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LOCAL INDICATORS FOR PLURISUBHARMONIC FUNCTIONS

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ABSTRACT. – The notion of index, classical in number theory and its calculation by P. Lelong (1997) for plurisubharmonic functions, allows to define an indicator which is applied to the study of the Monge–Ampère operator and a pluricomplex Green function. © Elsevier, Paris

1. Introduction

We recall some local notions wich are often used in various investigations.

(a) The index $I(F, x^0, a)$ of a zero $x^0 \in \Omega$ of a holomorphic function $F \in Hol(\Omega)$, where Ω is a domain of $\mathbf{C}^n$, is used in important results of number theory [13]; $I(F, x^0, a)$ is defined by means of the set $\omega \in \mathbf{N}^n$ of n-tuples $(i) = (i_1, \ldots, i_n)$ such that $D^{(i)}F(x^0) \neq 0$. Given a direction $(a) = (a_k > 0;\ 1 \leqslant k \leqslant n) \in \mathbf{R}^n$,

$$(1) \qquad I\big(F, x^0, a\big) = \inf_{(i)}(a, i) \quad \text{for } (a, i) = \sum_k a_k i_k \geqslant 0 \text{ and } (i) \in \omega.$$

(b) In fact, the index $I(F, x^0, a)$ is a property (function of a and x^0) of the current of integration $[W] = dd^c \log|F|$ over the analytic set $W = \{x \in \Omega:\ F(x) = 0\}$, where $d = \partial + \bar\partial$ and $d^c = (2\pi i)^{-1}(\partial - \bar\partial)$. We denote by $PSH(\Omega)$ the class of plurisubharmonic functions in a domain Ω of $\mathbf{C}^n$ and by $\Theta_p(\Omega)$ the class of positive closed currents represented by homogeneous forms of $dx_k, d\bar x_k$, $1 \leqslant k \leqslant n$, of bidegree $(n-p, n-p)$. The Lelong number $\nu(T, x^0)$ at $x^0 \in \Omega$ for $T \in \Theta_p(\Omega)$ is related to the trace measure of T,

$$(2) \qquad \sigma_T = T \wedge \beta_p,$$

where $\beta_p = (p!)^{-1}\beta_1^p$ is the volume element of $\mathbf{C}^p$. By the definition,

$$(3) \qquad \nu(T, x^0) = \lim_{r\to 0}\big(\tau_{2p} r^{2p}\big)^{-1}\sigma_T\big[B^{2n}\big(x^0, r\big)\big],$$

where τ_{2p} is the volume of the unit ball $B^{2p}(0,1)$ of $\mathbf{C}^p$. In (3), the trace measure σ_T belongs to the remarkable class $\mathcal{P}_p(\Omega)$ of positive measures characterized by the property that the quotient

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in the right hand side of (3) is an increasing function of r (see [16]). Another definition of the number $\nu(T,x^0)$ is derived from (2) and (3) by setting $\varphi_1(x)=\log|x-x^0|$:

$$\nu(T,x^0)=\lim_{t\to-\infty}\int_{\{\varphi_1<t\}} T\wedge\big(dd^c\varphi_1\big)^p. \tag{4}$$

(c) By replacing in (4) φ_1 with a function $\varphi\in PSH(\Omega)$ such that $\exp\varphi$ is continuous and the set $\{\varphi(x)=-\infty\}$ is relatively compact, C.O. Kiselman (see [10] and [11]) and J.-P. Demailly (see [4] and [5]) have found new important applications of (4). The choice of the function

$$\varphi_a(x)=\sup_k a_k^{-1}\log\left|x_k-x_k^0\right| \tag{5}$$

which is circled with the center x^0 and such that $\{\varphi_a(x)=-\infty\}$ is reduced to x^0, allows us to put into this framework the notion of index $I(F,x^0,a)$ of a zero of a holomorphic function F at x^0, see [5].

$$I\big(F,x^0,a\big)=(a_1\cdot a_2\ldots a_n)\nu\big[T,x^0,\varphi_a\big(x-x^0\big)\big] \tag{6}$$

for $T=dd^c\log|F|\in\Theta_1(\Omega)$. It can be extended to arbitrary plurisubharmonic functions by setting for $f\in PSH(\Omega)$ and $x^0\in\Omega$,

$$n\big(f,x^0,a\big)=(a_1\cdot a_2\ldots a_n)\,\nu\big[dd^cf,x^0,\varphi_a\big(x-x^0\big)\big], \tag{7}$$

so that

$$I\big(F,x^0,a\big)=n\big(f,x^0,a\big)\quad\text{for } f=\log|F|. \tag{8}$$

(d) In the recent paper [19], $n(f,x^0,a)$ has appeared in a simpler form given for $x^0=0$ by the relation

$$n(f,0,a)=\lim_{w\to0}(\log w)^{-1}f(w,x,a),\quad 0\leqslant w\leqslant1,\ x\in\Omega\setminus A, \tag{9}$$

where $f(w,x,a)=f(w^{a_1}x_1,\ldots,w^{a_n}x_n)$ and A is an algebric set for $f=\log|F|$. In the general case, for $f\in PSH(\Omega)$, A is of zero measure and

$$n(f,0,a)=\liminf_{w\to0}(\log w)^{-1}f(w,x,a)$$

outside a set $A'\subset A$ which is pluripolar in Ω.

In the first part of the present paper we will suppose $f\in PSH(\Omega)$ and $D\Subset\Omega$, where D is the unit polydisk $\{x\in\mathbf{C}^n\colon\ \sup|x_k|<1\}$; we denote by $PSH_-(D)$ the class of f satisfying $\sup\{f(x)\colon\ x\in D\}\leqslant0$ and $f\not\equiv-\infty$. The domain D as well as the weights φ_a in (5) being circled, it is natural to work with a circled image f_c of f and then with its convex image on the space $\mathbf{R}^n_-$ of $u_k=\log|x_k|$, $1\leqslant k\leqslant n$. Developing the results of J.-P. Demailly [5] and C.O. Kiselman [10] we get the value $n(f,0,a)$ which produces, via $a_k=-\log|y_k|$, a function $\Psi_{f,0}(y)$, the local indicator of f at 0, which is plurisubharmonic in $D_{(y)}$, the unit polydisk in the space $\mathbf{C}^n_y$. It satisfies the Monge–Ampère equation:

$$\big(dd^c\Psi_{f,0}\big)^n=\tau_f(0)\delta(0),\quad \tau_f(0)>0, \tag{10}$$

where $\delta(0)$ is the Dirac measure at the origin of $\mathbf{C}^n_y$, and

$$\big(dd^cf\big)^n\geqslant\tau_f\big(x^0\big)\delta\big(x^0\big) \tag{11}$$

for $f \in PSH(\Omega)$ such that $(dd^c f)^n$ is well defined, and $x^0 \in \Omega$. Moreover, there is the relation $\tau_f(x^0) \geqslant [\nu(T, x^0)]^n$ for $T = dd^c f \in \Theta_{n-1}(\Omega)$.

Then we consider a class of plurisubharmonic functions on Ω with singularities on a finite set $\{x^1, \dots, x^N\}$, controlled by given indicators Ψ_m, $1 \leqslant m \leqslant N$. We construct a plurisubharmonic function G vanishing on $\partial\Omega$ and such that $\Psi_{G,x^m} = \Psi_m$, $1 \leqslant m \leqslant N$, and

$$\left(dd^c G\right)^n = \sum_m \tau_m \delta\left(x^m\right)$$

with τ_m the mass of $(dd^c\Psi_m)^n$. We prove that G is the unique plurisubharmonic function with these properties. For the case $\Psi_m(x) = \nu_m \log|x - x^m|$, it coincides with the pluricomplex Green function with weighted poles at $x^1, \dots, x^N$ [17]. We prove a variant of comparison theorem for plurisubharmonic functions with controlled singularities and study a Dirichlet problem for this class of functions.

A part of the results of Section 3 are close to those from [20] where a weighted pluricomplex Green function with infinite singular set was introduced and the corresponding Dirichlet problem was studied.

2. Circled functions and convex projections

We will consider here the case $x^0 = 0$ and $f \in PSH(\Omega)$ supposing $0 \in D \Subset \Omega$, where D is the unit polydisk $\{x \in \mathbf{C}^n\colon\ \sup|x_k| < 1\}$, and $f \in PSH_-(D)$, that is $f(x) \leqslant 0$ for $x \in \overline{D}$ and $f \not\equiv -\infty$.

A set $A \subset \mathbf{C}^n$ is called 0-*circled* (or just *circled*) if $x = (x_k) \in A$ implies $x' = (x_k e^{i\theta_k}) \in A$ for $0 \leqslant \theta_k \leqslant 2\pi$, $1 \leqslant k \leqslant n$. We will say that a function $f(x)$ defined on A, is circled if it is invariant with respect to the rotations $x_k \mapsto x_k e^{i\theta_k}$, $1 \leqslant k \leqslant n$.

Given a function $f \in PSH(\Omega)$, Ω being a circled domain, we consider a circled function $f_c \in PSH(\Omega)$ equal to the mean value of $f(x_k e^{i\theta_k})$ with respect to $0 \leqslant \theta_k \leqslant 2\pi$, $1 \leqslant k \leqslant n$. In what follows, we will also use another circled function $f_c' \geqslant f_c$, equal to the maximum of $f(x_k e^{i\theta_k})$ for $0 \leqslant \theta_k \leqslant 2\pi$, $1 \leqslant k \leqslant n$. Note that the differential operators, namely ∂, $\bar\partial$, $d = \partial + \bar\partial$ and $d^c = (2\pi i)^{-1}(\partial - \bar\partial)$, commute with the mapping $f \mapsto f_c$, so $(\partial f)_c = \partial f_c$, however it is not the case for $f \mapsto f_c'$.

To a Radon measure σ on a circled domain Ω, we relate a circled measure σ_c defined by $\sigma_c(f) = \sigma(f_c)$ for continuous functions f. In the same way, to a current $T \in \Theta_p(\Omega)$ we associate a circled current T_c which is defined on homogeneous forms λ of bidegree (p, p) by $T_c(\lambda) = T(\lambda_c)$, where λ_c are obtained by replacing the coefficients of λ with their mean values with respect to θ_k. In particular, if $T = dd^c f$, $f \in PSH(\Omega)$ and Ω circled, $T_c = dd^c f_c$. It gives us a specific property of the value $n(f, 0, a)$ defined by (4) and (6), and of the index $I(F, 0, a)$.

PROPOSITION 1. – *Let Ω be a* 0*-circled domain and $T \in \Theta_p(\Omega)$. For every* 0*-circled weight φ, $\nu(T, \varphi) = \nu(T_c, \varphi)$. In particular, since the weight φ_a defined by* (5) *for $x^0 = 0$, is circled, the number $n(f, 0, a)$ in* (7) *for $f \in PSH(\Omega)$ at the origin can be calculated by replacing f with f_c: $n(f, 0, a) = n(f_c, 0, a)$, and the number $n(f, 0, a)$ is calculated on the convex image $g(u)$ of f_c, $g(u) = f[\exp(u_k + i\theta_k)]$:*

$$n(f, 0, a) = \lim_{v \to -\infty} v^{-1} g(u_k + a_k v).$$

For $f \in PSH_-(D)$, as was shown in [19],

$$n(f, 0, a) = \lim_{w \to 0} (\log w)^{-1} f\left(w^{a_1} x_1, \dots, w^{a_n} x_n\right),$$

for almost all $x \in D$. The limit exists for all $x \in D$ when replacing $f(x)$ by $f_c(x)$ or by $f'_c(x)$. Indeed, for $R > 1$

$$f_c(x) \leqslant f'_c(x) \leqslant \gamma_R\, f_c(Rx) \leqslant 0 \tag{12}$$

with $\gamma_R = (R-1)^n(R+1)^{-n}$ which satisfies $1-\epsilon \leqslant \gamma_R \leqslant 1$ for $R > R_0(\epsilon)$.

The calculation of $n(f,0,a)$ for $f \in PSH_-(D)$ uses the convex image $g_f(u) = f_c[\exp(u_k + i\theta_k)]$ or $g'_f(u) = f'_c[\exp(u_k + i\theta_k)]$ obtained by setting $x_k = \exp(u_k + i\theta_k)$, the functions g_f and g'_f being defined on $\mathbf{R}^n_- = \{-\infty \leqslant u_k \leqslant 0\}$.

PROPOSITION 2. – *In order that a function $h(u_1,\ldots,u_n)\colon \mathbf{R}^n_- \to \mathbf{R}_-$ be the image of $f \in PSH_-(D)$ obtained by $x_k = \exp(u_k + i\theta_k)$ and*

$$f(x) = f\big[\exp(u_k + i\theta_k)\big] = h(u),$$

it is necessary and sufficient that h be convex of $u \in \mathbf{R}^n_-$, increasing in each u_k, $-\infty \leqslant u_k \leqslant 0$, and $h(u) \not\equiv -\infty$, $h(u) \leqslant 0$.

The necessity condition results from the classic properties of $f \in PSH_-(D)$. To show the sufficiency, we remark that convexity of h implies its continuity on $\mathbf{R}^n_-$. On the other hand, we have (the derivatives being taken in the sense of distributions), for any $\lambda \in \mathbf{C}^n$,

$$4\sum \frac{\partial^2}{\partial x_k \partial \bar{x}_j}\lambda_k \overline{\lambda_j} = \sum \frac{\partial^2}{\partial u_k \partial u_j}\lambda'_k \overline{\lambda'_j} \tag{13}$$

where $\lambda'_k = x_k^{-1}\lambda_k$. Let $A \subset D$ be the union of the subspaces $\{x_k = 0\}$ in D. By (13), $f \in PSH(D \setminus A)$. The condition $f(x) \leqslant 0$ implies that f extends by upper semicontinuity to A, so $f \in PSH_-(D)$ for $f(x) = h(\log|x_1|,\ldots,\log|x_n|)$.

DEFINITION. – *We denote by $Conv(\mathbf{R}^n_-)$ the class of functions $h(u_1,\ldots,u_n) \leqslant 0$, $-\infty \leqslant u_k \leqslant 0$, satisfying the conditions listed in Proposition* 2.

PROPOSITION 3. – *Let $h \in Conv(\mathbf{R}^n_-)$ be the image of*

$$f(x_k) = h\big[\exp(u_k + i\theta_k)\big] \in PSH_-(D).$$

Then

(a)
$$\lim_{v\to-\infty} v^{-1}h(u_1+v,\ldots,u_n+v) = \lim_{v\to-\infty}\frac{\partial}{\partial v}h(u_1+v,\ldots,u_n+v) = \nu(f,0),$$

where $\nu(f,0)$ is the Lelong number of f at $x=0$. More generally,

$$\lim_{v\to-\infty} v^{-1}h(u_1+a_1v,\ldots,u_n+a_nv) = n(f,0,a) \tag{14}$$

is independent of u_k;

(b) $\lim_{v\to-\infty} v^{-1}h(u_1+v,u_2,\ldots,u_n) = \nu_1(f,0)$ *is independent of u_k and is the generic Lelong number* (*cf.* [7]) *of the current $T = dd^c f$ along the variety $D_1 = \{x \in D\colon\ x_1 = 0\}$. Moreover, for $x'_1 = (x_2,\ldots,x_n)$, the function*

$$h_1(r_1,x'_1) = (2\pi)^{-1}\int_0^{2\pi} f\big(r_1e^{i\theta_1},x'_1\big)\,\mathrm{d}\theta_1,$$

has the property

$$\lim_{w\to 0}(\log w)^{-1}h_1(wr_1,x_1')=\nu_1(f,0)$$

for $x_1'\in D_1$ *with exception of a pluripolar subset of* D_1;

(c) $\sum_1^n \nu_k(f,0)\leqslant \nu(f,0)$.

Proof. – Existence and equality of the limits in (a) follow from the increasing with respect to v, $-\infty<v\leqslant 0$, and from the condition $h\leqslant 0$. Moreover, if $l(\rho)$ is the mean value of $f(x)$ over the sphere $|x|=\rho$, then

$$\nu(f,0)=\lim_{\rho\to 0}\frac{\partial\, l(\rho)}{\partial\log\rho}=\lim_{\rho\to 0}(\log\rho)^{-1}l(\rho). \tag{15}$$

We compare the mean values with respect to θ_k over the circled domains $B(0,\rho)$ and $D(\rho)=\{\sup|x_k|\leqslant\rho<1\}$ for the image $h(u)$ of the circled function $f_c(x)$, for $u_k=\log\rho-\frac{1}{2}\log n$ and $u_k'=\log\rho,\ 1\leqslant k\leqslant n$:

$$h(u)\leqslant l(\log\rho)\leqslant h(u'),$$

since $D(\rho/\sqrt{n})\subset B(0,\rho)\subset D(\rho)$. This gives us (14) and (a).

Statement (b) is known (cf. [16]). The limit

$$-c(x_1')=\lim_{r\to 0}\left(\log\frac{1}{r}\right)^{-1}h\big(r,x_1'\big)\leqslant 0 \tag{16}$$

for $r\searrow 0$ exists and is obtained by increasing negative values, the second term of (16) belonging to $PSH(D_1)$ for $r>0$. If $c(\hat{x}_1')=0$ for a point $\hat{x}_1'\in D_1$, then $c(x_1')=0$ except for a pluripolar subset of D and the statement is proved. Otherwise, consider the set $\overline{D_1}(r)\Subset D_1$ and $c_0=\sup c(x_1')$ for $x_1'\in D_1(r)$ and apply the preceding argument to

$$\lim_{r\to 0}\left[\left(\log\frac{1}{r}\right)^{-1}h(r,x_1')+c_0\right].$$

The statement for $h\in Conv(\mathbf{R}_-^n)$ follows from this precise property of the plurisubharmonic image.

To establish (c), we observe that for $u\in\mathbf{R}_-^n$ and $h(u)$ the image in $Conv(\mathbf{R}_-^n)$ of $f\in PSH_-(\overline{D})$,

$$\frac{\partial}{\partial v}h(u_1+v,\dots,u_n+v)=\sum_1^n\frac{\partial h}{\partial u_k}(u_1+v,\dots,u_n+v),$$

the derivatives are positive and decreasing for $v\searrow-\infty$, and the limit of

$$\frac{\partial h}{\partial u_k}(u_1+v,u_2,\dots,u_n)$$

is equal to $\nu_1(f,0)$, the Lelong number of dd^cf along D_1. Therefore

$$\frac{\partial}{\partial v}h(u_1+v,\dots,u_n+v)\geqslant\sum_1^n\nu_k(f,0),$$

so taking $v \searrow -\infty$ we get by (a):

$$\nu(f,0) \geqslant \sum_1^n \nu_k(f,0). \tag{17}$$

□

Remark. – Actually, by a theorem of Y.T. Siu, (17) is a particular case of the following statement: the number $\nu(f,0)$ is at least equal to the sum of the generic numbers $\nu(W_i)$ for $T = dd^c f$ along analytic varieties W_j of codimension 1 containing the origin.

In what follows, we will use a special subclass of circled plurisubharmonic functions $f \in PSH_-(D)$ that have the following "conic" property: the convex image $g_f(u)$ of f satisfies the equation

$$g_f(cu) = c\, g_f(u) \quad \text{for every } c > 0. \tag{18}$$

Such a function f will be called an *indicator*. For example, the weights φ_a in (5) are indicators.

PROPOSITION 4. – *Let $f \in PSH_-(D)$ be an indicator. Then $(dd^c f)^n = 0$ on*

$$D_0 = \{x \in D\colon\ x_1 \dots x_n \neq 0\}.$$

Proof. – It is sufficient to show that the domain D_0 can be foliated by one-dimensional analytic varieties γ_y such that the restriction of f to each leaf γ_y is harmonic on γ_y. So, given $y = (|y_k| e^{i\theta_k}) \in D_0$, consider an analytic variety γ_y, the image of $\mathbf{C}$ under the holomorphic mapping $\lambda = (\lambda_1, \dots, \lambda_n)$ with $\lambda_k(\zeta) = |y_k|^\zeta e^{i\theta_k}$. Note that $y = \lambda(1) \in \gamma_y$. As f is circled, the function $f_y(\zeta) = f(\lambda(\zeta))$, the restriction of f to γ_y, depends only on Re ζ. By (18), $f_y(\zeta)$ satisfies $f_y(c\zeta) = c\, f_y(\zeta)$ for all $c > 0$. Therefore, it is linear and thus harmonic on γ_y. □

3. Indicator of a plurisubharmonic function

Given a function $f \in PSH(\Omega)$ and a point $x^0 \in \Omega$, we will construct a function $\Psi_{f,x^0}(y)$ related to local properties of f at x^0. We will have $\Psi_{f,x^0} \in PSH_-(D)$, D being the unit polydisk in the space $\mathbf{C}^n_{(y)}$, and $\Psi_{f,x^0}(y) < 0$ in D if and only if the Lelong number of f at x^0 is strictly positive, otherwise $\Psi_{f,x^0}(y) \equiv 0$.

DEFINITION. – *The* local indicator (*or just* indicator) $\Psi_{f,0}$ of a function $f \in PSH_-(D)$, $D \subset \mathbf{C}^n_{(x)}$, at $x^0 = 0$ *is defined for $y \in D \subset \mathbf{C}^n_{(y)}$ by*

$$\Psi_{f,0}(y) = -n\bigl(f, 0, -\log|y_k|\bigr).$$

Referring to (9) with $R = -\log w$, $0 < R < +\infty$, we rewrite this as:

$$\Psi_{f,0}(y) = \lim_{R \to +\infty} R^{-1} f\bigl[\exp\bigl(u_k + i\theta_k + R\log|y_k|\bigr)\bigr]. \tag{19}$$

The limit (19) exists *almost everywhere* for $x_k = \exp(u_k + i\theta_k)$, however (see Introduction) the value $n(f,0,a)$ can be calculated as well by replacing $f(x)$ with the circled functions $f_c(x)$ or $f_c'(x)$. One can then substitute them for f in (19) to get $\Psi_{f,0}$. At $x^0 \neq 0$, the function $\Psi_{f,x^0}(y)$ is defined by means of $f[x_k^0 + \exp(u_k + i\theta_k + R\log|y_k|)]$.

If f is replaced by $f_c[\exp(u_k+i\theta_k)] = g_f(u_k)$ or by $f'_c[\exp(u_k+i\theta_k)] = g'_f(u_k)$, the limit exists, by Proposition 3, for *every* $u=(u_k)\in \mathbf{R}^n_-$:

$$\Psi_{f,0}(y) = \lim_{R\to+\infty} R^{-1} g\bigl(u_k + R\log|y_k|\bigr), \tag{20}$$

and does not depend on u.

PROPOSITION 5. – *Let $f\in PSH_-(D)$. Then*

(a) $\Psi_{f,0}(y)\in PSH_-(D)$ *and is* 0*-circled*;

(b) *the convex image $g_\psi(u)$ in $\mathbf{R}^n_-$ has the conic property $g_\psi(cu) = c\,g_\psi(u)$ for every $c>0$, i.e., $\Psi_{f,0}$ is an indicator*;

(c)

$$\Psi_{f,0}(y) \geqslant f'_c(y) \geqslant f(y),\ \forall y\in D; \tag{21}$$

(d) *the mapping $f\mapsto \Psi_{f,0}$ is a projection, $\Psi_{f,0}(y)$ is its own indicator at the origin*;

(e) *the indicator $\Psi_{f,0}$ is the least indicator majorizing f on D*;

(f) *if $f_j(x_j)$ is the restriction of f to the complex subspace $\{x_s=0,\ \forall s\neq j\}$ and*

$$f_j(x_j)\not\equiv -\infty, \tag{22}$$

then $\Psi_{f,0}(y)\geqslant \nu_j\log|y_j|$, ν_j being the Lelong number of $f_j(x_j)$ at the origin;

(g) *if* (22) *holds for each j, then the Monge–Ampère operator $(dd^c\Psi_{f,0})^n$ is well defined on the whole polydisk D and*

$$\bigl(dd^c\Psi_{f,0}\bigr)^n = 0 \tag{23}$$

on $D\setminus\{0\}$.

Proof. – Statement (a) follows from (20), $g(R\log|y_k|)$ being a convex negative function for $R>0$, and the limit of the quotient is obtained by increasing negative values. When setting $v_k = \log|y_k| = -a_k$, the image of $\Psi_{f,0}$ belongs to *Conv*$(\mathbf{R}^n_-)$ and $\Psi_{f,0}(y)$ is a 0-circled plurisubharmonic function.

The property (b), essential for the indicator $\Psi_{f,0}$, results from the equality $n(f,0,ca) = c\,n(f,0,a)$ for all $c>0$.

Relations (c) are a consequence of (20) where $g(u)$ is the convex image $g'_f(u)$ of

$$f'_c(x) = \sup_{\theta_k} f\bigl(x_k e^{i\theta_k}\bigr).$$

We have $g'_f(\log|y_k|)\geqslant f(y)$. On the other hand, the quotient $m(R) = R^{-1}g'_f(R\log|y_k|)$, $R>R_0>1$, is a convex, negative and increasing function of R for $|y_k|<1$. Therefore,

$$\lim_{R\to+\infty} m(R)\geqslant m(1),$$

and by (20),

$$0\geqslant \Psi_{f,0}(y)\geqslant g'_f\bigl(\log|y_k|\bigr)\geqslant f(y)$$

for $y\in D$.

Statement (d) follows from (20) for $f=\Psi_{f,0}$ and from relation (b).

To prove (e), consider any indicator $\psi(y)\geqslant f(y)$ on D. Then $\Psi_{\psi,0}(y)\geqslant \Psi_{f,0}(y)$, and by (d), $\Psi_{\psi,0}=\psi$, so $\psi(y)\geqslant \Psi_{f,0}(y)$ $\forall y\in D$.

The bound in (f) results from (c) and the maximum principle for plurisubharmonic functions, since (for $j=1$)

$$\Psi_{f,0}(y) \geqslant \sup_{\theta_k} f\left(y_k e^{i\theta_k}\right) \geqslant \sup_{\theta_1} f\left(y_1 e^{i\theta_1}, 0, \dots, 0\right)$$

and for $|y_1| \searrow 0$ the quotient $(\log|y_1|)^{-1} \sup_{\theta_1} f_1(y_1 e^{i\theta_1})$ for the restriction f_1 to the complex subspace $\{x_s = 0,\ \forall s > 1\}$, decreases to ν_1.

Finally, in the assumptions of (g), the function $\Psi_{f,0}(y)$ is locally bounded on $D \setminus \{0\}$ by (f), so the operator $(dd^c \Psi_{f,0})^n$ is well defined on D. Equation (23) is valid on the domain $D \setminus \{y\colon\ y_1 y_2 \dots y_n = 0\}$ by Proposition 4 and then on $D \setminus \{0\}$, because the Monge–Ampère measure of a bounded plurisubharmonic function has zero mass on any pluripolar set (see [2]). □

Remark. – Statement (d) of Proposition 5 is, in other words, that all the directional numbers $\nu(dd^c \Psi_{f,0}, \varphi_a)$ of $\Psi_{f,0}$ coincide with the directional numbers $\nu(dd^c f, \varphi_a)$ of the original function f, $\forall a \in \mathbf{R}^n_+$.

The above construction is in fact of local character and Proposition 5 remains valid for the indicator Ψ_{f,x^0} of any function $f(x)$ plurisubharmonic in a neighbourhood ω of a point $x^0 \in \mathbf{C}^n$, with the the following change in the statement (c): (21) should be replaced by

$$\Psi_{f,x^0}\left(x - x^0\right) \geqslant f(x) + b\colon\ y \in D\left(x^0, r\right)\big\} \quad \forall x \in D\left(x^0, r\right), \tag{24}$$

where $D(x^0, r) = \{x\colon\ |x_k - x_k^0| < r,\ 1 \leqslant k \leqslant n\}$; b is a constant depending on f and r; $r > 0$ and is such that the polydisk $D(x^0, r) \Subset \omega$. And of course the restriction f_j in (22) should be taken to the subspaces $\{x_s = x_s^0,\ \forall s \neq j\}$.

Let now $f(x) \in PSH(\omega)$ be locally bounded on $\omega \setminus \{x^0\}$. Then its indicator Ψ_{f,x^0} satisfies the equation:

$$\left(dd^c \Psi_{f,x^0}\right)^n = \tau_f\left(x^0\right) \delta(0) \tag{25}$$

with some number $\tau_f(x^0) \geqslant 0$, $\delta(0)$ the Dirac measure at 0, and $\tau_f(x^0) > 0$ if and only if the Lelong number of the function f at x^0 is strictly positive. And now we relate this value to $(dd^c f)^n$.

THEOREM 1. – *Let $f \in PSH(\omega)$ be locally bounded out of a point $0 \in \omega$. Then*

$$\left(dd^c f\right)^n \geqslant \tau_f(0)\, \delta(0). \tag{26}$$

Proof. – In view of (24), the function f satisfies

$$\limsup_{x \to 0} \frac{\Psi_{f,0}(x)}{f(x)} \leqslant 1. \tag{27}$$

By the Comparison theorem of Demailly [7, Theorem 5.9] this implies

$$\left(dd^c \Psi_{f,0}\right)^n\big|_{\{0\}} \leqslant \left(dd^c f\right)^n\big|_{\{0\}}.$$

On the other hand,

$$\left(dd^c \Psi_{f,0}\right)^n\big|_{\{0\}} = \left(dd^c \Psi_{f,0}\right)^n = \tau_f(0)\delta(0)$$

by (25), that gives us (26) and the theorem is proved. □

Remark. – It is well known that for any plurisubharmonic function v with isolated singularity at 0, there is the relation

$$(dd^c v)^n \geqslant \left[\nu\left(dd^c v, 0\right)\right]^n \delta(0). \tag{28}$$

By the Remark after the proof of Proposition 5, $\nu(dd^c f, 0)$ is equal to $\nu(dd^c \Psi_{f,0}, 0)$. Applying (28) to $v = \Psi_{f,0}$ we get, in view of Theorem 1,

$$\left(dd^c f\right)^n \geqslant \left(dd^c \Psi_{f,0}\right)^n \geqslant \left[\nu\left(dd^c \Psi_{f,0}, 0\right)\right]^n = \left[\nu\left(dd^c f, 0\right)\right]^n,$$

so (26) is an improvement of (28).

For example, if $f(x) = \log(|x_1|^{k_1} + |x_2|^{k_2})$ with $0 < k_1 < k_2$, then

$$\left(dd^c f\right)^2 = \tau_f(0)\,\delta(0) = k_1 k_2\,\delta(0) > k_1^2\,\delta(0) = \left[\nu\left(dd^c f, 0\right)\right]^2 \delta(0),$$

and thus $\tau_f(0) > [\nu(dd^c f, 0)]^2$.

More generally, if F is a holomorphic mapping to $\mathbf{C}^n$ with an isolated zero at 0 of multiplicity μ_0, and $f = \log|F|$, then

$$\left[\nu\left(dd^c f, 0\right)\right]^n \leqslant \tau_f(0) \leqslant \mu_0.$$

In fact, relation (27) makes it possible to obtain extra bounds for $(dd^c f)^n$ in case of $\exp f \in C(\Omega)$. Such a function f can be then considered as a plurisubharmonic weight φ for Demailly's generalized numbers $\nu(T, \varphi)$ of a closed positive current T of bidimension (p, p), $1 \leqslant p \leqslant n-1$ [7]:

$$\nu(T, \varphi) = \lim_{s \to -\infty} \int_{\{\varphi < s\}} T \wedge \left(dd^c \varphi\right)^p = T \wedge \left(dd^c \varphi\right)^p\big|_{\{0\}}.$$

Moreover, the function $\Psi_{f,0}$ is such a weight, too. By Comparison theorem from [7], Theorem 5.1, relation (27) implies

$$\nu(T, \Psi_{f,0}) \leqslant \nu(T, f). \tag{29}$$

Take

$$T_k = \left(dd^c f\right)^k \wedge \left(dd^c \Psi_{f,0}\right)^{n-k-1}, \quad 1 \leqslant k \leqslant n-1.$$

These currents are well defined on a neighbourhood of 0 and are of bidimension $(1, 1)$. Applying (29) to $T = T_k$ we obtain

$$T_k \wedge dd^c f|_{\{0\}} \geqslant T_k \wedge dd^c \Psi_{f,0}|_{\{0\}},$$

that gives us

PROPOSITION 6. – *Let $f \in PSH_-(\Omega)$ be locally bounded out of* $\{0\}$ *and* $\exp f \in C(\Omega)$. *Then*

$$\begin{aligned}\left(dd^c f\right)^n\big|_{\{0\}} &\geqslant \left(dd^c f\right)^{n-1} \wedge dd^c \Psi_{f,0}\big|_{\{0\}} \geqslant \cdots \\ &\geqslant \left(dd^c f\right)^{n-k} \wedge \left(dd^c \Psi_{f,0}\right)^k\big|_{\{0\}} \geqslant \cdots \geqslant \left(dd^c \Psi_{f,0}\right)^n.\end{aligned}$$

4. Dirichlet problem with local indicators

Let Ω be a bounded pseudoconvex domain in $\mathbf{C}^n$ and K be a compact subset of Ω. By $PSH(\Omega, K)$ we denote the class of plurisubharmonic functions on Ω that are locally bounded on $\Omega \setminus K$.

Let $K=\{x^1,\dots,x^N\}\subset\Omega$ and $\{\Psi_m\}$ be N indicators, i.e., circled functions in $PSH_-(D)$ whose convex images satisfy (18). In the sequel we assume that $\Psi_m\in PSH(D,\{0\})$. Then by Proposition 4,

$$(dd^c\Psi_m)^n=\tau_m\delta(0),\quad 1\leqslant m\leqslant N. \tag{30}$$

Let us fix the system $\Phi=\{(x^1,\Psi_1),\dots,(x^N,\Psi_N)\}$ and consider a positive measure T_Φ on Ω, defined as

$$T_\Phi=\sum_{1\leqslant m\leqslant N}\tau_m\delta(x^m). \tag{31}$$

Each function Ψ_m can be extended from a neighbourhood of the origin to a function $\tilde\Psi_m\in PSH(\mathbf{C}^n,\{0\})$, and the indicators of the functions

$$\tilde\Psi(x)=\sum_m\tilde\Psi_m(x-x^m)+A, \tag{32}$$

at x^m are equal to Ψ_m, $1\leqslant m\leqslant N$, for any real number A. So the class

$$N_{\Phi,\Omega}=\{v\in PSH_-(\Omega,K)\colon\ \Psi_{v,x^m}\leqslant\Psi_m,\ 1\leqslant m\leqslant N\} \tag{33}$$

is not empty.

Theorem 1 implies:

THEOREM 2. – $(dd^cf)^n\geqslant T_\Phi\quad\forall f\in N_{\Phi,\Omega}$.

Now we introduce the function

$$G_{\Phi,\Omega}(x)=\sup\{v(x)\colon\ v\in N_{\Phi,\Omega}\}. \tag{34}$$

THEOREM 3. – *Let Ω be a hyperconvex domain in $\mathbf{C}^n$. Then the function $G=G_{\Phi,\Omega}$ has the following properties*:

(a) $G\in PSH_-(\Omega,K)$;
(b) $G(x)\to 0$ *as* $x\to\partial\Omega$;
(c) $\Psi_{G,x^m}=\Psi_m$, $1\leqslant m\leqslant N$;
(d) $(dd^cG)^n=T_\Phi$, *the measure* T_Φ *being defined by* (31);
(e) $G\in C(\overline{\Omega}\setminus K)$.

Remark. – In the case where $\Psi_m=\nu_m\log|x|$, the function $G_{\Phi,\Omega}$, the pluricomplex Green function with several weighted poles, was introduced in [17]. A situation with infinite number of poles was considered in [20], by A. Zeriahi, where a function $G_{f,\Omega}$ was introduced as the upper envelope of the class $\{v\in PSH_-(\Omega,K)\colon\ \nu(dd^cv,x)\geqslant\nu(dd^cf,x),\ \forall x\}$, f being a plurisubharmonic function with the following properties: e^f is continuous, $f^{-1}(-\infty)$ is a compact subset of Ω, and the set $\{x\colon\ \nu(dd^cf,x)>0\}$ is dense in $f^{-1}(-\infty)$. Our proof is much the same as of the corresponding statements of [20].

Proof of Theorem 3. – Since $N_{\Phi,\Omega}\neq\emptyset$, the function $G=G_{\Phi,\Omega}$ is well defined and

$$G^*=\limsup_{y\to x}G(y)\in PSH_-(\Omega,K).$$

The function $\tilde\Psi$ in (32) can be modified in a standard way to $\tilde\Psi'\in PSH_-(\Omega,K)$ such that $\tilde\Psi'(x)=\alpha\,\rho(x)$ in a neighbourhood of $\partial\Omega$, α being a positive number and ρ a bounded exhaustion function on Ω, and $\tilde{\Psi}'(x) = \tilde{\Psi}(x) - \beta$ on a neighbourhood of K. It shows us that

$$G^* \geqslant \tilde{\Psi}'. \tag{35}$$

It implies, in particular, that

$$\Psi_{G^*,x^m} \geqslant \Psi_{\tilde{\Psi}',x^m} = \Psi_m, \quad 1 \leqslant m \leqslant N. \tag{36}$$

Since $\Psi_{\sup\{v,w\},x} \leqslant \sup\{\Psi_{v,x}, \Psi_{w,x}\}$ for any plurisubharmonic functions v and w, there exists an increasing sequence of functions $v_j \in N_{\Phi,\Omega}$ such that $\lim_{j\to\infty} v_j = v \leqslant G$ and $v^* = G^*$.

The indicator of v_j at x^m is the limit of $R^{-1} g_{v_j,x^m}(R\log|y_k|)$ for $R \to +\infty$, the function $g_{v_j,x^m}(u)$ being the convex image of the mean value of $v_j(x_k^m + e^{u_k+i\theta_k})$ with respect to θ_k for $u_k < \log\operatorname{dist}(x^m, \partial\Omega)$, $1 \leqslant k \leqslant n$, and the limit is obtained by the increasing values. It gives us

$$R^{-1} g_{v_j,x^m}\big(R\log|y_k|\big) \leqslant \Psi_{v_j,x^m}(y) \leqslant \Psi_m(y). \tag{37}$$

The functions v_j increase to G^* out of a pluripolar set $X = \{x \in \Omega\colon\ v(x) < v^*(x)\}$. Since the restriction of X to the distinguished boundary of any polydisk is of zero Lebesgue measure [15], (37) implies that

$$R^{-1} g_{G^*,x^m}\big(R\log|y_k|\big) \leqslant \Psi_m(y)$$

and thus, taking $R \to +\infty$,

$$\Psi_{G^*,x^m} \leqslant \Psi_m, \quad 1 \leqslant m \leqslant N. \tag{38}$$

As $G^* \in PSH_-(\Omega)$, the function G^* belongs to the class $N_{\Phi,\Omega}$ and so

$$G^* \equiv G. \tag{39}$$

By (36) and (38), $\Psi_{G,x^m} = \Psi_m$. It proves statements (a) and (c); statement (b) follows from inequality (35).

Continuity of G can be proved exactly as in [20].

To prove (d), observe that in view of (35) and (39), $\tilde{\Psi}' \leqslant G$. By the Comparison theorem of Demailly [5, Theorem 2.1], this implies

$$\big(dd^cG\big)^n\big|_{\{x^m\}} \leqslant \big(dd^c\tilde{\Psi}'\big)^n\big|_{\{x^m\}} \leqslant \big[dd^c\Psi_m\big(x - x^m\big)\big]^n, \quad 1 \leqslant m \leqslant N,$$

and therefore

$$\big(dd^cG\big)^n\big|_K \leqslant T_\Phi. \tag{40}$$

On the other hand, by Theorem 2,

$$\big(dd^cG\big)^n \geqslant T_\Phi.$$

Being comparing to (40) this provides

$$\big(dd^cG\big)^n\big|_K = T_\Phi.$$

Finally, the equality $(dd^cG)^n = 0$ on $\Omega \setminus K$ can be proved in a standard way by showing it is maximal on $\Omega \setminus K$ (see [1,6]), that proves (d).

The theorem is proved. □

As a consequence, we get an “indicator” variant of the Schwarz type lemma (see [17,20]):

THEOREM 4. – *Let the indicator of a function* $g \in PSH(\Omega)$ *at* x^m *does not exceed* Ψ_m, $1 \leqslant m \leqslant N$, *and let* $g(x) \leqslant M$ *on* Ω. *Then* $g(x) \leqslant M + G_{\Phi,\Omega}(x)$, $\forall x \in \Omega$.

Now we are going to show that the function $G_{\Phi,\Omega}$ is the unique plurisubharmonic function with the properties (a)–(d) of Theorem 3. It is known that for unbounded plurisubharmonic functions u, the Dirichlet problem

$$\begin{cases} (dd^c v)^n = \mu \geqslant 0 & \text{on } \Omega, \\ v = h & \text{on } \partial\Omega, \end{cases} \tag{41}$$

need not have a unique solution even in a simple case $\mu = \delta(0)$, $h \equiv 0$. However, a solution is unique under some regularity assumptions on the functions v. For example, as was established in [20], (41) has a unique solution for

$$\mu = \sum \left[\nu\left(dd^c f, x^m\right)\right]^n \delta\left(x^m\right), \tag{42}$$

with $f(x)$ specified in the remark after the statement of Theorem 3, if the functions $v(x) \in PSH_-(\Omega, K)$ have to satisfy

$$\nu\left(dd^c v, x^m\right) = \nu\left(dd^c f, x^m\right), \quad 1 \leqslant m \leqslant N. \tag{43}$$

These additional relations mean that

$$v(x) \sim \nu\left(dd^c v, x^m\right) \log\left[x - x^m\right] \quad \text{near } x^m \tag{44}$$

(v has regular densities at its poles, in the terminology of [17]).

In our situation,

$$\mu = T_f = \sum_m \tau_m \delta\left(x^m\right), \tag{45}$$

where τ_m are defined by (30) with $\Psi_m = \Psi_{f,x^m}$, and we are going to replace condition (43) by $\Psi_{v,x^m} = \Psi_{f,x^m}$, $1 \leqslant m \leqslant N$.

To prove the uniqueness, we need a variant of the comparison theorem for unbounded plurisubharmonic functions (see [13,3,6,12,20] for different classes of plurisubharmonic functions).

THEOREM 5. – *Let* $f \in PSH(\Omega, K)$, $K = \{x^1, \dots, x^m\}$, *and*

$$\left(dd^c f\right)^n\big|_K = T_f, \tag{46}$$

the measure T_f *being given by (45). Let* $v \in PSH(\Omega, K)$ *satisfy the conditions*

(1) $\liminf_{x\to\partial\Omega}(f(x) - v(x)) \geqslant 0$;

(2) $(dd^c v)^n \geqslant (dd^c f)^n$ *on* $\Omega \setminus K$;

(3) $\Psi_{v,x^m} \leqslant \Psi_{f,x^m}$, $1 \leqslant m \leqslant N$.

Then $v \leqslant f$ *on* Ω.

The proof is just as of Theorem 3.3 of [20], and we omit it here.

COROLLARY. – *Under the conditions of Theorem* 3, *the function* $G_{\Phi,\Omega}$ *is the unique plurisubharmonic function with the properties* (a)–(d) *of that theorem.*

Remarks. – 1. Condition (46) is essential. Indeed, let

$$f(x) = \frac{1}{2}\log\left(|x_1|^4 + \left|x_1 + x_2^2\right|^2\right), \qquad v(x) = \frac{1}{2}\log\left(|x_1|^2 + |x_2|^4\right) + m$$

with $m = \inf\{f(x)\colon |x| = 1\} > -\infty$. Then $v(x) \leqslant f(x)$ for $|x| = 1$,

$$\left(dd^c f\right)^2 = \left(dd^c v\right)^2 = 0 \text{ on } \left\{0 < |x| < 1\right\},$$

and $\Psi_{v,0}(x) = \Psi_{f,0}(x) = \log \max\{|x_1|, |x_2|^2\}$. However, for $x_2 = t \in (0, e^m)$, $x_1 = -x_2^2$, $f(x) = 2\log t < \log t + m < v(x)$. The reason here is that $(dd^c f)^2 = 4\delta(0) > 2\delta(0) = T_f$.

2. By Comparison theorem of Demailly [5], relation (46) means that

$$f(x) \sim \Psi_{f,x^m}\left(x - x^m\right) \quad \text{near } x^m,$$

a weaker than (44) but still controlled regularity.

3. By Proposition 5, an indicator Ψ possesses the properties (a)–(d) of $G_{\Phi,\Omega}$ from Theorem 3 with $\Omega = D$, the unit polydisk, and $\Phi = (\{0\}, \Psi)$. Therefore, $\Psi = G_{D,\Phi}$.

Theorems 3 and 5 allow us also to state the following result.

THEOREM 6. – *Let Ω be a bounded strictly pseudoconvex domain, $K = \{x^1, \ldots, x^m\}$, and let a function $f \in PSH(\Omega, K)$ satisfy*

$$\left(dd^c f\right)^n = T_f.$$

Then the Dirichlet problem:

$$\begin{cases} (dd^c v)^n = T_f & \text{on } \Omega, \\ \Psi_{v,x^m} = \Psi_{f,x^m} & \text{for } 1 \leqslant m \leqslant N, \\ v = h & \text{on } \partial\Omega, \end{cases}$$

has a unique solution in the class $PSH(\Omega, K)$ for each function $h \in C(\partial\Omega)$. This solution is continuous on $\overline{\Omega} \setminus K$.

Proof. – Let $\Phi = \{(x^m, \Psi_m)\}$ with $\Psi_m = \Psi_{f,x^m}$. Consider the class

$$N_{f,h} = \left\{v \in PSH(\Omega, K)\colon \Psi_{v,x^m} \leqslant \Psi_{f,x^m}\ \forall m,\ \lim_{x \to y} v(x) = h(y)\ \forall y \in \partial\Omega\right\}.$$

Let $u_0(x)$ be the unique solution of the corresponding homogeneous problem

$$\begin{cases} (dd^c u)^n = 0 & \text{on } \Omega, \\ u = h & \text{on } \partial\Omega. \end{cases}$$

Then $u_0 + G_{\Phi,\Omega} \in N_{f,h}$, so $N_{f,h} \neq \emptyset$.

The desired solution v_0 is given as

$$v_0(x) = \sup\left\{v(x)\colon v \in N_{f,h}\right\}.$$

Just as in the proof of Theorem 3, one can show that v_0 does solve the problem and is continuous on $\overline{\Omega} \setminus K$. The uniqueness follows from Theorem 5. □

Remark. – Theorem 6 can be related to the following question wich was one of the motivations of the present study. Let $F\colon \overline{\Omega} \to \mathbf{C}^n$ be a holomorphic mapping with isolated zeros $\{x^m\} \subset \Omega$

of multiplicities μ_m. Then the function $f(x) = \log|F(x)|$ solves the Dirichlet problem

$$\begin{cases} (dd^c v)^n = \sum_m \mu_m \delta(x^m) & \text{on } \Omega, \\ v = f & \text{on } \partial\Omega. \end{cases}$$

Under what extra conditions on v, the function f is the unique solution of the problem? By Theorem 6, if f has regular behaviour at x^m with respect to its indicators, i.e., if

$$\left(dd^c \Psi_{f,x^m}\right)^n = \mu_m \delta\left(x^m\right), \quad 1 \leqslant m \leqslant N,$$

it gives the unique solution for the problem

$$\begin{cases} (dd^c v)^n = \sum \mu_m \delta(x^m) & \text{on } \Omega, \\ \Psi_{v,x^m} = \Psi_{f,x^m} & 1 \leqslant m \leqslant N, \\ v = f & \text{on } \partial\Omega. \end{cases}$$

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