J.-A. Dieudonné, Parc Valrose, F-06108 Nice Cédex 02, France
E-mail: burnol@math.unice.fr

Communicated by J.-L. Colliot-Thélène

Received February 10, 2000

tion on a theme initiated by Nyman and Beurling. The adelic aspects are related to previous work by Tate, Iwasawa, and Connes.

1. ADELES, IDELES, AND ZEROS

We express the Riemann hypothesis for abelian L-functions as a Hilbert space closure property (Theorem 1.1). This takes place within the adelic set-up used by Tate [20] and Iwasawa [15] to establish the functional equations of these L-functions. We treat simultaneously the number field and function field cases (the Tate–Iwasawa ideas have been adapted by Weil to the function field case in [22]). Our approach is Hilbert space-theoretical. We take our hint from Nyman's equivalent formulation of the original Riemann hypothesis [19]. Beurling [5] (for the disc) and Lax [16] (for the half-plane) described the invariant subspaces of the Hardy spaces and, as is explained in [1] (see also [3] and [4] for Beurling's L^p-extension [6]), this description is the conceptual element behind Nyman's thorem. We devote a section to explain (without mention of adeles and ideles) what our construction amounts to for the Riemann zeta function. It is technically of a very straightforward nature, its only deeper aspects being embedded in the Beurling–Lax theory.

We associate to the global field an adelic Lax–Phillips scattering [17]. All axioms (where the idele class group replaces the more usual $\mathbb{Z}$ or $\mathbb{R}$) are satisfied, except possibly the *causality axiom* which we show to be equivalent to the Riemann hypothesis (this is our main result, Theorem 1.7).

0022-314X/01 $35.00

The validity of one of the axioms is related to an observation of Connes [9, proof of VIII.5]. The study of connections between the Riemann zeta function and scattering theory is at least 30 years old. In particular the Faddeev–Pavlov study of scattering for automorphic functions [13], further developed by Lax and Phillips in their book [18], has attracted widespread attention. In their approach the scattering matrix is directly related to the values taken by the Riemann zeta function on the line $\mathrm{Re}(s) = 1$, and the Riemann hypothesis itself is equivalent to some decay properties of scattering waves. Another well-known instance is the approach of De Branges [11, 12] within the theory of Hilbert spaces of entire functions, also related to scattering (Conrey and Li recently pointed out some difficulties of this approach [10]). The connection between our scattering process and the Riemann zeta function (or more generally an abelian L-function) is the following: each bad zero ($\mathrm{Re}(\rho) > \frac{1}{2}$) appears as a pole of the scattering operator, where there should be none, if the process was causal. But if the Riemann hypothesis holds, then the scattering itself is of a trivial nature and says absolutely nothing on the zeros on the critical line. We point out that the same holds with the positivity criterion of Weil [21, 23]: the Weil distribution are of positive type if and only if the Riemann hypothesis holds, but beyond that, positivity tells nothing on the location of the zeros except that they are indeed on the critical line. Our formulation applies equally well to function fields and number fields: this is as in Weil's positivity approach (especially when formulated as in [8]) and as in the work of Connes [9]. The infinite places cause us less trouble than in [21] and [9]. Our sole motivation in formulating the Riemann hypothesis in a novel manner is the hope that creators of other tools, of a deeper nature than those used here, would incorporate the gained insight in their design constraints. An obvious deficiency of this paper is its inability to achieve an alternative proof of the Riemann hypothesis in the function field case, where it is not a hypothesis but a well-known theorem.

Let K be a global field (an **A**-field in the terminology of Weil [22]), either an algebraic number field or a field finitely generated and of transcendence degree 1 over a finite field. We briefly review some normalizations. The adele ring $\mathbb{A}_K$ is its own Pontrjagin dual. The set of characters (additive, unitary) for which K (diagonally embedded) is its own annihilator is nonempty (and a single orbit under the action of $K^\times$). We pick one such good character and let the additive Fourier transform $\mathscr{F}$ be defined with respect to it (and the corresponding self-dual Haar measure, which is in fact independent of the choice made). On each local multiplicative group $K_v^\times$ we write d^*v_v for the multiplicative measure which assigns volume 1 to the units (finite place) or is $\frac{dx}{2\,|x|}$ (real place) or $\frac{dr\,d\theta}{\pi r}$ (complex place). On the idele group $\mathbb{A}_K^\times$ (also seen as a subset of $\mathbb{A}_K$) we use $d^*v = \prod_v d^*v_v$, and on the idele class group $\mathscr{C}_K = \mathbb{A}_K^\times / K^\times$ we use the

Haar measure d^*u which (function field case) assigns volume 1 to the units or (number field case) is pushed down to $\frac{dt}{t}$ under $t = |u| = |v| = \prod_\nu |v_\nu|_\nu$ ($v \in \mathbb{A}_K^\times$, $u = \bar{v}$).

Let $\mathscr{S}(\mathbb{A}_K)$ be the vector space of Bruhat–Schwartz functions.

DEFINITION 1.1.

$$E: \mathscr{S}(\mathbb{A}_K) \to (\mathscr{C}_K \to \mathbb{C})$$

$$\varphi(x) \mapsto f(\bar{v}) = \sqrt{|v|} \sum_{q \in K^\times} \varphi(qv) - \frac{\int_{\mathbb{A}_K} \varphi(x)\, dx}{\sqrt{|v|}}$$

For functions satisfying the additional conditions $\varphi(0) = \int_{\mathbb{A}_K} \varphi(x)\, dx = 0$, E is a tool at the heart of the constructions of Connes in [9]. For technical, class-field theoretical, reasons, we do not impose any vanishing condition. The map E is related to the ideas of Tate [20] and Iwasawa [15] and is especially tuned for Hilbert space matters, as expressed in the following lemma:

LEMMA 1.2. *$E(\mathscr{S}(\mathbb{A}_K)) \subset L^2(\mathscr{C}_K, d^*u)$ and is dense in it. The Fourier–Mellin transform of $E(\varphi)$, as a function of the unitary characters of $\mathscr{C}_K$, is, up to a multiplicative constant depending only on K, equal to the Tate L-functions associated to φ (restricted to the critical line).*

Note 1.3. As has already been noted by Connes [9, proof of VIII.5], $E(\mathscr{S}_{00})$ is dense in $L^2(\mathscr{C}_K, d^*u)$, where $\mathscr{S}_{00} = \{\varphi \in \mathscr{S}(\mathbb{A}_K) \mid \varphi(0) = \int_{\mathbb{A}_K} \varphi(x)\, dx = 0\}$.

The idele group acts on $\mathscr{S}(\mathbb{A}_K)$ ($U(v) \cdot \varphi(x) = (1/\sqrt{|v|})\, \varphi(x/v)$) and on $L^2(\mathscr{C}_K, d^*u)$ ($U(v) \cdot f(u) = f(u/\bar{v})$), and E intertwines the two actions. Furthermore the Poisson–Tate summation formula shows that E intertwines the Fourier transform $\mathscr{F}$ on $\mathbb{A}_K$ with the inversion I ($f(u) \mapsto f(\frac{1}{u})$) on $\mathscr{C}_K$. Each idele v defines an adelic parallelepiped

$$P(v) = \{x = (x_\nu) \in \mathbb{A}_K \mid \forall \nu\ |x_\nu|_\nu \leqslant |v_\nu|_\nu\}$$

whose volume is proportional to $|v|$.

DEFINITION 1.4.

$$\mathscr{S}_{\leqslant 1} = \{\varphi \in \mathscr{S}(\mathbb{A}_K) \mid \exists v \in \mathbb{A}_K^\times : |v| = 1 \text{ and } \operatorname{supp}(\varphi) \subset P(v)\}$$

$$\widetilde{\mathscr{S}_{\leqslant 1}} = \{\varphi \in \mathscr{S}(\mathbb{A}_K) \mid \exists v \in \mathbb{A}_K^\times : |v| = 1 \text{ and } \operatorname{supp}(\mathscr{F}(\varphi)) \subset P(v)\}$$

$$\mathscr{D}_+ = E(\mathscr{S}_{\leqslant 1})^\perp$$

$$\mathscr{D}_- = E(\widetilde{\mathscr{S}_{\leqslant 1}})^\perp$$

LEMMA 1.5. *The Lax–Phillips scattering axioms* ([17], *with* $\mathbb{Z}$ *or* $\mathbb{R}$ *replaced with* $\mathscr{C}_K$) *are satisfied for the incoming subspace* $\mathscr{D}_-$

$$|\lambda| \leqslant 1 \Rightarrow U(\lambda)\,\mathscr{D}_- \subset \mathscr{D}_- \qquad \bigwedge U(\lambda)\,\mathscr{D}_- = \{0\} \qquad \overline{\bigvee U(\lambda)\,\mathscr{D}_-} = L^2(\mathscr{C}_K, d^*u)$$

and for the outgoing subspace $\mathscr{D}_+$

$$|\lambda| \geqslant 1 \Rightarrow U(\lambda)\,\mathscr{D}_+ \subset \mathscr{D}_+ \qquad \bigwedge U(\lambda)\,\mathscr{D}_+ = \{0\} \qquad \overline{\bigvee U(\lambda)\,\mathscr{D}_+} = L^2(\mathscr{C}_K, d^*u)$$

Note 1.6. The property $\bigwedge U(\lambda)\,\mathscr{D}_+ = \{0\}$ is cousin to the density property $\overline{E(\mathscr{S}_{00})} = L^2(\mathscr{C}_K, d^*u)$ noted by Connes. The property $\overline{\bigvee U(\lambda)\,\mathscr{D}_+} = L^2(\mathscr{C}_K, d^*u)$ is an easy corollary of the Artin–Whaples product formula. As $\mathscr{D}_- = I(\mathscr{D}_+)$ and as I is an isometry which interchanges dilations and contractions, the axioms for $\mathscr{D}_-$ and $\mathscr{D}_+$ are equivalent.

Our main result is:

THEOREM 1.7 (A causality criterion). *The Riemann hypothesis holds for all abelian L-functions of K if and only if* $\mathscr{D}_- \perp \mathscr{D}_+$.

We also express the Riemann hypothesis as a closure property. We need a slightly technical definition first:

DEFINITION 1.8. Let A be the convolution operator

$$(A \cdot f)(u_0) = \int_{\mathscr{C}_K} a\left(\frac{u_0}{u}\right) f(u)\, d^*u,$$

where in the number field case

$$a(w) = \sqrt{|w|} \cdot \mathbf{1}_{|w| \leqslant 1}$$

and in the function field case (q the cardinality of the field of constants)

$$a(w) = \sqrt{|w|} \cdot \left(\sqrt{q} - \frac{1}{\sqrt{q}}\right) \cdot \mathbf{1}_{|w|<1} + \left(1 - \frac{1}{\sqrt{q}}\right) \mathbf{1}_{|w|=1}.$$

DEFINITION 1.9.

$$\mathbb{H}^2 = \{ f \in L^2(\mathscr{C}_K, d^*u) \mid \text{ess} - \text{supp}(f) \subset \{|u| \leqslant 1\}\}$$

LEMMA 1.10. *The operator* $V = 1 - A$ *is a unitary operator on* $L^2(\mathscr{C}_K, d^*u)$, *commuting with the regular action of* $\mathscr{C}_K$, *and sending* $\mathbb{H}^2$ *to* (*a subspace of*) *itself.*

THEOREM 1.11 (A closure criterion). *$V(\overline{E(\mathcal{S}_{\leqslant 1})}) \subset \mathbb{H}^2$ with equality if and only if the Riemann hypothesis holds for all abelian L-functions of K.*

2. THE CRITERION FOR THE RIEMANN ZETA FUNCTION

When considering only the Riemann zeta function, Theorem 1.11 boils down to a variant of Nyman's criterion [19]. Let us recall this criterion (see also [1, 3, 4]):

Let $\rho_\alpha(u) = \{\frac{\alpha}{u}\} - \alpha\{\frac{1}{u}\}$, for $0 < \alpha < 1$, and $u \in (0, 1)$ (with $\{\cdot\}$ the fractional part). Let N be the closed span in $L^2((0, 1), du)$ of the functions ρ_α. We also consider both N and $L^2((0, 1), du)$ as closed subspaces of $L^2((0, \infty), du)$.

THEOREM 2.1 (Nyman [19]). *The constant function* **1** *on* $(0, 1)$ *belongs to N if and only if the Riemann hypothesis holds.*

Note that N is invariant under the semigroup of unitary contractions $U(\lambda)$: $f(u) \mapsto \sqrt{1/\lambda}\, f(\frac{u}{\lambda})$, $\lambda \leqslant 1$, $u > 0$ as $U(\lambda) \cdot \rho_\alpha = \sqrt{1/\lambda}\,(\rho_{\alpha\lambda} - \alpha\rho_\lambda)$. So, it will contain the constant function **1** (hence all step functions) if and only if it actually coincides with all of $L^2((0, 1), du)$.

For the proof one considers the Mellin transform

$$f(u) \mapsto \hat{f}(s) = \int_0^1 f(u)\, u^{s-1}\, du$$

which by a Paley–Wiener theorem establishes an isometry between $L^2((0, 1), du)$ and the Hardy space $\mathbb{H}^2(\mathrm{Re}(s) > \frac{1}{2})$ of analytic functions with bounded norm

$$\|F\|^2 = \sup_{\sigma > 1/2} \int_{\mathrm{Re}(s) = \sigma} |F(s)|^2 \frac{|ds|}{2\pi}.$$

Such functions $\hat{f}(s)$ have (a.e.) boundary values also obtained as

$$\hat{f}\left(\frac{1}{2} + i\tau\right) = \operatorname*{l.i.m}_{\epsilon \to 0} \int_\epsilon^1 \sqrt{u}\, f(u)\, u^{i\tau} \frac{du}{u}$$

equivalently as the Fourier–Plancherel transform of $e^{t/2} f(e^t)$, $t \leqslant 0$.

The unitary semigroup considered above acts on $\mathbb{H}^2(\mathrm{Re}(s) > \frac{1}{2})$ as $F(s) \mapsto \lambda^{s-1/2} F(s)$, and Lax [16] has described the closed subspaces invariant under this action. It can be directly shown (see [14]) that the conformal representation

$$w = \frac{s-1}{s}$$

$$g(w) = s \cdot F(s)$$

establishes an isometry between $\mathbb{H}^2(\mathrm{Re}(s) > \frac{1}{2})$ and $\mathbb{H}^2(|w| < 1)$ which identifies the invariant subspaces of the former with closed subspaces invariant under the shift $g(w) \mapsto w \cdot g(w)$ for the latter. These were described by Beurling [5] and we learn that the continuous case (Lax) and discrete case (Beurling) are completely equivalent (this equivalence is also a corollary to the conformal invariance of Brownian motion on the complex numbers).

The Beurling–Lax recipe to determine an invariant closed subspace such as N is to look at the Mellin transforms of the functions $\rho_\alpha(u)$'s

$$\widehat{\rho_\alpha}(s) = \frac{\alpha - \alpha^s}{s} \zeta(s)$$

and at the "greatest lower bound of their inner factors": first there will be the Blaschke product

$$B(s) = \prod_{\zeta(\rho)=0,\ \mathrm{Re}(\rho) > 1/2} \frac{s-\rho}{s-(1-\bar{\rho})} \frac{1-\bar{\rho}}{\rho} \left| \frac{\rho}{1-\rho} \right|,$$

where the zeros appear according to their multiplicities, then an inner factor associated to a singular measure on the critical line (the analytic continuation of $\zeta(s)$ implies its nonexistence), and a final inner factor $\lambda^{s-1/2}$ $(0 < \lambda \leqslant 1)$. We argue that $\lambda = 1$ as follows: $\lambda^{s-1/2}\mathbb{H}^2$ is the Mellin transform of $L^2((0, \lambda), du)$ which contains N only if $\lambda = 1$ (obviously).

Bercovici and Foias [3, 2.1] prove $\lambda = 1$ in the following manner: if $\widehat{\rho_\alpha}(s) = \lambda^{s-1/2} f(s)$ for some $f(s) \in \mathbb{H}^2(\mathrm{Re}(s) > \frac{1}{2})$ then $\widehat{\rho_\alpha}(\sigma) = O(\lambda^\sigma)$ for $\sigma \to +\infty$. Indeed[1] $f(s)$ is $O(1)$ in any half-plane $\mathrm{Re}(s) \geqslant \frac{1}{2} + \varepsilon$, $\varepsilon > 0$ (this follows from its Cauchy integral representation or from $\hat{f}(s) = \int_0^1 f(u)\, u^{s-1}\, du$ and Cauchy–Schwarz). But obviously $\lim_{\sigma \to +\infty} \sigma \cdot \widehat{\rho_\alpha}(\sigma) \neq 0$, thus giving a contradiction if $\lambda < 1$. The following lemma, of independent interest, could also have been used:

LEMMA 2.2. *If $F(s) \in \mathbb{H}^2$ is $O(|s|^K)$ on the critical line, then its outer factor $F_{\mathrm{out}}(s)$ is $O(|s|^K)$ on the entire closed half-plane.*

[1] I thank the referee for correcting my incomplete understanding of the Bercovici–Foias proof at this point.

Proof. One has

$$\log(|s\,F_{\text{out}}(s)|) = \int_{\text{Re}(s_0)=1/2} \log(|s_0 F(s_0)|)\,\frac{2\,\text{Re}(s)-1}{|s-s_0|^2}\,\frac{|ds_0|}{2\pi}$$

and

$$\log(|s|) = \int_{\text{Re}(s_0)=1/2} \log(|s_0|)\,\frac{2\,\text{Re}(s)-1}{|s-s_0|^2}\,\frac{|ds_0|}{2\pi};$$

hence the result. ∎

Let us add a few more words to this discussion of Nyman's theorem. As

$$\int_0^1 \left\{\frac{1}{u}\right\} u^{s-1}\,du = \frac{1}{s-1} - \frac{\zeta(s)}{s}$$

(for $\text{Re}(s) > 0$) and $\frac{s-1}{s}\cdot\frac{1}{s-1} = \frac{1}{s} = \int_0^1 u^{s-1}\,du$ we see that $\frac{s-1}{s}\frac{\zeta(s)}{s}$ belongs to $\mathbb{H}^2(\text{Re}(s) > \frac{1}{2})$. The unitary operator V on $L^2((0,\infty), du)$ given by the multiplier $\frac{s-1}{s}$ in the spectral representation acts as

$$f(u) \mapsto f(u) - \int_u^\infty \frac{1}{t} f(t)\,dt.$$

As $\frac{\zeta(s)}{s} = -\int_0^\infty \left\{\frac{1}{u}\right\} u^{s-1}\,du$ (for $0 < \text{Re}(s) < 1$) we obtain after a straightforward computation:

$$\frac{s-1}{s}\,\frac{\zeta(s)}{s} = \int_0^1 A(u)\,u^{s-1}\,du$$

$$A(u) = \left[\frac{1}{u}\right]\log(u) + \log\left(\left[\frac{1}{u}\right]!\right) + \left[\frac{1}{u}\right].$$

Stirling's formula implies $A(u) = \frac{1}{2}\log(\frac{1}{u}) + O(1)$ so this integral representation is valid for $\text{Re}(s) > 0$. As $A(u) = 1 + \log(u)$ for $\frac{1}{2} < u \leqslant 1$ there is no inner factor of the type $\lambda^{s-1/2}$ with $\lambda < 1$. There is no other singular factor thanks to the analytic continuation, so $\frac{s-1}{s}\frac{\zeta(s)}{s}$ is the product of an outer factor with the Blaschke product $B(s)$. Hence

THEOREM 2.4. *The Riemann hypothesis holds if and only if* $\frac{s-1}{s}\frac{\zeta(s)}{s}$ *is an outer function or equivalently if the functions* $U(\lambda)\cdot A(u)$ $(0 < \lambda \leqslant 1)$ *span* $L^2((0,1), du)$.

The generalized Jensen's formula (see [14]) then implies a formula first derived by Balazard, Saias, and Yor:

THEOREM 2.5 [2].

$$\frac{1}{2\pi}\int_{\mathrm{Re}(s)=1/2}\frac{\log|\zeta(s)|}{|s|^2}\,|ds| = \sum_{\zeta(\rho)=0,\ \mathrm{Re}(\rho)>1/2}\log\left|\frac{\rho}{1-\rho}\right|$$

The only difference with the proof of Balazard, Saias, and Yor is that we do not need the general theory of Hardy spaces beyond that of $\mathbb{H}^2$, which is of a more elementary nature. This concludes our discussion of Nyman's theorem. We now turn to some variations on this theme (other variations have been considered by Bercovici and Foias in [3] and [4]).

Let $\phi(x)$ be a smooth function on the real line with compact support in $[0, 1]$, and $\int_0^1 \phi(x)\,dx=0$. The Mellin transform

$$\hat{\phi}(s)=\int_0^\infty \phi(u)\,u^{s-1}\,du$$

is an entire function, vanishing at 1. We consider

$$T(\phi)(u)=\sum_{n\geqslant 1}\phi(nu)\qquad (u>0)$$

which is a smooth function of u on $(0, \infty)$ with support in $(0, 1]$. Its behavior when $u\to 0$ is governed by the Poisson summation formula:

$$T(\phi)(u)=\frac{1}{|u|}\sum_{n\in\mathbb{Z}}\psi\left(\frac{n}{u}\right),$$

where ψ is the Fourier transform $\int \phi(y)\,e^{2\pi i\,xy}\,dy$ of ϕ (hence belongs to the Schwartz space of rapidly decreasing functions). So

$$\forall K\qquad T(\phi)(u)=_{u\to 0}O(u^K)$$

and the Mellin transform

$$\widehat{T(\phi)}(s)=\int_0^1 T(\phi)(u)\,u^{s-1}\,du$$

is an entire function. For $\mathrm{Re}(s)>1$

$$\widehat{T(\phi)}(s)=\zeta(s)\,\hat{\phi}(s);$$

hence by analytic continuation this holds true for all s.

Let $\mathcal{S}^0_{\leqslant 1}$ be the vector space consisting of these functions ϕ, $\overline{\mathcal{S}^0_{\leqslant 1}}$ its closure in $L^2((0, 1), du)$, and K the closure of the vector space of functions $T(\phi)$. Both $\overline{\mathcal{S}^0_{\leqslant 1}}$ and K are invariant under contractions, and hence described by the Beurling–Lax theory. One just has to take the greatest lower bound of the inner factors of the $\hat{\phi}(s)$'s (resp. the $\widehat{T(\phi)}(s)$'s). Obviously $\overline{\mathscr{S}^0_{\leqslant 1}}$ is the subspace perpendicular to the constant $\mathbf{1}$ and this shows that the greatest lower bound for the zeros of the $\hat{\phi}(s)$'s is simply $s=1$ with multiplicity 1. This cancels exactly the pole of the zeta function. For the $\widehat{T(\phi)}(s)$'s the analytic continuation across the critical line implies that the only possible singular factor is of the type $\lambda^{s-1/2}$ with $\lambda \leqslant 1$. For a suitably chosen ϕ, $T(\phi)$ does not vanish in $(\frac{1}{2}, 1)$ so necessarily $\lambda = 1$. The conclusion is that K coincides with the space N considered by Nyman. Thus:

THEOREM 2.6. *The Riemann hypothesis holds if and only if the constant function* $\mathbf{1}$ *belongs to the closure of* $\{T(\varphi)\colon \varphi \in \mathscr{S}^0_{\leqslant 1}\}$.

We describe one more variation. Let $\mathscr{S}^{ev}$ be the vector space of even Schwartz functions on $\mathbb{R}$. Let, for $u > 0$:

$$E(\varphi)(u) = \sum_{n \geqslant 1} \varphi(nu) - \frac{\int_0^\infty \varphi(x)\, dx}{u}.$$

The Poisson summation formula gives

$$E(\varphi)(u) = \frac{1}{u} \sum_{n \geqslant 1} \mathscr{F}(\varphi)\left(\frac{n}{u}\right) - \frac{1}{2}\varphi(0)$$

so that $E(\varphi)(u)$ is $0(1)$ when $u \to 0$ and is $O(\frac{1}{u})$ when $u \to \infty$ and belongs to $L^2(\mathbb{R}_+, du)$. Its Mellin transform

$$\widehat{E(\varphi)}(s) = \int_0^\infty E(\varphi)(u)\, u^{s-1}\, du$$

is absolutely convergent and analytic for $0 < \mathrm{Re}(s) < 1$. It can be rewritten as

$$\int_0^1 E(\varphi)(u)\, u^{s-1}\, du + \int_1^\infty \sum_{n \geqslant 1} \varphi(nu)\, u^{s-1}\, du + \frac{\int_0^\infty \varphi(x)\, dx}{s-1}$$

which is then valid in the half-plane $\mathrm{Re}(s) > 0$. Then, for $\mathrm{Re}(s) > 1$, as

$$\int_0^\infty \sum_{n \geqslant 1} \varphi(nu)\, u^{s-1}\, du - \int_0^1 \frac{\int_0^\infty \varphi(x)\, dx}{u}\, u^{s-1}\, du + \frac{\int_0^\infty \varphi(x)\, dx}{s-1}$$

hence simply as

$$\sum_{n\geqslant 1} n^{-s} \int_0^\infty \varphi(u)\, u^{s-1}\, du = \zeta(s)\, \hat{\varphi}(s)$$

which remains valid for $\mathrm{Re}(s) > 0$.

We now need to get rid of the pole of $\zeta(s)$ with the help of the operator V (which on $L^2(\mathbb{R}_+, du)$ acts as $\frac{s-1}{s}$ in the spectral representation):

$$V \cdot f(u) = f(u) - \int_u^\infty \frac{1}{v} f(v)\, dv$$

One checks $V \cdot \frac{1}{u} = 0$ so

$$VE(\varphi)(u) = \sum_{n\geqslant 1} \varphi(nu) - \int_u^\infty \sum_{n\geqslant 1} \varphi(nv) \frac{dv}{v} = \sum_{n\geqslant 1} \varphi(nu) - \int_0^\infty \left[\frac{v}{u}\right] \varphi(v) \frac{dv}{v}.$$

Let $\mathcal{S}_{\leqslant 1}$ be the vector space of smooth even functions with support in $[-1, 1]$. For $\varphi \in \mathcal{S}_{\leqslant 1}$, $VE(\varphi)$ has support in $(0, 1]$ and its Mellin transform $\frac{s-1}{s} \zeta(s)\, \hat{\varphi}(s)$ thus belongs to $\mathbb{H}^2$. As in the previous discussions, the Mellin transform of the (closure of) $VE(\mathcal{S}_{\leqslant 1})$ is the space of multiples of the Blaschke product $B(s)\, \mathbb{H}^2$. Hence:

THEOREM 2.7. *$\overline{VE(\mathcal{S}_{\leqslant 1})} \subset \mathbb{H}^2$ with equality if and only if the Riemann hypothesis holds.*

Let B be the unitary operator on $L^2(\mathbb{R}_+, du)$ which acts in the spectral representation as multiplication with $B(s)$. Let

$$\mathcal{D}_+ = E(\mathcal{S}_{\leqslant 1})^\perp = V^{-1}B \cdot (\mathbb{H}^2)^\perp = V^{-1}BI \cdot \mathbb{H}^2$$

(where I is the inversion $f(u) \mapsto \frac{1}{u} f(\frac{1}{u})$, or spectrally $s \mapsto 1-s$). Let

$$\mathcal{D}_- = E(\mathcal{F}(\mathcal{S}_{\leqslant 1}))^\perp = I(\mathcal{D}_+) = IV^{-1}BI \cdot \mathbb{H}^2 = VB^{-1} \cdot \mathbb{H}^2$$

Then, in the terminology of Lax and Phillips [17], $\mathcal{D}_+$ (resp. $\mathcal{D}_-$) is an outgoing (resp. incoming) space for the action of $\mathbb{R}_+^\times$ on $L^2(\mathbb{R}_+, du)$. The scattering operator associated to them is

$$S = (V^{-1}B)^{-1} \cdot VB^{-1} = V^2 B^{-2}.$$

It is an invariant operator whose spectral multiplier is $(\frac{s-1}{s})^2 \cdot B(s)^{-2}$ and is an inner function if and only if $B(s)$ has no zero in $\mathrm{Re}(s) > \frac{1}{2}$, that is if the Riemann hypothesis holds. The scattering multiplier is inner if and only if $\mathcal{D}_+ \perp \mathcal{D}_-$. So:

THEOREM 2.8. $E(\mathcal{S}_{\leqslant 1})^{\perp} \subset \overline{E(\mathcal{F}(\mathcal{S}_{\leqslant 1}))}$ *if and only if the Riemann hypothesis holds.*

3. AN ADELIC SCATTERING

We now prove Theorems 1.7 and 1.11. Let $\mathcal{C}_K^1$ be the (compact) subgroup of idele classes of unit modulus. There is some (noncanonical) isomorphism $\mathcal{C}_K = \mathcal{C}_K^1 \times N$, $N = \{|u| : u \in \mathcal{C}_K\} \subset \mathbb{R}_+^\times$. If K has positive characteristic we let q be the cardinality of the field of constants. It is known that the module group N is $q^{\mathbb{Z}}$. Each character χ of $\mathcal{C}_K^1$ extends to a character of $\mathcal{C}_K$ trivial on N, which we still denote by χ. At each place v there is a local character χ_v from the embedding $K_v^\times \to \mathcal{C}_K$. And χ is said to be ramified at v if the restriction of χ_v to the unit subgroup is non-trivial.

We start with the properties of

$$E: \mathcal{S}(\mathbb{A}_K) \to (\mathcal{C}_K \to \mathbb{C})$$

$$\varphi(x) \mapsto f(\bar{v}) = \sqrt{|v|} \sum_{q \in K^\times} \varphi(qv) - \frac{\int_{\mathbb{A}_K} \varphi(x)\, dx}{\sqrt{|v|}}$$

From the definition one has $E(\varphi)(u) = O(1/\sqrt{|u|})$ when $|u| \to \infty$, and as the Poisson–Tate formula gives

$$E \cdot \mathcal{F} = I \cdot E$$

one also has $E(\varphi)(u) = O(\sqrt{|u|})$ when $|u| \to 0$. So indeed

$$E(\mathcal{S}(\mathbb{A}_K)) \subset L^2(\mathcal{C}_K, d^*u).$$

Let χ be a unitary character on $\mathcal{C}_K$ (trivial on N). The Fourier–Mellin transform (for $\mathrm{Re}(s) = \frac{1}{2}$)

$$\widehat{E(\varphi)}(\chi, s) = \int_{\mathcal{C}_K} E(\varphi(u))\, \chi(u)\, |u|^{s-1/2}\, d^*u$$

is in fact absolutely convergent and analytic for $0 < \mathrm{Re}(s) < 1$. It can be rewritten (with $u = \bar{v}$, $v \in \mathbb{A}_K^\times$) as

$$\int_{|u| \leqslant 1} E(\varphi)(u)\, \chi(u)\, |u|^{s-1/2}\, d^*u + \int_{|u|>1} \sum_{q \in K^\times} \varphi(qv)\, \chi(u)\, |u|^s\, d^*u$$

$$-\int_{\mathbb{A}_K} \varphi(x)\, dx \int_{|u|>1} \chi(u)\, |u|^{s-1}\, d^*u.$$

The integral $\int_{|u|>1} \chi(u)\,|u|^{s-1}\,d^*u$ (which vanishes if $\chi \neq \mathbf{1}$) is a meromorphic function $F_\chi(s)$, which can be evaluated explicitly. One obtains (both in the number field and in the function field cases)

$$(\mathrm{Re}(s)>1) \Rightarrow F_\chi(s) = -\int_{|u|\leqslant 1} \chi(u)\,|u|^{s-1}\,d^*u.$$

So $\widehat{E(\varphi)}(\chi, s)$ has a meromorphic continuation to $\mathrm{Re}(s)>0$ which, for $\mathrm{Re}(s)>1$, coincides with

$$\int_{|u|\leqslant 1} E(\varphi)(u)\,\chi(u)\,|u|^{s-1/2}\,d^*u + \int_{|u|>1} \sum_{q\in K^\times} \varphi(qv)\,\chi(u)\,|u|^s\,d^*u$$
$$+\int_{\mathbb{A}_K} \varphi(x)\,dx \int_{|u|\leqslant 1} \chi(u)\,|u|^{s-1}\,d^*u$$
$$=\int_{\mathscr{C}_K} \sum_{q\in K^\times} \varphi(qv)\,\chi(u)\,|u|^s\,d^*u$$
$$=C(K)\int_{\mathbb{A}_K^\times} \varphi(v)\,\chi(v)\,|v|^s\,d^*v,$$

the constant $C(K)$ being as in Tate's thesis [20] related to the way the measures d^*u on $\mathscr{C}_K$ and d^*v on $\mathbb{A}_K^\times$ differ. We recognize in the last integral the Tate L-function $L(\varphi, \chi, s)$. The identity

$$\widehat{E(\varphi)}(\chi, s) = C(K)\,L(\varphi, \chi, s)$$

for $\mathrm{Re}(s) = \frac{1}{2}$ holds by analytic continuation. With this, Lemma 1.2 is proven.

We turn to the description of $\Delta = \overline{E(\mathscr{S}_{\leqslant 1})}$. The crucial thing is that it is invariant (obviously) under the (unitary) action of the semigroup of contractions $\{|u| \leqslant 1\}$, in particular under the action of the compact group $\mathscr{C}_K^1$. It thus decomposes as a Hilbert space sum of isotypical components Δ_χ, which we wish to compare to the isotypical components of $\mathbb{H}^2 = \{f \in L^2(\mathscr{C}_K, d^*u) \mid \mathrm{ess}-\mathrm{supp}(f) \subset \{|u| \leqslant 1\}\}$. We do this in the spectral representation using the Fourier–Mellin transform (in the function field case we write $z = q^{-(s-1/2)}$).

First, it is a straightforward check that the A-operator (1.8) is an invariant operator whose action on L^2 is given by the following spectral multipliers $A(\chi, s)$:

$$\chi \neq 1 \Rightarrow A(\chi, s) = 0$$

$$A(1, s) = \frac{1}{s} \qquad \text{(number field case)}$$

$$A(1, z) = 1 - \frac{1-\sqrt{q}z}{\sqrt{q}-z} \qquad \text{(function field case)}$$

so that $V = 1 - A$ is indeed a unitary (on L^2) invariant operator with multipliers

$$\chi \neq 1 \Rightarrow V(\chi, s) = 1$$

$$V(1, s) = \frac{s-1}{s} \qquad \text{(number field case)}$$

$$V(1, z) = \frac{1-\sqrt{q}\,z}{\sqrt{q}-z} \qquad \text{(function field case)}$$

From this spectral representation or with a direct computation we also find the important identity

$$V\left(\frac{1}{\sqrt{|u|}} \cdot \mathbf{1}_{|u|>1}\right) = -\alpha(K)\sqrt{|u|} \cdot \mathbf{1}_{|u| \leqslant 1}$$

with $\alpha(K) = 1$ (resp. $1/\sqrt{q}$) in the number field case (resp. function field case). From the Artin–Whaples product formula we obtain $E(\varphi)(u) = -(\int_{\mathbb{A}_K} \varphi(x)\, dx)/\sqrt{|u|}$ for $|u| > 1$ and $\varphi \in \mathscr{S}_{\leqslant 1}$. So we see that $V(\Delta)$ is a subspace of $\mathbb{H}^2$. We now describe it exactly with the help of the Beurling–Lax theory.

Let S_f be the set of finite places of K, and let S_∞ be the (possibly empty) set of infinite places. Let q_v be the cardinality of the residue field at the finite place v and π_v a uniformizer element of $K_v^\times$, which we also consider as an element of $\mathbb{A}_K^\times$. The value $\chi(\pi_v)$ is independent of the choice of π_v if the character χ is unramified at v. The ("incomplete" in the number field case) L-function associated to χ is

$$L(\chi, s) = \prod_{v \in S_f,\ \text{unramified}} \frac{1}{1-\chi(\pi_v)\, q_v^{-s}}$$

The Bruhat–Schwartz function φ is built from local components, all of them except finitely many being equal to the characteristic function of the local integers, so its Tate L-function $L(\varphi, \chi, s)$ is a multiple of $L(\chi, s)$ by a function holomorphic in $\mathrm{Re}(s) > 0$. By Lemma 1.2 this implies that the Paley–Wiener transform $\widehat{E(\varphi)}(\chi, s)$ $(\mathrm{Re}(s) > \frac{1}{2})$ vanishes at each bad zero with at least the same multiplicity as $L(\chi, s)$.

DEFINITION 3.1. Let B be the unitary invariant operator whose spectral multiplier in the χ-isotypical component of $L^2(\mathscr{C}_K, d^*u)$ is the Blaschke product on the zeros (with multiplicity) of the L-function $L(\chi, s)$ in the half-plane $\mathrm{Re}(s) > \frac{1}{2}$ (number field case) or the open disc $|z| < 1$ $(z = q^{-(s-1/2)}$, function field case).

We will soon show that one can indeed build a convergent Blaschke product with the bad zeros so that B exists! (The function field case is trivial as there are only finitely many.) This being temporarily admitted we have obtained $V(\Delta) \subset B \cdot \mathbb{H}^2$. And we prove

THEOREM 3.2.

$$V(\Delta) = B \cdot \mathbb{H}^2$$

We treat the function-field case first. We choose φ_v to be $\mathbf{1}_{|x|_v \leqslant 1}$ at a non-ramified place and $\overline{\chi_v(x)} \cdot \mathbf{1}_{|x|_v = 1}$ at a ramified place. With these choices we obtain $\varphi = \prod_v \varphi_v$ which belongs to $\mathscr{S}_{\leqslant 1}$ and for which (at first for $\mathrm{Re}(s) > 1$):

$$L(\varphi, \chi, s) = L(\chi, s).$$

We do not claim that $E(\varphi)$ is χ-equivariant; nevertheless this identity combined with Lemma 1.2 and the inclusion $V(\Delta) \subset \mathbb{H}^2$ shows that $V(\chi, s)$ $L(\chi, s)$ belongs to $\mathbb{H}^2(|z| < 1)$. It is clear from the product representation that it does not vanish at $z = 0$, and it is known for $\chi = 1$ that the pole at $s = 1$ of the zeta function $Z_K(s)$ is of order 1. Analytic continuation across $|z| = 1$ implies the nonexistence of a singular inner factor. So the smallest closed subspace of $\mathbb{H}^2(|z| < 1)$ containing $V(\chi, s)\, L(\chi, s)$, and invariant under shifts, is exactly $B(\chi, s)\, \mathbb{H}^2$. The conclusion follows.

Let us now consider the case where K is an algebraic number field. We define $\varphi_v(x_v)$ exactly as in the function field case when v is finite and as $\overline{\chi_v(x)} \cdot g_v(|x|_v)$ at each infinite place, with g_v a smooth function on $\mathbb{R}_+^\times$ with compact support in $(0, 1)$. The product function $\varphi(x) = \prod_v \varphi_v(x_v)$ then belongs to $\mathscr{S}_{\leqslant 1}$ and $E(\varphi)$ has a Paley–Wiener transform

$$\int_{\mathscr{C}_K} E(\varphi)(u) \cdot \chi(u)\, |u|^{s-1/2}\, d^*u = C(K) \int_{\mathbb{A}_K^\times} \varphi(v)\, \chi(v)\, |v|^s\, d^*v$$

$$= C(K)\, L(\chi, s) \cdot \prod_{v \in S_\infty} \hat{g}_v(s).$$

From this and the inclusion $V(\Delta) \subset \mathbb{H}^2$ follows the existence of the Blaschke product $B(\chi, s)$ as promised above. Furthermore it is clearly possible to choose the g_v in such a manner that $\widehat{g_v}(s)$ does not vanish at any s prescribed in advance, and the existence of analytic continuation accross the critical line then reduces the possibility of an inner factor to $\lambda^{s-1/2}$ with $\lambda \leqslant 1$. The Bercovici–Foias argument implies as in our discussion of Nyman's theorem that $\lambda = 1$. Finally it is known that the pole of the zeta function ($\chi = 1$) has exact order 1. With all this the identity $V(\Delta) = B \cdot \mathbb{H}^2$ is proven. This completes the proof of the closure criterion 1.11.

Let $\mathscr{D}_+ = E(\mathscr{S}_{\leqslant 1})^\perp = \Delta^\perp = V^{-1}B \cdot (\mathbb{H}^2)^\perp$. Let Z be the unitary operator which is just 1 in the number field case and z (in each isotypical component) in the function field case. Then $(\mathbb{H}^2)^\perp = Z^{-1}I \cdot \mathbb{H}^2$ and $\mathscr{D}_+ = V^{-1}BZ^{-1}I \cdot \mathbb{H}^2$. From this follows

$$\bigwedge U(\lambda)\, \mathscr{D}_+ = \{0\} \quad \overline{\bigvee U(\lambda)\, \mathscr{D}_+} = L^2(\mathscr{C}_K, d^*u)$$

so that $\mathscr{D}_+$ indeed qualifies as an outgoing subspace and $\mathscr{D}_-$ as an incoming subspace. One has $\mathscr{D}_- = IV^{-1}BZ^{-1}I \cdot \mathbb{H}^2 = VB^{-1}Z \cdot \mathbb{H}^2$. The Lax–Phillips scattering operator associated to the pair $(\mathscr{D}_+, \mathscr{D}_-)$ is an invariant unitary operator, unique up to a multiplicative constant in each isotypical component. It is:

$$S = (V^{-1}B)^{-1} \cdot VB^{-1}Z = ZV^2B^{-2}.$$

With the help of S the pair $(\mathscr{D}_+, \mathscr{D}_-)$ is unitarily equivalent to $((\mathbb{H}^2)^\perp, S \cdot \mathbb{H}^2)$. So it is an orthogonal pair if and only if $S \cdot \mathbb{H}^2 \subset \mathbb{H}^2$, if and only if $B = 1$, if and only if the Riemann hypothesis holds for all abelian L-functions of K. With this the proof of the causality criterion 1.7 is complete.

Note 3.3. The reader of the monograph of Lax and Phillips [17, Chap. 2] will perhaps be perplexed by the fact that "causal" means there "inner with respect to the exterior domain $|z| > 1$" (in the discrete case). But this is because they represent the semigroup leaving invariant the outgoing space with the help of the non-negative powers of z. In our case we represent it with the help of the non-negative powers of $\frac{1}{z}$. So "causal" is to be understood to mean "inner with respect to the domain $|\frac{1}{z}| > 1$" (that is $|z| < 1$).

Note 3.4. We have used $IBI = B^{-1}$. This follows from $\overline{L(\chi, \bar{s})} = L(\bar{\chi}, s)$ which implies $B(\bar{\chi}, \bar{s}) = \overline{B(\chi, s)}$ $(= B(\chi, s)^{-1}$ for $\mathrm{Re}(s) = \frac{1}{2})$.

REFERENCES

1. M. Balazard and E. Saias, The Nyman–Beurling equivalent form for the Riemann hypothesis, *Expo. Math.* **18** (2000), 131–138.
2. M. Balazard, E. Saias, and M. Yor, Notes sur la fonction ζ de Riemann, 2, *Adv. in Math.* **143** (1999), 284–287.
3. H. Bercovici and C. Foias, A real variable restatement of Riemann's Hypothesis, *Israel J. of Math.* **48** (1984), 57–68.
4. H. Bercovici and C. Foias, On the Zorn spaces in Beurling's approach to the Riemann hypothesis, *in* "Analysis and Topology," pp. 143–149, World Scientific, River Edge, NJ, 1998.
5. A. Beurling, On two problems concerning linear transformations in Hilbert space, *Acta Math.* **81** (1949), 239–255.
6. A. Beurling, A closure problem related to the Riemann Zeta-function, *Proc. Nat. Acad. Sci.* **41** (1955), 312–314.
7. J.-F. Burnol, Scattering on the p-adic field and a trace formula, *Internat. Math. Res. Notices* **2000:2** (2000), 57–70.
8. J.-F. Burnol, The explicit formula and the conductor operator, available at http://arXiv.org/abs/math/9902080 (1999).
9. A. Connes, Trace formula in non-commutative Geometry and the zeros of the Riemann zeta function, *Selecta Math. (N.S.)* **5** (1999), 29–106.
10. J. B. Conrey and X.-J. Li, "A Note on Some Positivity Conditions Related to Zeta- and *L*-Functions," American Institute of Mathematics preprint series, available at www.aimath.org/preprints_98.html (1998).
11. L. De Branges, The Riemann hypothesis for Hilbert spaces of entire functions, *Bull. Amer. Math. Soc. (N.S.)* **15** (1986), 1–17.
12. L. De Branges, A conjecture which implies the Riemann hypothesis, *J. Funct. Anal.* **121** (1994), 117–184.
13. L. D. Faddeev and B. S. Pavlov, Scattering theory and automorphic functions, *Sem. Steklov Math. Inst. Leningrad* **27** (1972), 161–193.
14. K. Hoffman, "Banach Spaces of Analytic Functions," Dover, New York, 1988. [First published 1962]
15. K. Iwasawa, Letter to J. Dieudonné (1952), *in* "Zeta Functions in Geometry," Adv. Stud. in Pure Math. (N. Kurokawa and T. Sunada, Eds.), Vol. 21, Kinokuniya, Tokyo, 1992.
16. P. Lax, Translation invariant subspaces, *Acta Math.* **101** (1959), 163–178.
17. P. Lax and R. S. Phillips, "Scattering Theory," rev. ed., Pure and Applied Mathematics, Vol. 26, Academic Press, San Diego, 1989.
18. P. Lax and R. S. Phillips, "Scattering Theory for Automorphic Functions," Annals of Mathematics Studies, Princeton University Press, Princeton, NJ, 1976.
19. B. Nyman, "On Some Groups and Semigroups of Translations," Thesis, Uppsala, 1950.

20. J. Tate, “Fourier Analysis in Number Fields and Hecke’s Zeta Function,” Princeton, 1950, reprinted in “Algebraic Number Theory” (J. W. S. Cassels and A. Fröhlich, Eds.), Academic Press, San Diego, 1967.
21. A. Weil, “Sur les ‘formules explicites’ de la théorie des nombres premiers,” Comm. Sem. Math. Univ. Lund, pp. 252–265. Oeuvres, Vol. II, Marcel Riesz, 1952.
22. A. Weil, “Basic Number Theory,” 3rd ed., Springer-Verlag, Berlin/New York, 1974.
23. A. Weil, Sur les formules explicites de la théorie des nombres, *Izv. Mat. Nauk. (Ser. Mat.)* **36** (1972), 3–18. Oeuvres, Vol. III.