Distance-regular Graphs with $\Gamma(x) \simeq 3 \ast K_{a + 1}$

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We show that a distance-regular graph $\Gamma$ with $\Gamma(x) = 3 \ast K_{a+1}$ (at least two for every $x \in \Gamma$ and $d \geq r(\Gamma) + 3$ is a distance-2 graph of a distance-biregular graph with vertices of valency 3. In particular, intersection numbers $c_i, a_i, b_i (0 \leq i \leq d)$ can be denoted by at most four parameters.

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1. INTRODUCTION

Let $\Gamma$ be a connected undirected finite graph without loops or multiple edges.

In the following if $\alpha$ is a vertex of $\Gamma$, then we write $\alpha \in \Gamma$. For $\alpha, \beta \in \Gamma$, let $\delta(\alpha, \beta)$ be the distance between $\alpha$ and $\beta$. Let $d$ be the maximal distance in $\Gamma$ and we call it the diameter of $\Gamma$.

Let $\Gamma_i(\alpha) = \{\beta \in \Gamma \mid \delta(\alpha, \beta) = i\}$, and write $\Gamma(\alpha) = \Gamma_1(\alpha)$. For vertices $\alpha, \beta$ in $\Gamma$ with $\delta(\alpha, \beta) = i$, let

\begin{align*}
&c_i(\alpha, \beta) = |\Gamma_{i-1}(\alpha) \cap \Gamma(\beta)|, \\
&a_i(\alpha, \beta) = |\Gamma_i(\alpha) \cap \Gamma(\beta)|, \\
&b_i(\alpha, \beta) = |\Gamma_{i+1}(\alpha) \cap \Gamma(\beta)|.
\end{align*}

$\Gamma$ is said to be distance-regular if for each $i$ the numbers $c_i(\alpha, \beta), a_i(\alpha, \beta)$ and $b_i(\alpha, \beta)$ depend only on $i = \delta(\alpha, \beta)$. In this case, we write $c_i, a_i$ and $b_i$ for the corresponding numbers and we call them the intersection numbers of $\Gamma$. Note that $c_0 = a_0 = b_0 = 0$. We call $k = b_0$ the valency of $\Gamma$.

A bipartite graph $\Gamma$ is said to be distance-biregular if for each $i$ the numbers $c_i(\alpha, \beta)$ and $b_i(\alpha, \beta)$ depend only on $i$ and the part the base point $\alpha$ belongs to.

For the basic properties of the parameters $c_i, a_i,$ and $b_i$ of distance-regular graphs, see [6]. For example, we have the following:

1. $c_i + a_i + b_i = k$, for $0 \leq i \leq d$;
2. $1 = c_1 \leq \cdots \leq c_d$;
3. $k = b_0 > b_1 \geq \cdots \geq b_{d-1} > 0$.

Let

\[ r(\Gamma) = \#\{i \mid (c_i, a_i, b_i) = (c_1, a_1, b_1)\}. \]

In the following we denote $a = a_1$.

We shall investigate the distance-regular graphs which satisfy the following:

(*) For every $\alpha$ and $\beta$ in $\Gamma$ with $\delta(\alpha, \beta) = 1$, an induced subgraph $\Gamma(\alpha) \cap \Gamma(\beta)$ is a complete graph or a graph without vertices.

In this case, it follows that

\[ \Gamma(\alpha) = m \ast K_{a + 1} \]

for some $m$ and for all $\alpha \in \Gamma$, where $K_{a+1}$ is the complete graph with $(a + 1)$ vertices and $m \ast K_{a+1}$ means the disjoint union of $m K_{a+1}$'s. Clearly, we have $k = m(a + 1)$.

For example, if $r(\Gamma) \geq 2$ or $a \leq 1$, then $\Gamma$ satisfies the condition (*) (see [6]). If $m = 1$, then it is easy to see that $\Gamma$ is a complete graph.
The case \( m = 2 \) is treated by Mohar and Shawe-Taylor [11]. They showed that if \( k > 2 \), then \( \Gamma \) is isomorphic to the line graph of a Moore graph or the point graph of a generalized \( 2d \)-gon of order \((a + 1, 1)\).

Suppose that \( m = 3 \). If \( a = 0 \), then \( k = 3 \). Distance-regular graphs of valency 3 have been completely classified by several authors (see [4]).

Recently the case \( m = 3 \) and \( a = 1 \) was treated by Hiraki et al. [8]. They showed that there are only four graphs in this class.

This paper is devoted to the case \( m = 3 \) and \( a > 2 \). In this class, the author only knows the following examples: the Hamming graph \( H(3, a + 1) \); and the point graph of a generalized \( 2d \)-gon of order \((a + 1, 2)\).

We see that \( d (= 3) = r(\Gamma) + 2 \) in the first case, and that \( d (\leq 4) = r(\Gamma) + 1 \) in the second case.

We now state our main result of this paper.

**Theorem 1.1.** Let \( \Gamma \) be a distance-regular graph with \( \Gamma(x) = 3 \cdot K_{a+1} \) for all \( x \in \Gamma \) and \( a > 2 \). If \( d > r(\Gamma) + 3 \), then \( \Gamma \) is a distance-2 graph of a distance-biregular graph with vertices of valency 3.

This result means that all intersection numbers can be denoted by at most four parameters. (In the case \( d \geq r(\Gamma) + 3 \), see Theorem 4.1. In the case \( d \leq r(\Gamma) + 2 \), this is clear.) We hope that our result gives a key to a classification of distance-regular graphs with \( \Gamma(x) = 3 \cdot K_{a+1} \).

For a subset \( S \subset \Gamma \), we sometimes recognize it as the induced subgraph with the vertex set \( S \). In particular, for \( \alpha, x \in \Gamma \) with \( \delta(\alpha, x) = i \), we call the induced subgraph \( \Gamma_{i-1}(\alpha) \cap \Gamma(x) \) the \( c_i \)-graph with respect to \((\alpha, x)\) and \( \Gamma_{i+1}(\alpha) \cap \Gamma(x) \) the \( b_i \)-graph with respect to \((\alpha, x)\). We call a complete subgraph a **clique** and a subgraph without edges a **coclique**.

For vertices \( x, y \) with \( x \neq y \), we write \( x \sim y \) when \( \delta(x, y) = 1 \), and \( x \not\sim y \) otherwise.

In the rest of the paper, we assume that \( \Gamma \) is a distance-regular graph with \( \Gamma(x) = 3 \cdot K_{a+1} \) for all \( x \in \Gamma \), and that \( r = r(\Gamma) \).

### 2. Intersection Diagram

Let \( e(A, B) \) denote the number of edges between subsets \( A, B \) of \( \Gamma \), and let \( e(\{x\}, A) = e(x, A) \) for \( x \in \Gamma \).

For \( \alpha, \beta \in \Gamma \) with \( \alpha \sim \beta \), we set \( D_i^\alpha(\alpha, \beta) = \Gamma_i(\alpha) \cap \Gamma_j(\beta) \). In particular, we denote \( D_i^\alpha = D_i^\alpha(\alpha, \alpha) \) with a fixed pair of vertices \((\alpha, \alpha)\). The **intersection diagram** with respect to \((\alpha, \beta)\) is the collection \( \{D_i^\alpha\}_{i,j}^\beta \) with lines between \( D_i^\alpha \)'s and \( D_i^\beta \)'s. Basic properties of the intersection diagram are summarized in [5] and [7]. In particular, the following holds for every pair of the base points \((\alpha, \beta)\) with \( \alpha \sim \beta \).

**Lemma 2.1.** The following hold:

1. \( e(D_i^{-1}, D_i^\alpha) = e(D_i^{-1}, D_i^\beta) = e(D_i^{-1}, D_i^{-1}) = e(D_i^{-1}, D_i^{-1}) = 0 \) for \( 2 \leq i \leq r \).
2. \( e(D_j^{r+1}, D_j^{-1}) = 0 \).
3. \( e(D_j^\ast, D_j^\ast) = 0 \).
4. \( e(D_j^\ast, D_j^\ast) = 0 \).
5. \( e(D_{r+1}^\ast, D_{r+1}^\ast) = 0 \).

**Proof.** See [7].
REMARK. In [7], these properties depend on the assumption that \( r \geq 2 \), but we can also have them under the assumption that \( \Gamma \) satisfies the condition (*).

If \( c_{r+1} \geq 2 \), the shape of the intersection diagram is as shown in Figure 1.

\[
\{ \alpha \} = D_0^0 \rightarrow D_1^1 \rightarrow \ldots \rightarrow D_r^r \rightarrow D_{r+1}^{r+1} \rightarrow \cdots
\]

\[
\{ \beta \} = D_0^0 \rightarrow D_1^1 \rightarrow \ldots \rightarrow D_r^r \rightarrow D_{r+1}^{r+1} \rightarrow \cdots
\]

**FIGURE 1.**

If \( c_{r+1} = 1 \), the shape of the intersection diagram is as shown in Figure 2.

\[
\{ \alpha \} = D_0^0 \rightarrow D_1^1 \rightarrow \ldots \rightarrow D_r^r \rightarrow D_{r+1}^{r+1} \rightarrow \cdots
\]

\[
\{ \beta \} = D_0^0 \rightarrow D_1^1 \rightarrow \ldots \rightarrow D_r^r \rightarrow D_{r+1}^{r+1} \rightarrow \cdots
\]

**FIGURE 2.**

If \( c_{r+1} \geq 2 \), we say that the numerical girth \( g \) of \( \Gamma \) is \( 2r + 2 \). If \( c_{r+1} = 1 \), we say that \( g = 2r + 3 \). If \( \Gamma \) satisfies the condition (*), then the length of a shortest circuit without a triangle is equal to the numerical girth \( g \) (see [6]).

We easily obtain the following.

**LEMMA 2.2.** Suppose \( c_{r+1} = 1 \). Let \( \alpha, \beta \in \Gamma_{r+1}(x) \) with \( \alpha \sim \beta \), \( \alpha' \in \Gamma(x) \cap \Gamma_r(\alpha) \), and \( \beta' \in \Gamma(x) \cap \Gamma_r(\beta) \). If \( \Gamma(\alpha) \cap \Gamma(\beta) \cap \Gamma_r(x) = \emptyset \), then \( \delta(\alpha', \beta') = 2 \).

**PROOF.** We consider the intersection diagram with respect to \((\alpha, \beta)\). Clearly, \( x \in D_r^{r+1} \). Let \( \alpha'' \in \Gamma(\alpha) \cap \Gamma_{r-1}(\alpha') \) and \( \beta'' \in \Gamma(\beta) \cap \Gamma_{r-1}(\beta') \). Note that \( \delta(x, \alpha'') = \delta(x, \beta'') = r \). By our assumption, we have \( \alpha'' \in D_2^2 \) and \( \beta'' \in D_2^2 \). So \( \alpha' \in D_{r+1}^{r+1} \) and \( \beta' \in D_{r+1}^{r+1} \). Hence \( \alpha' \sim \beta' \) by Lemma 2.1(5). \( \Box \)

3. Case of Even Numerical Girth

In this section we shall consider the case \( c_{r+1} \geq 2 \). We note that our argument works for \( a = 1 \) as well, in this case.

**THEOREM 3.1.** Let \( \Gamma \) be a distance-regular graph with \( \Gamma(x) = 3 \ast K_{a+1} \) for all \( x \in \Gamma \). If \( a = 1 \) and \( c_{r+1} \geq 2 \), then one of the following holds:

1. \( c_{r+1} = 3 \) and \( d = r + 1 \); or
2. \( c_{r+1} = 2 \) and \( d = r + 2 \).

Firstly, we show the following.
LEMMA 3.2. The following hold:
(1) \( c_{r+1} \leq 3 \). In particular, if \( c_{r+1} = 3 \), then \( d = r + 1 \).
(2) If \( c_{r+1} = 2 \), then \( a_{r+1} \geq 2a \).

PROOF. (1) Let \( \alpha, x \in \Gamma \) with \( \delta(\alpha, x) = r + 1 \). Since \( \Gamma(x) = 3 \ast K_a+1 \), it follows from Lemma 2.1(4) that \( c_{r+1} \leq 3 \). Moreover, if \( c_{r+1} = 3 \), then \( \Gamma(x) \subset \Gamma(\alpha) \cup \Gamma_{r+1}(\alpha) \); that is, \( d \leq r + 1 \).

(2) Let \( \alpha, x \) be as in (1). Let \( x_1, x_2 \in \Gamma(\alpha) \cap \Gamma(x) \). By Lemma 2.1(4), \( x_1 \neq x_2 \). As \( c_{r+1} = 2 \), we obtain \( \Gamma(x) \cap \Gamma(x_i) \subset \Gamma_{r+1}(\alpha) \) for \( i = 1, 2 \). Since \( \Gamma(x) \cap \Gamma(x_1) \) and \( \Gamma(x) \cap \Gamma(x_2) \) are disjoint, we obtain \( a_{r+1} \geq |\Gamma(x) \cap \Gamma(x_1)| + |\Gamma(x) \cap \Gamma(x_2)| = 2a \).

By the previous lemma, we may assume that \( c_{r+1} = 2 \).

LEMMA 3.3. Let \( c_{r+1} = 2 \). If \( d \geq r + 3 \), then every \( c_{r+2} \) graph is a disjoint union of exactly two cliques. In particular, if \( c_{r+2} = 2 \), then every \( c_{r+2} \) graph is a coclique.

PROOF. By Lemma 2.1(4), every \( c_{r+1} \) graph is a coclique of size 2. Let \( \alpha, x \in \Gamma \) with \( \delta(\alpha, x) = r + 2 \). Since every \( c_{r+2} \) graph contains a \( c_{r+1} \) graph, \( \Gamma_{r+1}(\alpha) \cap \Gamma(x) \) contains a coclique of size 2. If \( d \geq r + 3 \), then \( \Gamma_{r+1}(\alpha) \cap \Gamma(x) \neq \emptyset \). So the assertion follows from our assumption that \( \Gamma(x) = 3 \ast K_a+1 \).

Now we consider the intersection diagram with respect to \((\alpha, \beta)\), with \( \alpha \sim \beta \).

LEMMA 3.4. If \( c_{r+1} = 2 \) and \( d \geq r + 3 \), then \( e(D^{r+1}_{r+1}, D^{r+2}_{r+1}) = 0 \).

PROOF. Assume that \( e(D^{r+1}_{r+1}, D^{r+2}_{r+1}) \neq 0 \). Let \( x \in D^{r+2}_{r+1} \) and \( y \in D^{r+1}_{r+2} \), with \( x \sim y \). Since \( c_{r+2} = 2 \), there exist \( z_1, z_2 \in D^{r+1}_{r+1} \) such that \( z_1, z_2 \in \Gamma(x) \). Note that \( \delta(z_1, z_2) \neq 1 \) by Lemma 2.1(4). By Lemma 3.3, it must hold that \( y \sim z_1 \) or \( y \sim z_2 \). This is impossible.

To prove Theorem 3.1, we consider two cases, as set out below.

Case 1: \( c_{r+1} = 2 \) and \( a_{r+1} = 2a \). We need the following result by Hiraki [7].

PROPOSITION 3.5 [7]. Let \( \Gamma \) be a distance-regular graph with \( a = a_1 \geq 1 \), \( c_{r+1} = 2 \) and \( a_{r+1} = 2a \). Then \( (c_{r+1}, a_{r+1}, b_{r+1}) \neq (c_{r+2}, a_{r+2}, b_{r+2}) \).

LEMMA 3.6. If \( c_{r+1} = 2 \) and \( a_{r+1} = 2a \), then the following holds:
(1) \( e(x, D^r_\alpha) = 1 \) for all \( x \in D^{r+1}_{r+1} \).
(2) Let \( x \in D^{r+1}_{r+1} \), \( x_1 \in D^{r+1}_{r+1} \) and \( x_2 \in D^{r+1}_{r+1} \) with \( x_1 \sim x \sim x_2 \). Then \( x_1 \sim x_2 \).
(3) \( e(D^{r+1}_{r+1}, D^{r+2}_{r+1}) = e(D^{r+1}_{r+1}, D^{r+2}_{r+1}) = 0 \) and \( c_{r+2} = 2 \).
PROOF. For (1) and (2), see Lemma 3.4 and Lemma 3.5 in [7].

(3) Let \( x \in D_{r+1}^{r+1} \). By (1), there exist \( y_1 \in D_r \), \( y_2 \in D_{r+1}^{r+1} \) and \( y_3 \in D_r^{r+1} \), with \( y_1, y_2, y_3 \in \Gamma(x) \). By Lemma 2.1(3), we have \( y_2 \not\sim y_1 \not\sim y_3 \). On the other hand, we have \( y_2 \sim y_3 \) by (2). Since

\[
2a = |\Gamma(x) \cap \Gamma_{r+1}(\alpha)| \\
\geq |\Gamma(x) \cap (D_r^{r+1} \cup D_{r+1}^{r+1})| + |\Gamma(x) \cap D_r^{r+1}| \\
\geq |\Gamma(x) \cap (D_r^{r+1} \cup D_{r+1}^{r+1})| \\
= |\Gamma(x) \cap \Gamma(y_1) \cup (\Gamma(x) \cap \Gamma(y_2))| \\
= 2a,
\]

we have \( e(x, D_{r+1}^{r+1}) = 0 \). This implies that \( e(D_r^{r+1}, D_{r+1}^{r+1}) = 0 \). Hence we have \( e(D_r^{r+1}, D_{r+1}^{r+1}) = 0 \) as well.

If \( z \in D_{r+1}^{r+1} \), then \( e(z, D_{r+1}^{r+1}) = 0 \) by Lemma 3.4. Since \( e(z, D_{r+1}^{r+1}) = 0 \), we have \( \Gamma(z) \cap \Gamma_{r+1}(\alpha) = \Gamma(z) \cap \Gamma_{r+1}(\beta) \). Thus we have \( c_{r+2} = 2 \).

Now we prove Theorem 3.1 under the assumption that \( a_{r+1} = 2a \). Since \( c_{r+2} = 2 \) by the previous lemma, it follows from Proposition 3.5 that \( b_{r+2} < b_{r+1} \). Therefore there exist \( x \in D_{r+1}^{r+1} \) and \( y \in D_{r+2}^{r+2} \) such that \( x \sim y \).

We claim that there exists \( z \in D_{r+1}^{r+1} \) such that \( y \sim z \). Suppose not. Then there exist \( z_1 \in D_{r+1}^{r+1} \) and \( z_2, z_3 \in D_{r+1}^{r+1} \) such that \( x \not\sim z_1, z_2 \not\sim z_3 \) and \( \{z_1, z_2, z_3\} \subseteq \Gamma(y) \). Since \( \{x, z_1\} \) and \( \{z_2, z_3\} \) are cocliques by Lemma 3.3 and \( e(D_{r+1}^{r+1}, D_{r+1}^{r+1}) = 0 \) by Lemma 3.4, it follows that \( \{x, z_1, z_2, z_3\} \subseteq \Gamma(y) \) is a coclique of size 4. This is impossible.

Since we have \( e(z, D_r) = 1 \) by Lemma 3.6(1), there is a vertex \( y \in D_1 \) such that \( \delta(y, x) = r + 1 \). Since \( e(D_{r+1}^{r+1}) = 0 \) for \( 1 \leq i \leq r \), we have \( \delta(y, z) = r + 2 \). Thus,

\[
D_{r+2}^{r+2}(y, \beta) \ni y \in D_{r+2}^{r+2}(y, \beta).
\]

This contradicts Lemma 3.4. Therefore we obtain the desired result when \( a_{r+1} = 2a \).

Case 2: \( c_{r+1} = 2 \) and \( a_{r+1} > 2a \). Let \( S(\alpha, \beta) \) be the set defined as follows:

\[
S(\alpha, \beta) = \{x \in D_{r+1}^{r+1} \mid e(x, D_{r+1}^{r+1}) = e(x, D_{r+1}^{r+1}) = 2\}.
\]

**LEMMA 3.7.** If \( c_{r+1} = 2, a_{r+1} > 2a \) and \( d \geq r + 2 \), then \( S(\alpha, \beta) \neq \emptyset \) for some edge \( \alpha \beta \).

**PROOF.** Let \( \alpha, \beta, y \in D_1 \) such that \( \delta(\alpha, x) = r + 1, \gamma_1 \not\sim \gamma_2 \) and \( \gamma_1, \gamma_2 \in \Gamma(\alpha) \cap \Gamma(y) \). We note that \( \gamma_1 \not\sim \gamma_2 \) by Lemma 2.1(4), and that \( \Gamma(\alpha) \cap \Gamma(y) \subseteq \Gamma_{r+1}(x) \) and \( \Gamma(\alpha) \cap \Gamma(y) \subseteq \Gamma_{r+1}(x) \) as \( c_{r+1} = 2 \). Since \( a_{r+1} > 2a \), there exists a vertex \( \beta \in \Gamma_{r+1}(x) \cap \Gamma(\alpha) \) such that \( \beta \not\sim \gamma_1 \) and \( \beta \not\sim \gamma_2 \). Considering the intersection diagram of rank 1 with respect to \( (\alpha, \beta) \), we see that \( \gamma_1, \gamma_2 \in D_2 \) and \( x \in D_{r+1}^{r+1} \). Let \( \gamma_i \in \Gamma_{r-1}(\gamma_i) \cap \Gamma(x) \) for \( i = 1, 2 \). Then \( \gamma_1 \not= \gamma_2 \). Otherwise, it follows that \( \gamma_1, \gamma_2 \in \Gamma_{r-1}(\gamma_i) \cap \Gamma(\alpha) \) and it is contradictory that \( c_{r+1} = 1 \). Since it is clear that \( \gamma_i \in D_{r+1}^{r+1} \) for \( i = 1, 2 \), we have \( e(x, D_{r+1}^{r+1}) > 2 \); that is, \( x \in S(\alpha, \beta) \).

**LEMMA 3.8.** Let \( x \in S(\alpha, \beta) \), \( \{y_1, y_2\} = D_{r+1}^{r+1} \cap \Gamma(x) \) and \( \{y_3, y_4\} = D_{r+1}^{r+1} \cap \Gamma(x) \). If \( d \geq r + 3 \), then the following hold:

1. There exists a vertex \( \gamma \in D_1 \) such that \( \delta(\gamma, x) = r + 2 \).
2. \( y_1 \not\sim y_3 \) and \( y_2 \not\sim y_4 \), or \( y_1 \not\sim y_2 \) and \( y_3 \not\sim y_4 \).
PROOF. (1) Since $e(x, D_1') = 0$, we have $D_1' \cap \Gamma_r(x) = \emptyset$. Hence the clique $D_1'$ contains a $b_{r+1}$-graph $\Gamma_{r+2}(x) \cap \Gamma(\alpha)$. Thus we have (1).

(2) Let $\gamma$ be as in (1). Then $\{y_1, y_2, y_3, y_4\} \subset \Gamma(x) \cap \Gamma_{r+1}(\gamma)$. Since $d \geq r + 3$, every $c_{r+2}$-graph is a union of two cliques. On the other hand, we have $y_1 \not\sim y_2$ and $y_3 \not\sim y_4$ by Lemma 2.1(4). Hence either $y_1 \sim y_3, y_2 \not\sim y_4$ or $y_1 \not\sim y_4, y_2 \sim y_3$. 

Let $\alpha, \beta, \gamma, x$ and $y_i (i = 1, 2, 3, 4)$ be as in Lemma 3.8. We assume that $y_1 \sim y_3$ and $y_2 \sim y_4$. We set a circuit of length $2r + 3$ as follows:

$\{x_0 = \alpha, x_1 = \beta, \ldots, x_{r+1} = y_1, x_{r+2} = x, x_{r+3} = y_4, x_{r+4}, \ldots, x_{2r+3} = x_0\}$

where $x_{r+2} \in S(x_0, x_1)$. Let $\delta_{r+1} = y_3$ and $\delta_{r+2} = y_2$ (see Figure 3). Note that $\delta_{r+1} \in \Gamma(x_{r+1}) \cap \Gamma(x_{r+2}) \cap D_{r+1}$ and $\delta_{r+2} \in \Gamma(x_{r+2}) \cap \Gamma(x_{r+3}) \cap D_{r+1}$. We change the base points to $(x_1, x_2)$: see Figure 4.

Since $e(x_{r+3}, D_{r+1}) = 2$, $x_{r+3} \in S(x_1, x_2)$. By Lemma 3.8(2), it follows that there exists $\delta_{r+3} \in D_{r+1}^{r+1}(x_1, x_2)$ such that $x_{r+3} \sim \delta_{r+3} \sim x_{r+4}$.

By induction, we have that $x_0 \in S(x_{r+1}, x_{r+2})$, $x_1 \in D_{r+1}(x_{r+1}, x_{r+2})$ and that there exists $\delta_0 \in D_{r+1}^{r+1}(x_{r+1}, x_{r+2}) \cap \Gamma(x_0) \cap \Gamma(x_1)$

$\subset \Gamma_r(x) \cap D_1(\alpha, \beta)$.

On the other hand, there exists $\gamma \in \Gamma_{r+2}(x) \cap D_1(\alpha, \beta)$ by Lemma 3.8(1), and it follows that $\delta_0 \not\sim \gamma$. This contradicts the fact that $D_1(\alpha, \beta)$ is a clique. This completes the proof of Theorem 3.1.

4. CASE OF ODD NUMERICAL GIRTH

In this section we treat the case $c_{r+1} = 1$.

THEOREM 4.1. Let $\Gamma$ be a distance-regular graph with $\Gamma(x) = 3 \cdot K_{a+1}$ for all $x \in \Gamma$. Let $r = r(\Gamma)$. If $a \geq 2$ and $c_{r+1} = 1$, then one of the following holds:

(1) $d \leq r + 2$; or
(2) $\Gamma$ is a distance-2 graph of a distance-biregular graph with vertices of valency 3.

FIGURE 4.
In particular,

\[ (c_i, a_i, b_i) = \begin{cases} 
(0, 0, 3(a + 1)) & \text{for } i = 0 \\
(1, a, 2(a + 1)) & \text{for } 1 \leq i \leq r \\
(1, a + 2, 2a) & \text{for } i = r + 1 \\
(4, 2a - 1, a) & \text{for } r + 2 \leq i \leq d - 2 \\
(4, 2a + c - 1, a - c) & \text{for } i = d - 1 \\
(3(c + 2), 3(a - 1 - c), 0) & \text{for } i = d 
\end{cases} \]

with \( c = 0 \) or \( 1 \).

We consider the intersection diagram with respect to \((\alpha, \beta)\) with \( \alpha \sim \beta \). Let \( A(\alpha, \beta) \) and \( B(\alpha, \beta) \) be the sets defined as follows:

\[
A(\alpha, \beta) = \{ x \in D_{r+1}^\bullet | e(x, D_{r+1}^\bullet) = e(x, D_{r}^\bullet) = 1 \}, \\
B(\alpha, \beta) = \{ x \in D_{r+1}^\bullet | e(x, D_{r}) = 1 \}.
\]

In the following we denote \( A = A(\alpha, \beta) \) and \( B = B(\alpha, \beta) \) for a fixed pair of vertices \((\alpha, \beta)\).

It is clear that \( A \cap B = \emptyset, A \neq \emptyset, B \neq \emptyset \) and \( A \cup B = D_{r+1}^\bullet \).

Let \( D_1^\bullet = \{ \gamma_1, \ldots, \gamma_\alpha \}, \alpha = \gamma_0 \) and \( \beta = \gamma_{a+1} \). For \( \delta \in \Gamma \) and \( 0 \leq i_1, \ldots, i_r \leq a + 1 \), we denote

\[ \delta = (\delta_{(\gamma_0, \delta)} - r, \ldots, \delta_{(\gamma_{a+1}, \delta)} - r), \]

and

\[ \delta(i_1, \ldots, i_r) = (\delta_{(\gamma_{i_1}, \delta)} - r, \ldots, \delta_{(\gamma_{i_r}, \delta)} - r). \]

It is easy to see the following:

(1) \( \gamma_i \sim \gamma_m \) for \( 0 \leq l \neq m \leq a + 1 \).

(2) \( x \in D_{r+1}^\bullet (\gamma_i, \gamma_m) \) is equivalent to \( x(l, m) = (i - r, j - r) \).

(3) Let \( \delta \) be the vertex such that \( \max(\delta(l) | l = 0, 1, \ldots, a + 1) = i \) and let \( \delta(m) = i \). If \( \# \{ n | \delta(n) = i - 1 \} = s \), then the \( c_{r+1} \)-graph \( \Gamma_{r+i-1}(\delta) \cap \Gamma(\gamma_m) \) contains the complete graph \( K_s \).

The following is a key lemma in this section.

**Lemma 4.2.** Suppose that \( x \in B \) and \( x' \in D_r^\bullet \cap \Gamma(x) \). If \( d \geq r + 3 \), then \( \Gamma(x) \cap D_{r+1}^\bullet \subseteq \Gamma(x') \). In particular, \( \epsilon(A, B) = 0 \).

**Proof.** We assume that \( \delta(\gamma_1, x) = r \) and \( \delta(\gamma_1, x') = r - 1 \). Note that \( \bar{x}(0, 1, a + 1) = (1, 0, 1) \) and \( \bar{x}'(0, 1, a + 1) = (0, -1, 0) \). As \( c_r = c_{r+1} = 1 \), we obtain \( \delta(\gamma_i, x) = r + 1 \) and \( \delta(\gamma_i, x') = r \) for \( 2 \leq i \leq a \); that is, \( \bar{x} = (1, 0, 1, \ldots, 1) \) and \( \bar{x}' = (0, -1, 0, \ldots, 0) \).

Let \( y \in (\Gamma(x) \cap D_{r+1}^\bullet) \setminus \Gamma(x') \). Suppose that \( y \in B \). If \( \delta(y, \gamma_1) = r \), then \( \bar{y}(0, 1) = (1, 0) \). This means that \( x, y \in D_{r+1}^\bullet (\gamma_0, \gamma_1), x' \in D_{r-1}^\bullet (\gamma_0, \gamma_1), y \sim x \sim x' \) and \( y \sim x' \). This contradicts Lemma 2.1(2). Suppose that \( \delta(y, \gamma_1) = r + 1 \) and \( \delta(y, \gamma_2) = r \). Note that \( \bar{y}(1, 2) = (0, 1) \) and \( \bar{y}(1, 2) = (1, 0) \). However, as \( x \sim y \), this contradicts Lemma 2.1(5).

Suppose that \( y \in A \). Let \( y_1 \in D_{r+1}^\bullet \) and \( y_2 \in D_{r+1}^\bullet \) with \( y_1, y_2 \in \Gamma(y) \). Clearly, we see that \( \delta(y, y_i) \geq r + 1 \) for \( 1 \leq i \leq a \). As \( \delta(x, \gamma_1) = r \), \( \delta(y, \gamma_1) = r + 1 \). Since \( a \geq 2 \), suppose that \( \delta(y, y_2) = r + 1 \). Let \( y' \in \Gamma(y_2) \cap \Gamma(y) \). Note that \( y' \in B \). Since both \( \{y_1, y_2, x\} \) and
\{y_1, y_2, y'\} are cocliques of size three in \(\Gamma(y)\), we have \(x \sim y'\). However, since \(\bar{x}(1, 2) = (0, 1)\) and \(\bar{y}(1, 2) = (1, 0)\), this contradicts Lemma 2.1(5). Hence we can assume that \(\delta(y_1, y_2) = r + 2\) and \(\{y_1, y_2, x\} \subseteq \Gamma_{r+1}(y_2) \cap \Gamma(y)\). But, as \(\{y_1, y_2, x\}\) is a coclique, we must have \(b_{r+2} = 0\), or \(d \leq r + 2\). This contradicts our assumption.

By this lemma, we see the types of the vertices in \(A\) and \(B\) under the assumption that \(d \geq r + 3\). If \(x \in A\), then \(\bar{x} = (1, 2, \ldots, 2, 1)\). If \(x \in B\), then \(\bar{x}(i) = 0\) for some \(1 \leq i \leq a\), and \(\bar{x}(j) = 1\) for \(j \neq i\).

**Lemma 4.3.** Suppose that \(d \geq r + 3\). Let \(x \in D_{r+1}^r\) and \(y, z \in D_{r+1}^{r+1} \cap \Gamma(x)\). Then \(y \not\sim z\). In particular, \(a_{r+1} \leq a + 2\).

*Proof.* Suppose that \(y \sim z\). By considering the intersection diagram with respect to \((x, y)\), we have \(x \in A(x, y)\) and \(z \in D_1^2(x, y)\). On the other hand, \(\delta(\alpha, z) = r + 1\). This contradicts Lemma 4.2. Hence \(y \not\sim z\). Since \(e(x, D_{r+1}^{r+1}) = a_{r+1} - a\) and \(\Gamma(x) = 3 \ast K_{a+1}\), we have \(a_{r+1} \leq a + 2\).

**Lemma 4.4.** Suppose that \(d \geq r + 3\). Let \(x \in A, x_1 \in D_{r+1}^r \cap \Gamma(x), x_2 \in D_{r+1}^r \cap \Gamma(x)\). Then \(\Gamma(x) \cap D_{r+1}^{r+1} = \Gamma(x) \cap \Gamma(x_1)\) and \(\Gamma(x) \cap D_{r+1}^{r+2} = \Gamma(x) \cap \Gamma(x_2)\).

*Proof.* By the previous lemma, we see that \(\Gamma(x) \cap \Gamma(x_1) \subseteq D_{r+1}^{r+1}\) and \(\Gamma(x) \cap \Gamma(x_2) \subseteq D_{r+1}^{r+1}\). Suppose that there exists \(y \in (\Gamma(x) \cap D_{r+1}^{r+2}) \setminus \Gamma(x_1)\). Since \(e(x, D_{r+1}^{r+2}) = e(x, D_{r+1}^{r+2})\), there exists \(z \in \Gamma(x) \cap D_{r+1}^{r+2}\) such that \(y \sim z\). Let \(u \in \Gamma(x) \cap \Gamma(x_1)\) and \(v \in \Gamma(x) \cap \Gamma(x_2)\). We see that \(\{x, u, v\} \not\subseteq \Gamma_{r+1}(\alpha)\) and \(\{u, v\} \subseteq D_2^{r+1}\). Since \(x \in \Gamma_{r+1}(\gamma)\) and \(y \not\sim x_1\), we obtain \(\delta(u, v) = 2\) by Lemma 2.2. Hence a \(c_{r+2}\)-graph \(\Gamma_{r+1}(\gamma) \cap \Gamma(\alpha)\) contains a coclique of size 3, \(\{u, v, \beta\}\), which contradicts \(d \geq r + 3\).

**Lemma 4.5.** Let \(x \in D_{r+2}^{r+1}\). If \(d \geq r + 3\), then \(e(x, D_{r+2}^{r+1}) = 1\).

*Proof.* Suppose that \(e(x, D_{r+2}^{r+1}) \neq 0\). Let \(y\) and \(z\) in \(\Gamma(x) \cap D_{r+2}^{r+2}\). Since \(d \geq r + 3\), a \(c_{r+2}\)-graph \(\Gamma(x) \cap \Gamma_{r+1}(\alpha)\) cannot contain a coclique of size 3. Hence we have that \(y \not\sim z\). Let \(u \in \Gamma(\alpha) \cap \Gamma(x)\) and \(v \in \Gamma(\alpha) \cap \Gamma(z)\). Note that \(\Gamma(\alpha) \cap \Gamma(\gamma) \subseteq D_{r+1}^{r+1}\), \(\Gamma(\alpha) \cap \Gamma(z) \subseteq D_{r+1}^{r+1}\), and \(\Gamma(\gamma) \cap \Gamma(x) \subseteq D_{r+1}^{r+1}\). Since \(x \in \Gamma(y) \cap \Gamma(z)\), \(\Gamma(y) \cap \Gamma(x) \subseteq \Gamma_{r+1}(\alpha)\) and \(\Gamma(z) \cap \Gamma(\alpha) = \emptyset\). Hence we see that \(\delta(u, v) = 2\) by Lemma 2.2, and that \(u\) and \(v\) form a coclique in \(\Gamma(\alpha) \cap \Gamma_{r+1}(x)\), a contradiction.

By Lemma 4.3, \(a_{r+1} = a + 1\) or \(a + 2\). We first treat the case \(a_{r+1} = a + 1\).

**Lemma 4.6.** If \(a_{r+1} = a + 1\) and \(d \geq r + 3\), then the following hold:

1. \(c_{r+2} \geq 3\).
2. If \(x \in A\) and \(y \in D_{r+2}^{r+2}\) with \(x \sim y\), then \(\Gamma(x) \cap \Gamma(y) \subseteq D_{r+2}^{r+2}\). In particular, \(e(A, A) = 0\).

*Proof.* (1) Let \(x \in \Gamma_{r+1}(\alpha)\). Since \(a_{r+1} = a + 1\), there exist vertices \(y \in \Gamma_{r+2}(\alpha) \cap \Gamma(x)\) and \(z \in \Gamma_{r+1}(\alpha) \cap \Gamma(x)\) such that \(y \sim z\). Hence \(c_{r+2} \geq 2\).

Assume that \(c_{r+2} = 2\). Let \(\bar{y} = (2, 1, 2, \ldots, 2)\). As \(A(\gamma_1, \gamma_2) \neq \emptyset\), the existence of a vertex of type \(y\) is guaranteed by Lemma 4.2. Let \(x_1, x_2 \in \Gamma(y)\) such that \(\bar{x}(i) = 0\) for \(i = 1, 2\). It is clear that \(\bar{x}(0, a + 1) = (1, 1)\) and \(x_i \in B\). Moreover, by Lemma 4.2, we see that \(\bar{x}_1(0, 1, 2, a + 1) = (1, 0, 1, 1)\), \(\bar{x}_2(0, 1, 2, a + 1) = (1, 1, 0, 1)\), and that \(\delta(x_1, x_2) = 2\). Since \(a_{r+1} = a + 1\), it follows from Lemma 4.2 that there exist vertices \(y_1 \in \Gamma(\gamma_1) \cap D_{r+1}^{r+1}\) and \(y_2 \in \Gamma(\gamma_1) \cap D_{r+1}^{r+1}\). By the assumption that \(c_{r+2} = 2\), \(y \not\sim y_1\) and \(y \not\sim y_2\). Hence it must
hold that \( y_1 \sim y_2 \) as \( \Gamma(y) = 3 \ast K_{a+1} \). This implies that
\[
c_{r+2} = |\Gamma_{r+1}(\alpha) \cap \Gamma(y_2)|
= e(y_2, D_{r+1}^{\ast +1}) + e(y_2, D_{r+1}^{\ast +1} \cup D_{r+2}^{\ast +1})
\geq c_{r+1} + \#\{x_1, y_1\}
= 3.
\]

This is a contradiction.

(2) This is a direct consequence of Lemma 4.4, as \( a_{r+1} = a + 1 \). □

**Lemma 4.7.** If \( a_{r+1} = a + 1 \), then \( d \leq r + 2 \).

**Proof.** Suppose \( d \geq r + 3 \). Let \( x \in D_{r+1}^{\ast +1} \) and \( y \in D_{r+1}^{\ast +1} \) with \( x \sim y \). Let \( C = (D_{r+2}^{\ast +2} \cap \Gamma(x)) \Gamma(y) \). Clearly \( |C| = a + 1 \).

We claim that there exists \( z \in C \) such that \( z = (2, \ldots, 2, 1) \). Suppose that every \( z \in C \) has the property \( \#\{i \mid z(i) = 1\} \geq 2 \). Since \( |C| = a + 1 \), we can find some \( i \neq a + 1 \) and \( z_1, z_2 \in C \) (\( z_1 \neq z_2 \)) such that \( z_1(0, i, a + 1) = (2, 1, 1) = z_2(0, i, a + 1) \). Note that \( z_1, z_2 \in A(\gamma_i, \gamma_{x+1}) \) and \( z_1 \sim z_2 \). This is impossible by Lemma 4.6(2).

Assume \( z \) as above. Since \( z(i) = 2 \) for \( 1 \leq i \leq a \), we have \( e(z, B) = 0 \). Suppose \( e(z, A) \neq 0 \). Let \( u \in \Gamma(z) \cap A \) and \( u' \in \Gamma(u) \cap D_{r+1}^{\ast +1} \). Since \( z \in C \), we have \( x \neq u' \). On the other hand, \( z \neq u' \), as \( x \sim z \) and \( c_{r+1} = 1 \). For \( u \in A \), this contradicts Lemma 4.4. Hence \( e(z, A) = 0 \). Therefore we must have \( e(x, D_{r+2}^{\ast +2}) \geq 2 \) as \( c_{r+2} \geq 3 \). This contradicts Lemma 4.5. □

Now we consider the case \( a_{r+2} = a + 2 \).

**Lemma 4.8.** Suppose \( d \geq r + 3 \) with \( a_{r+1} = a + 2 \). Then the following hold:

1. Every \( c_{r+2} \)-graph is a disjoint union of two \( K_2 \)'s. In particular, \( c_{r+2} = 4 \).
2. If \( x \in A, x_1 \in \Gamma(x) \cap D_{r+1}^{\ast +1} \) and \( x_2 \in \Gamma(x) \cap D_{r+1}^{\ast +1} \), then \( \Gamma(x) \cap \Gamma(x_1) = D_{r+2}^{\ast +1} \Gamma(x) \cap \Gamma(x) = D_{r+1}^{\ast +2} \) and there exists a vertex \( y \in \Gamma(x) \cap D_{r+2}^{\ast +1} \) such that \( \Gamma(x) \cap \Gamma(y) \in D_{r+2}^{\ast +2} \).
3. There exists no triangle \( \{x, y, z\} \) such that \( x \sim y, y \sim z \), and \( z \sim x \).

**Proof.** (1) For \( x \in \Gamma_{r+1}(\alpha) \), we consider the type of cliques containing \( x \). Let \( x' \in \Gamma_r(\alpha) \cap \Gamma(x) \). As \( c_{r+1} = 1 \), \( \Gamma(x) \cap \Gamma(x') \subset \Gamma_{r+1}(\alpha) \). As \( a_{r+1} = a + 2 \), there exist \( y \) and \( z \) in \( \Gamma_{r+1}(\alpha) \cap \Gamma(x) \) such that \( y \neq x' \) and \( z \neq x' \). By Lemma 4.3, we obtain \( y \neq z \). As \( a_{r+1} = a + 2 \), \( \Gamma(x) \cap \Gamma(y) \subset \Gamma_{r+2}(\alpha) \) and \( \Gamma(x) \cap \Gamma(z) \subset \Gamma_{r+2}(\alpha) \). Thus the clique type of \( x \in \Gamma_{r+1}(\alpha) \) is uniquely determined. Therefore we see that \( C_{r+2} = m \ast K_2 \) for every \( c_{r+2} \)-graph \( C_{r+2} \) and some \( m \). Since \( d \geq r + 3 \), we have \( m \leq 2 \). Now we claim \( m = 2 \); that is, \( c_{r+2} = 4 \). Suppose that \( m = 1 \). We consider the intersection diagram of rank 1 with respect to \( (\alpha, \beta) \). Let \( \delta \in A, \delta_1 \in D_{r+1}(\alpha) \cap \Gamma(\delta) \) and \( \delta_2 \in D_{r+1}(\gamma) \cap \Gamma(\delta) \). By Lemma 4.2, there exists \( y \in D_{r+1}(\gamma) \) such that \( \delta(y, \delta) = r + 2 \). Note that \( \{\delta_1, \delta_2\} \subset \Gamma_{r+1}(\gamma) \cap \Gamma(\delta) \). Since \( \delta_1 \neq \delta_2 \), this contradicts \( m = 1 \).

(2) This is a direct consequence of Lemmas 4.3 and 4.4, as \( a_{r+2} = a + 2 \).

(3) We note that \( x \in A(\gamma_j, \gamma_j), y \in D_{r+1}(\gamma_j, \gamma_j) \) and \( z \in D_{r+2}(\gamma_j, \gamma_j) \). Let \( x' \in \Gamma(x) \cap D_{r+1}(\gamma_j, \gamma_j) \). By (2), we have \( x' \sim y \). On the other hand, we clearly have \( x' \sim z \). This contradicts the fact that \( \Gamma(x) \cap \Gamma(y) \) is a clique. □
LEMMA 4.9. Suppose that $d \geq r + 3$ with $a_{r+1} = a + 2$. For every $i$ with $r + 2 \leq i \leq d$, the following hold:

1. Let $\bar{x}(m,n) = (i-1-r,i-r)$ and $\bar{y}(m,n) = (i-r,i-1-r)$. If $x \sim y$, then $i = r + 2$ or $d$.
2. If $u \in D_i((\gamma_m, \gamma_n))$, then $e(u, D_i^{-1}(\gamma_m, \gamma_n)) \neq 0$, except the case $i = d$, and $e(D_d^{-1}(\gamma_m, \gamma_n), D_{d-1}(\gamma_m, \gamma_n)) \neq 0$. In other words if $\bar{u}(m,n) = (i-r,i-r)$, then there exists a vertex $v \in \Gamma(u)$ such that $\bar{v}(m,n) = (i-r-1,i-r-1)$, except the case as above.
3. If $\min(\bar{x}(j) \mid j = 0, 1, \ldots, a+1) = i-1-r$ and $\max(\bar{x}(j) \mid j = 0, 1, \ldots, a+1) = i-1-r$, then $s_i = \# \{ j \mid \bar{x}(j) = i-1-r \} \leq 2$ or $3$.

Moreover, if $s_i = 3$, then $i = d$.

4. If $r + 2 \leq i \leq d - 1$, every $c_i$-graph is a disjoint union of two $K_2$'s and every $b_i$-graph is a $K_{a+2}$.

PROOF. (1) We note that $x \in D_i^{-1}(\gamma_m, \gamma_n)$ and $y \in D_i^{-1}(\gamma_m, \gamma_n)$. Since a $c_i$-graph $\Gamma_{i-1}(\gamma_n) \cap \Gamma(x)$ contains a $c_{i-1}$-graph $D_i^{-1}(\gamma_m, \gamma_n) \cap \Gamma(x)$, it must hold that the $c_{i-1}$-graph is a clique or the $c_i$-graph is a disjoint union of three cliques. Hence we have $i = r + 2$ or $d$.

(2), (3) Let $i = r + 2$. The assertion (3) follows from Lemma 4.8(1).

Suppose that there exists $u \in D_{r+2}^{-1}(\gamma_m, \gamma_n)$ such that $e(u, D_{r+2}^{-1}(\gamma_m, \gamma_n)) = 0$. Then there exist $y_1, y_2 \in \Gamma(x) \cap D_{r+2}^{-1}(\gamma_m, \gamma_n)$ and $y_3, y_4 \in D_{r+2}^{-1}(\gamma_m, \gamma_n)$ such that $y_1 \neq y_2$ and $y_3 \neq y_4$. Since $\Gamma(u) = 3 \ast K_a$, we may assume that $y_1 \sim y_3$ and it follows from Lemma 4.8(1) that there is a vertex $z \in \Gamma_{r+1}(\gamma_n) \cap \Gamma(u) \cap \Gamma(y_1) \cap \Gamma(y_3)$. By Lemma 4.5, we have $z \in D_{r+2}^{-1}(\gamma_m, \gamma_n)$, a contradiction. Thus we have (2) in the case $i = r + 2$.

Let $i-1 \geq r + 2$. Suppose that there exists $u \in D_i^{-1}(\gamma_m, \gamma_n)$ such that $e(u, D_i^{-1}(\gamma_m, \gamma_n)) = 0$. Then there exist $y_1, y_2 \in \Gamma(x) \cap D_i^{-1}(\gamma_m, \gamma_n)$ and $y_3, y_4 \in D_{i-1}^{-1}(\gamma_m, \gamma_n)$ such that $y_1 \neq y_2$ and $y_3 \neq y_4$. Since $\Gamma(u) = 3 \ast K_a$, we may assume that $y_1 \sim y_3$, which contradicts (1). By the same argument, it must also hold that $e(u, D_i^{-1}(\gamma_m, \gamma_n)) \neq 0$ for every $u \in D_i^{-1}(\gamma_m, \gamma_n)$ in the case $e(D_{d-1}^{-1}(\gamma_m, \gamma_n), D_{d-1}^{-1}(\gamma_m, \gamma_n)) = 0$. Thus we have (2).

Suppose that (3) holds for $i-1 \geq r + 2$. Let $\bar{x}(j_1,j_2,j_3,j_4) = (i-r,i-r-1,i-r-1,i-r-1)$. Then, by (2) and (3) with $i-1$, we can find vertices $y_1, y_2, y_3 \in \Gamma(x) \cap \Gamma_{i-1}(\gamma_n)$ such that

\[
\begin{align*}
\bar{y}_1(j_1,j_2,j_3,j_4) &= (i-r-1,i-r-2,i-r-2,i-r-1), \\
\bar{y}_2(j_1,j_2,j_3,j_4) &= (i-r-1,i-r-2,i-r-2,i-r-2), \\
\bar{y}_3(j_1,j_2,j_3,j_4) &= (i-r-1,i-r-1,i-r-2,i-r-2). 
\end{align*}
\]

Now, by (1) in this lemma for $i \neq r + 3$ and Lemma 4.8(3) for $i = r + 3$, $\{ y_1, y_2, y_3 \} \cap \Gamma_{i-1}(\gamma_n) \cap \Gamma(x)$ is a coclique of size 3. Hence $i = d$. By the same argument, we know that $\# \{ j \mid \bar{x}(j) = i-r-1 \} \leq 3$. Thus we have (3).

(4) This is a direct consequence of (3).

LEMMA 4.10. Suppose that $d \geq r + 3$, with $a_{r+1} = a + 2$. Then the intersection number of $\Gamma$ is the one in Theorem 4.1.

PROOF. By Lemma 4.9(4), we only need to determine $c_d$. We note that $C_d = t \ast K_{s_d}$ for every $c_d$-graph $C_d$ and $t = 2$ or $3$ by Lemma 4.9(3).

Suppose that $s_d = 2$ and $t = 2$. Note that $c_{d-1} = c_d$ in this case. If $x \in \Gamma_d(\alpha)$, it follows from the assumption $t = 2$ that there exists $\beta \in \Gamma_d(x) \cap \Gamma(\alpha)$ such that $\Gamma(\alpha) \cap \Gamma(\beta) \subset$
A DRG with $\Gamma(x) = 3 \ast K_{a+1}$

Hence we may assume that $\vec{x} = (d, \ldots, d)$. Since $c_{d-1} = c_d$, we have $e(D_{d-1}^d(y_m, y_n)) = 0$ for each $m$ and $n$. Hence we have $e(x, D_{d-1}^d(y_m, y_n)) \neq 0$ for each $m$ and $n$ by Lemma 4.9(2). We can easily find a coclique of size 4 $\{y_1, y_2, y_3, y_4\} \subset \Gamma(x)$ such that

$\vec{y}_1(i_1, i_2, i_3, i_4) = (d - r - 1, d - r - 1, d - r, d - r)$,
$\vec{y}_2(i_1, i_2, i_3, i_4) = (d - r - 1, d - r, d - r - 1, d - r)$,
$\vec{y}_3(i_1, i_2, i_3, i_4) = (d - r, d - r - 1, d - r - 1, d - r)$,
$\vec{y}_4(i_1, i_2, i_3, i_4) = (d - r, d - r, d - r - 1, d - r - 1)$.

This is a contradiction. Thus we obtain $t = 3$ and $c_d = 6$.

Suppose that $s_d = 3$ and $t = 2$. There exists a vertex $x \in \Gamma$ such that $\vec{x}(i_1, i_2, i_3, i_4) = (d - r - 1, d - r - 1, d - r - 1, d - r)$. By Lemma 4.9(2), we can find three vertices $\{y_1, y_2, y_3\} \subset \Gamma(x)$ such that

$\vec{y}_1(i_1, i_2, i_3, i_4) = (d - r - 2, d - r - 2, d - r - 1, d - r)$,
$\vec{y}_2(i_1, i_2, i_3, i_4) = (d - r - 2, d - r - 1, d - r - 2, d - r)$,
$\vec{y}_3(i_1, i_2, i_3, i_4) = (d - r - 1, d - r - 2, d - r - 2, d - r)$.

By Lemma 4.8(3) for $d = r + 3$ and Lemma 4.9(1) for $d \geq r + 4$, we see that $\{y_1, y_2, y_3\} \subset \Gamma(x) \cap \Gamma_{d-1}(y_m)$ becomes a coclique of size 3, a contradiction. Thus $t = 3$ and $c_d = 9$.

To show that, if $a_{r+1} = a + 2$ and $d \geq r + 3$, $\Gamma$ is a distance-2 graph of a distance-biregular graph with a vertex of valency 3, let $\Delta$ be the set of all maximal cliques in $\Gamma$. Let $\bar{\Gamma} = \Gamma \cup \Delta$. We view $\bar{\Gamma}$ as an incidence graph, i.e. for $\alpha \in \Gamma$ and $x \in \Delta$, $\alpha \sim x$ in $\bar{\Gamma}$ iff $\alpha \in x$ in $\Gamma$.

We use the notation for the graph $\bar{\Gamma}$.

Note that for $\alpha, \beta \in \Gamma$ and $x, y \in \Delta$,

$\vec{\delta}(\alpha, \beta) = 2i$ iff $\delta(\alpha, \beta) = i$
$\vec{\delta}(\alpha, x) = 2i + 1$ iff $\delta(\alpha, x) = i$
$\vec{\delta}(x, y) = 2i + 2$ iff $\delta(x, y) = i$ and $x \neq y$.

It is easy to check the following. For $\alpha \in \Gamma$ and $x \in \Delta$:

$\vec{e}_2(\alpha) + \vec{b}_2(\alpha) = \vec{e}_{2i+1}(x) + \vec{b}_{2i+1}(x) = 3$,
$\vec{e}_2(x) + \vec{b}_2(x) = \vec{e}_{2i+1}(\alpha) + \vec{b}_{2i+1}(\alpha) = a + 2$,

$\vec{e}(\alpha) = \begin{cases} 
1 & \text{for } 1 \leq i \leq 2r + 2 \\
2 & \text{for } 2r + 3 \leq i \leq 2(d - 1) \\
3 & \text{for } i = 2d \\
3 & \text{for } i = 2d - 1, 2d \\
\end{cases}$

$\vec{e}(x) = \begin{cases} 
1 & \text{for } 1 \leq i \leq 2r + 2 \\
2 & \text{for } 2r + 3 \leq i \leq 2(d - 1) \\
3 & \text{for } i = 2d \\
3 & \text{for } i = 2d - 1, 2d \\
\end{cases}$
Therefore $\Gamma$ is a distance-2 graph of a distance-biregular graph $\Gamma$. This completes the proof of Theorem 4.1. Theorem 1.1 follows directly from Theorems 3.1 and 4.1.

5. CONCLUDING REMARKS

(1) Bounding the diameter of a distance-regular graph by some function of its valency $k$ is generally an open problem. As a partial answer to this, E. Bannai and T. Ito showed in [2] that $d$ (or $r$) is theoretically bounded by a function of $k$ and $d - r$. Recently, H. Suzuki showed that, if $\Gamma$ is a distance-regular graph with $\Gamma(x) = 3^*K_{a+1}$ for every $x \in \Gamma$ and with $d \leq r + 2$, then $d \leq 41$.

(2) We have not had the classification of distance-biregular graphs of valency 3 and $a + 2$ ($a \geq 2$). (Note that, if $r$ is odd, H. Suzuki [12] gave the classification.) As the next step, we want the absolute bound of diameter $d$ (not depending on the parameter $a$) in the case $d \geq r + 3$.

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Received 28 April 1993 and accepted 27 February 1995

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