New fermionic formula
for unrestricted Kostka polynomials ☆

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Abstract

A new fermionic formula for the unrestricted Kostka polynomials of type $A_{n-1}^{(1)}$ is presented. This formula is different from the one given by Hatayama et al. and is valid for all crystal paths based on Kirillov–Reshetikhin modules, not just for the symmetric and antisymmetric case. The fermionic formula can be interpreted in terms of a new set of unrestricted rigged configurations. For the proof a statistics preserving bijection from this new set of unrestricted rigged configurations to the set of unrestricted crystal paths is given which generalizes a bijection of Kirillov and Reshetikhin.

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1. Introduction

The Kostka numbers $K_{\lambda\mu}$, indexed by the two partitions λ and μ, play an important role in symmetric function theory, representation theory, combinatorics, invariant theory and mathematical physics. The Kostka polynomials $K_{\lambda\mu}(q)$ are q-analogs of the Kostka numbers. There are several combinatorial representations of the Kostka polynomials. For example, Lascoux and Schützenberger [17] proved that the Kostka polynomials are generating functions of semistandard tableaux of shape λ and content μ with charge statistic. In [19] the Kostka polynomials are expressed as generating function over highest-weight crystal paths with energy statistics. Crystal paths are elements in tensor products of finite-dimensional crystals. Dropping

the highest-weight condition yields unrestricted Kostka polynomials [6–8,26]. In the $A_1^{(1)}$ setting, unrestricted Kostka polynomials or q-supernomial coefficients were introduced in [25] as q-analogs of the coefficient of x^a in the expansion of $\prod_{j=1}^{N}(1+x+x^2+\cdots+x^j)^{L_j}$. An explicit formula for the $A_{n-1}^{(1)}$ unrestricted Kostka polynomials for completely symmetric and completely antisymmetric crystals was proved in [7,11]. This formula is called fermionic as it is a manifestly positive expression.

In this paper we give a new explicit fermionic formula for the unrestricted Kostka polynomials for all Kirillov–Reshetikhin crystals of type $A_{n-1}^{(1)}$. This fermionic formula can be naturally interpreted in terms of a new set of unrestricted rigged configurations for type $A_{n-1}^{(1)}$. Rigged configurations are combinatorial objects originating from the Bethe Ansatz, that label solutions of the Bethe equations. The simplest version of rigged configurations appeared in Bethe's original paper [3] and was later generalized by Kerov, Kirillov and Reshetikhin [12,13] to models with GL(n) symmetry. Since the solutions of the Bethe equations label highest weight vectors, one expects a bijection between rigged configurations and semi-standard Young tableaux in the GL(n) case. Such a bijection was given in [13,14]. Here we extend this bijection to a bijection Φ between the new set of unrestricted rigged configurations and unrestricted paths. The definition of Φ is given by an algorithm which recursively builds a path from a given rigged configuration by extending the construction of [13,14]. In [22] the bijection is established in a different manner by constructing a crystal structure on the set of rigged configurations. Using that a bijection exists for highest weight elements by [13,14] the equivalent crystal structure forces the bijection for all unrestricted elements. To prove that Φ preserves the statistics, we use the analogous result for highest weight vectors [14] and show that the crystal structures of [22] are compatible under the algorithmically defined Φ.

Recently, fermionic expressions for generating functions of unrestricted paths for type $A_1^{(1)}$ have also surfaced in connection with box-ball systems. Takagi [28] establishes a bijection between box-ball systems and a new set of rigged configurations to prove a fermionic formula for the q-binomial coefficient. His set of rigged configurations coincides with our set in the type $A_1^{(1)}$ case. There is a generalization of Takagi's bijection to type $A_{n-1}^{(1)}$ case [16]. Hence this generalization gives a box-ball interpretation of the unrestricted rigged configurations.

One of the motivations to seek an explicit expression for unrestricted Kostka polynomials is their appearance in generalizations of the Bailey lemma [2]. Bailey's lemma is a very powerful method to prove Rogers–Ramanujan-type identities. In [26] a type A_n generalization of Bailey's lemma was conjectured which was subsequently proven in [29]. A type A_2 Bailey chain, which yields an infinite family of identities, was given in [1]. The new fermionic formulas of this paper might trigger further progress towards generalizations of the Bailey lemma.

The bijection Φ has been implemented as a C++ program [4] and has been incorporated into the combinatorics package of MuPAD-Combinat by Francois Descouens [18].

This paper is structured as follows. In Section 2 we review crystals of type $A_{n-1}^{(1)}$, unrestricted paths and the definition of unrestricted Kostka polynomials as generating functions of unrestricted paths with energy statistics. In Section 3 we give our new definition of unrestricted rigged configurations (see Definition 3.3) and derive from this a fermionic expression for the generating function of unrestricted rigged configurations graded by cocharge (see Section 3.2). The statistic preserving bijection between unrestricted paths and unrestricted rigged configurations is established in Section 4 (see Definition 4.6 and Theorem 4.1). As a corollary this yields the equality of the unrestricted Kostka polynomials and the fermionic formula of Section 3 (see Corollary 4.2). The result that the crystal structures on paths and rigged configurations are compatible under Φ

is stated in Theorem 4.13. Most of the technical proofs are relegated to three appendices. An extended abstract of this paper can be found in [5].

2. Unrestricted paths and Kostka polynomials

2.1. Crystals $B^{r,s}$ of type $A_{n-1}^{(1)}$

Kashiwara [9] introduced the notion of crystals and crystal graphs as a combinatorial means to study representations of quantum algebras associated with any symmetrizable Kac–Moody algebra. In this paper we only consider the Kirillov–Reshetikhin crystal $B^{r,s}$ of type $A_{n-1}^{(1)}$ and hence restrict to this case here.

As a set, the crystal $B^{r,s}$ consists of all column-strict Young tableaux of shape (s^r) over the alphabet $\{1, 2, \ldots, n\}$. As a crystal associated to the underlying algebra of finite type A_{n-1}, $B^{r,s}$ is isomorphic to the highest weight crystal with highest weight (s^r). We will define the classical crystal operators explicitly here. The affine crystal operators e_0 and f_0 are given explicitly in [27]. Since we do not use these operators in this paper we will omit the details.

Let $I = \{1, 2, \ldots, n-1\}$ be the index set for the vertices of the Dynkin diagram of type A_{n-1}, P the weight lattice, $\{\Lambda_i \in P \mid i \in I\}$ the fundamental roots, $\{\alpha_i \in P \mid i \in I\}$ the simple roots, and $\{h_i \in \mathrm{Hom}_{\mathbb{Z}}(P, \mathbb{Z}) \mid i \in I\}$ the simple coroots. As a type A_{n-1} crystal, $B = B^{r,s}$ is equipped with maps $e_i, f_i : B \to B \cup \{0\}$ and $\mathrm{wt} : B \to P$ for all $i \in I$ satisfying

$$f_i(b) = b' \quad \Leftrightarrow \quad e_i(b') = b \quad \text{if } b, b' \in B,$$
$$\mathrm{wt}(f_i(b)) = \mathrm{wt}(b) - \alpha_i \quad \text{if } f_i(b) \in B,$$
$$\langle h_i, \mathrm{wt}(b) \rangle = \varphi_i(b) - \varepsilon_i(b),$$

where $\langle \cdot, \cdot \rangle$ is the natural pairing. The maps f_i, e_i are known as the Kashiwara operators. Here for $b \in B$,

$$\varepsilon_i(b) = \max\{k \geqslant 0 \mid e_i^k(b) \neq 0\},$$
$$\varphi_i(b) = \max\{k \geqslant 0 \mid f_i^k(b) \neq 0\}.$$

Note that for type A_{n-1}, $P = \mathbb{Z}^n$ and $\alpha_i = \epsilon_i - \epsilon_{i+1}$ where $\{\epsilon_i \mid i \in I\}$ is the standard basis in P. Here $\mathrm{wt}(b) = (\mu_1, \ldots, \mu_n)$ is the weight of b where μ_i counts the number of letters i in b.

Following [10] let us give the action of e_i and f_i for $i \in I$. Let $b \in B^{r,s}$ be a tableau of shape (s^r). The row word of b is defined by $\mathrm{word}(b) = w_r \cdots w_2 w_1$ where w_k is the word obtained by reading the kth row of b from left to right. To find $f_i(b)$ and $e_i(b)$ we only consider the subword consisting of the letters i and $i + 1$ in the word of b. First view each $i + 1$ in the subword as an opening bracket and each i as a closing bracket. Then we ignore each adjacent pair of matched brackets successively. At the end of this process we are left with a subword of the form $i^p(i + 1)^q$. If $p > 0$ (respectively $q > 0$) then $f_i(b)$ (respectively $e_i(b)$) is obtained from b by replacing the unmatched subword $i^p(i + 1)^q$ by $i^{p-1}(i + 1)^{q+1}$ (respectively $i^{p+1}(i + 1)^{q-1}$). If $p = 0$ (respectively $q = 0$) then $f_i(b)$ (respectively $e_i(b)$) is undefined and we write $f_i(b) = 0$ (respectively $e_i(b) = 0$).

A crystal B can be viewed as a directed edge-colored graph whose vertices are the elements of B, with a directed edge from b to b' labeled $i \in I$, if and only if $f_i(b) = b'$. This directed graph is known as the crystal graph.

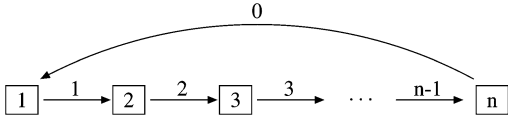**Example 2.1.** The crystal graph for $B = B^{1,1}$ is given in Fig. 1.

Fig. 1. Crystal $B^{1,1}$.

Given two crystals B and B', we can also define a new crystal by taking the tensor product $B \otimes B'$. As a set $B \otimes B'$ is just the Cartesian product of the sets B and B'. The weight function wt for $b \otimes b' \in B \otimes B'$ is $\mathrm{wt}(b \otimes b') = \mathrm{wt}(b) + \mathrm{wt}(b')$ and the Kashiwara operators e_i, f_i are defined as follows:

$$e_i(b \otimes b') = \begin{cases} e_i b \otimes b' & \text{if } \varepsilon_i(b) > \varphi_i(b'), \\ b \otimes e_i b' & \text{otherwise,} \end{cases}$$

$$f_i(b \otimes b') = \begin{cases} f_i b \otimes b' & \text{if } \varepsilon_i(b) \geqslant \varphi_i(b'), \\ b \otimes f_i b' & \text{otherwise.} \end{cases}$$

This action of f_i and e_i on the tensor product is compatible with the previously defined action on $\mathrm{word}(b \otimes b') = \mathrm{word}(b)\mathrm{word}(b')$, but is different from Kashiwara's convention [9].

Example 2.2. Let $i = 2$ and

$$b = \boxed{\begin{array}{cc} 1 & 2 \\ 2 & 3 \end{array}} \otimes \boxed{\begin{array}{cc} 2 & 3 \\ 3 & 4 \\ 4 & 5 \end{array}}.$$

Then $\mathrm{word}(b) = 2312453423$, the relevant subword is $23 - 2 - -3 - 23$, and the unmatched subword is $2 - - - - - - - - 3$. Hence

$$f_2(b) = \boxed{\begin{array}{cc} 1 & 2 \\ 3 & 3 \end{array}} \otimes \boxed{\begin{array}{cc} 2 & 3 \\ 3 & 4 \\ 4 & 5 \end{array}} \quad \text{and} \quad e_2(b) = \boxed{\begin{array}{cc} 1 & 2 \\ 2 & 3 \end{array}} \otimes \boxed{\begin{array}{cc} 2 & 2 \\ 3 & 4 \\ 4 & 5 \end{array}}.$$

2.2. Unrestricted paths

$A_{n-1}^{(1)}$-unrestricted Kostka polynomials or supernomial coefficients were first introduced in [26] as generating functions of unrestricted paths graded by an energy function. An unrestricted path is an element in the tensor product of crystals $B = B^{r_k, s_k} \otimes B^{r_{k-1}, s_{k-1}} \otimes \cdots \otimes B^{r_1, s_1}$.

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ be an n-tuple of nonnegative integers. The set of *unrestricted paths* is defined as

$$\mathcal{P}(B, \lambda) = \{ b \in B \mid \mathrm{wt}(b) = \lambda \}.$$

Example 2.3. For $B = B^{1,1} \otimes B^{2,2} \otimes B^{3,1}$ of type A_3 and $\lambda = (2, 3, 1, 2)$ the path

$$b = \boxed{2} \otimes \boxed{\begin{array}{cc} 1 & 2 \\ 2 & 4 \end{array}} \otimes \boxed{\begin{array}{c} 1 \\ 3 \\ 4 \end{array}}$$

is in $\mathcal{P}(B, \lambda)$.

There exists a crystal isomorphism $R : B^{r,s} \otimes B^{r',s'} \to B^{r',s'} \otimes B^{r,s}$, called the *combinatorial R-matrix*. Combinatorially it is given as follows. Let $b \in B^{r,s}$ and $b' \in B^{r',s'}$. The product $b \cdot b'$ of two tableaux is defined as the row Schensted insertion of b' into b. Then $R(b \otimes b') = \tilde{b}' \otimes \tilde{b}$ is the unique pair of tableaux such that $b \cdot b' = \tilde{b}' \cdot \tilde{b}$.

The *local energy function* $H : B^{r,s} \otimes B^{r',s'} \to \mathbb{Z}$ is defined as follows. For $b \otimes b' \in B^{r,s} \otimes B^{r',s'}$, $H(b \otimes b')$ is the number of boxes of the shape of $b \cdot b'$ outside the shape obtained by concatenating (s^r) and $(s'^{r'})$.

Example 2.4. For

$$b \otimes b' = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & 4 \\ \hline \end{array} \otimes \begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline 4 \\ \hline \end{array}$$

we have

$$b \cdot b' = \begin{array}{|c|c|c|}\hline 1 & 1 & 3 \\ \hline 2 & 2 & 4 \\ \hline 4 & & \\ \cline{1-1} \end{array} = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 4 \\ \hline \end{array} \cdot \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array} = \tilde{b}' \cdot \tilde{b}$$

so that

$$R(b \otimes b') = \tilde{b}' \otimes \tilde{b} = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 4 \\ \hline \end{array} \otimes \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array}.$$

Since the concatenation of $\begin{array}{|c|c|}\hline & \\ \hline & \\ \hline \end{array}$ and $\begin{array}{|c|}\hline \\ \hline \\ \hline \end{array}$ is $\begin{array}{|c|c|c|}\hline & & \\ \hline & & \\ \cline{1-1} \end{array}$, the local energy function $H(b \otimes b') = 0$.

Now let $B = B^{r_k,s_k} \otimes \cdots \otimes B^{r_1,s_1}$ be a k-fold tensor product of crystals. The *tail energy function* $\overleftarrow{D} : B \to \mathbb{Z}$ is given by

$$\overleftarrow{D}(b) = \sum_{1 \leqslant i < j \leqslant k} H_{j-1} R_{j-2} \cdots R_{i+1} R_i(b),$$

where H_i (respectively R_i) is the local energy function (respectively combinatorial R-matrix) acting on the ith and $(i+1)$th tensor factors of $b \in B$.

Definition 2.5. The *q-supernomial coefficient* or the *unrestricted Kostka polynomial* is the generating function of unrestricted paths graded by the tail energy function

$$X(B, \lambda) = \sum_{b \in \mathcal{P}(B,\lambda)} q^{\overleftarrow{D}(b)}.$$

3. Unrestricted rigged configurations and fermionic formula

Rigged configurations are combinatorial objects invented to label the solutions of the Bethe equations, which give the eigenvalues of the Hamiltonian of the underlying physical model [3]. Motivated by the fact that representation theoretically the eigenvectors and eigenvalues can also be labeled by Young tableaux, Kirillov and Reshetikhin [13] gave a bijection between tableaux and rigged configurations. This result and generalizations thereof were proven in [14].

In terms of crystal base theory, the bijection is between highest weight paths and rigged configurations. The new result of this paper is an extension of this bijection to a bijection between unrestricted paths and a new set of rigged configurations. The new set of unrestricted rigged configurations is defined in this section, whereas the bijection is given in Section 4. In [22], a crystal structure on the new set of unrestricted rigged configurations is given, which provides a different description of the bijection.

3.1. Unrestricted rigged configurations

Let $B = B^{r_k, s_k} \otimes \cdots \otimes B^{r_1, s_1}$ and denote by $L = (L_i^{(a)} \mid (a, i) \in \mathcal{H})$ the multiplicity array of B, where $L_i^{(a)}$ is the multiplicity of $B^{a,i}$ in B. Here $\mathcal{H} = I \times \mathbb{Z}_{>0}$ and $I = \{1, 2, \ldots, n-1\}$ is the index set of the Dynkin diagram A_{n-1}. The sequence of partitions $\nu = \{\nu^{(a)} \mid a \in I\}$ is a (L, λ)-*configuration* if

$$\sum_{(a,i)\in\mathcal{H}} i m_i^{(a)} \alpha_a = \sum_{(a,i)\in\mathcal{H}} i L_i^{(a)} \Lambda_a - \lambda, \tag{3.1}$$

where $m_i^{(a)}$ is the number of parts of length i in partition $\nu^{(a)}$. Note that we do not require λ to be a dominant weight here. The (*quasi-*)*vacancy number* of a configuration is defined as

$$p_i^{(a)} = \sum_{j \geqslant 1} \min(i, j) L_j^{(a)} - \sum_{(b,j)\in\mathcal{H}} (\alpha_a \mid \alpha_b) \min(i, j) m_j^{(b)}.$$

Here $(\cdot \mid \cdot)$ is the normalized invariant form on the weight lattice P such that $(\alpha_i \mid \alpha_j)$ is the Cartan matrix. Let $C(L, \lambda)$ be the set of all (L, λ)-configurations. We call $p_i^{(a)}$ quasivacancy number to indicate that they can actually be negative in our setting. For the rest of the paper we will simply call them vacancy numbers.

When the dependence of $m_i^{(a)}$ and $p_i^{(a)}$ on the configuration ν is crucial, we also write $m_i^{(a)}(\nu)$ and $p_i^{(a)}(\nu)$, respectively.

In the usual setting a rigged configuration (ν, J) consists of a configuration $\nu \in C(L, \lambda)$ together with a double sequence of partitions $J = \{J^{(a,i)} \mid (a, i) \in \mathcal{H}\}$ such that the partition $J^{(a,i)}$ is contained in a $m_i^{(a)} \times p_i^{(a)}$ rectangle. In particular this requires that $p_i^{(a)} \geqslant 0$. For unrestricted paths we need a bigger set, where the lower bound on the parts in $J^{(a,i)}$ can be less than zero.

To define the lower bounds we need the following notation. Define the partition λ^{part} as $\lambda^{\text{part}} = (c_1, c_2, \ldots, c_{n-1})^t$ where $c_k = \lambda_{k+1} + \lambda_{k+2} + \cdots + \lambda_n$ and t denotes the transpose of the partition $(c_1, c_2, \ldots, c_{n-1})$. We also set $c_0 = c_1$. Let $\mathcal{A}(\lambda^{\text{part}})$ be the set of tableaux of shape λ^{part} such that the entries in column k are from the set $\{1, 2, \ldots, c_{k-1}\}$ and are strictly decreasing along each column.

Example 3.1. For $n = 4$ and $\lambda = (0, 1, 1, 1)$, the set $\mathcal{A}(\lambda^{\text{part}})$ consists of the following tableaux:

$$
\begin{array}{ccc}
\boxed{3\,3\,2} & \boxed{3\,3\,2} & \boxed{3\,2\,2} \\
\boxed{2\,2} & \boxed{2\,1} & \boxed{2\,1} \\
\boxed{1} & \boxed{1} & \boxed{1}
\end{array}
\quad
\begin{array}{ccc}
\boxed{3\,3\,1} & \boxed{3\,3\,1} & \boxed{3\,2\,1} \\
\boxed{2\,2} & \boxed{2\,1} & \boxed{2\,1} \\
\boxed{1} & \boxed{1} & \boxed{1}
\end{array}\ .
$$

Note that each $t \in \mathcal{A}(\lambda^{\text{part}})$ is weakly decreasing along each row. This is due to the fact that $t_{j,k} \geqslant c_k - j + 1$ since column k of height c_k is strictly decreasing and $c_k - j + 1 \geqslant t_{j,k+1}$ since the entries in column $k + 1$ are from the set $\{1, 2, \ldots, c_k\}$.

Given $t \in \mathcal{A}(\lambda^{\text{part}})$, we define the *lower bound* as

$$M_i^{(a)}(t) = -\sum_{j=1}^{c_a} \chi(i \geqslant t_{j,a}) + \sum_{j=1}^{c_{a+1}} \chi(i \geqslant t_{j,a+1}),$$

where $t_{j,a}$ denotes the entry in row j and column a of t, and $\chi(S) = 1$ if the statement S is true and $\chi(S) = 0$ otherwise.

Definition 3.2. Let $M, p, m \in \mathbb{Z}$ such that $m \geqslant 0$. A (M, p, m)-*quasipartition* μ is a tuple of integers $\mu = (\mu_1, \mu_2, \ldots, \mu_m)$ such that $M \leqslant \mu_m \leqslant \mu_{m-1} \leqslant \cdots \leqslant \mu_1 \leqslant p$. Each μ_i is called a part of μ. Note that for $M = 0$ this would be a partition with at most m parts each not exceeding p.

Definition 3.3. An *unrestricted rigged configuration* (ν, J) associated to a multiplicity array L and weight λ is a configuration $\nu \in C(L, \lambda)$ together with a sequence $J = \{J^{(a,i)} \mid (a, i) \in \mathcal{H}\}$ where $J^{(a,i)}$ is a $(M_i^{(a)}(t), p_i^{(a)}, m_i^{(a)})$-quasipartition for some $t \in \mathcal{A}(\lambda^{\text{part}})$. Denote the set of all unrestricted rigged configurations corresponding to (L, λ) by $\text{RC}(L, \lambda)$.

Remark 3.4.

(1) Note that this definition is similar to the definition of level-restricted rigged configurations [23, Definition 5.5]. Whereas for level-restricted rigged configurations the vacancy number had to be modified according to tableaux in a certain set, here the lower bounds are modified.

(2) For type A_1 we have $\lambda = (\lambda_1, \lambda_2)$ so that $\mathcal{A} = \{t\}$ contains just the single tableau

$$t = \begin{array}{|c|} \hline \lambda_2 \\ \hline \lambda_2 - 1 \\ \hline \vdots \\ \hline 1 \\ \hline \end{array}.$$

In this case $M_i(t) = -\sum_{j=1}^{\lambda_2} \chi(i \geqslant t_{j,1}) = -i$. This agrees with the findings of [28].

The quasipartition $J^{(a,i)}$ is called *singular* if it has a part of size $p_i^{(a)}$. It is often useful to view an (unrestricted) rigged configuration (ν, J) as a sequence of partitions ν where the parts of size i in $\nu^{(a)}$ are labeled by the parts of $J^{(a,i)}$. The pair (i, x) where i is a part of $\nu^{(a)}$ and x is a part of $J^{(a,i)}$ is called a *string* of the ath rigged partition $(\nu, J)^{(a)}$. The label x is called a *rigging*.

Example 3.5. Let $n = 4$, $\lambda = (2, 2, 1, 1)$, $L_1^{(1)} = 6$ and all other $L_i^{(a)} = 0$. Then

$$(\nu, J) = \begin{array}{c} \square\square\square \\ \square \end{array} \begin{array}{c} -2 \\ 0 \end{array} \quad \square\square \;\; 0 \quad \square \;\; -1$$

is an unrestricted rigged configuration in $\text{RC}(L, \lambda)$, where we have written the parts of $J^{(a,i)}$ next to the parts of length i in partition $\nu^{(a)}$. For example, $\nu^{(1)} = (3, 1)$, $\nu^{(2)} = (2)$, $\nu^{(3)} = (1)$, $J^{(1,1)} = (0)$, $J^{(1,3)} = (-2)$, $J^{(2,2)} = (0)$, and $J^{(3,1)} = (-1)$. To see that the riggings form quasipartitions, let us write the vacancy numbers $p_i^{(a)}$ next to the parts of length i in partition $\nu^{(a)}$:

$$\begin{array}{c} \square\square\square \\ \square \end{array} \begin{array}{c} 0 \\ 3 \end{array} \quad \square\square \;\; 0 \quad \square \;\; -1 \cdot$$

This shows that the labels are indeed all weakly below the vacancy numbers. For

$$
\begin{array}{|c|c|c|}
\hline
4 & 4 & 1 \\
\hline
3 & 3 \\
\cline{1-2}
2 \\
\cline{1-1}
1 \\
\cline{1-1}
\end{array}
\in \mathcal{A}(\lambda^{\mathrm{part}})
$$

we get the lower bounds

$$
\boxed{}\ \ -2 \quad \boxed{}\ 0 \quad \boxed{}\ -1,
$$

which are less or equal to the riggings in (ν, J).

Let $B = B^{r_k, s_k} \otimes \cdots \otimes B^{r_1, s_1}$ and L the corresponding multiplicity array. Let $(\nu, J) \in$ RC(L, λ). Note that rewritting (3.1) we get

$$
\left| \nu^{(a)} \right| = \sum_{j>a} \lambda_j - \sum_{j=1}^{k} s_j \max(r_j - a, 0). \tag{3.2}
$$

Hence for large i, by definition of vacancy numbers we have

$$
p_i^{(a)} = \left| \nu^{(a-1)} \right| - 2 \left| \nu^{(a)} \right| + \left| \nu^{(a+1)} \right| + \sum_j \min(i, j) L_j^{(a)} = \lambda_a - \lambda_{a+1} \tag{3.3}
$$

and

$$
M_i^{(a)}(t) = -\sum_{j=1}^{c_a} \chi(i \geqslant t_{j,a}) + \sum_{j=1}^{c_{a+1}} \chi(i \geqslant t_{j,a+1}) = -c_a + c_{a+1} = -\lambda_{a+1}. \tag{3.4}
$$

For a given $t \in \mathcal{A}(\lambda^{\mathrm{part}})$ define

$$
\Delta p_i^{(a)}(t) = p_i^{(a)} - M_i^{(a)}(t).
$$

We write $\Delta p_i^{(a)}$ for $\Delta p_i^{(a)}(t)$ when there is no cause of confusion. For large i, $\Delta p_i^{(a)}(t) = \lambda_a$.

From the definition of $p_i^{(a)}$ one may easily verify that

$$
-p_{i-1}^{(a)} + 2p_i^{(a)} - p_{i+1}^{(a)} \geqslant m_i^{(a-1)} - 2m_i^{(a)} + m_i^{(a+1)}. \tag{3.5}
$$

Let $t_{\cdot, a}$ denote the ath column of t. Then it follows from the definition of $M_i^{(a)}(t)$ that

$$
M_i^{(a)}(t) = M_{i-1}^{(a)}(t) - \chi(i \in t_{\cdot, a}) + \chi(i \in t_{\cdot, a+1}).
$$

Hence (3.5) can be rewritten as

$$
\begin{aligned}
-\Delta p_{i-1}^{(a)} &+ 2\Delta p_i^{(a)} - \Delta p_{i+1}^{(a)} - \chi(i \in t_{\cdot, a}) + \chi(i \in t_{\cdot, a+1}) \\
&+ \chi(i+1 \in t_{\cdot, a}) - \chi(i+1 \in t_{\cdot, a+1}) \geqslant m_i^{(a-1)} - 2m_i^{(a)} + m_i^{(a+1)}.
\end{aligned} \tag{3.6}
$$

Lemma 3.6. *Suppose that for some $t \in \mathcal{A}(\lambda^{\mathrm{part}})$, $\Delta p_i^{(a)}(t) \geqslant 0$ for all $a \in I$ and i such that $m_i^{(a)} > 0$. Then there exists a $t' \in \mathcal{A}(\lambda^{\mathrm{part}})$ such that $\Delta p_i^{(a)}(t') \geqslant 0$ for all i and a.*

Proof. By definition $\Delta p_0^{(a)}(t) = 0$ and $\Delta p_i^{(a)}(t) = \lambda_a \geqslant 0$ for large i. By (3.6)

$$\Delta p_i^{(a)}(t) \geqslant \frac{1}{2}\big\{\Delta p_{i-1}^{(a)}(t) + \Delta p_{i+1}^{(a)}(t) + \chi(i \in t_{.,a}) - \chi(i \in t_{.,a+1})$$
$$- \chi(i+1 \in t_{.,a}) + \chi(i+1 \in t_{.,a+1}) + m_i^{(a-1)} + m_i^{(a+1)}\big\} \tag{3.7}$$

when $m_i^{(a)} = 0$. Hence $\Delta p_i^{(a)}(t) < 0$ is only possible if $m_i^{(a-1)} = m_i^{(a+1)} = 0$, column a of t contains $i+1$ but no i, and column $a+1$ of t contains i but no $i+1$. Let k be minimal such that $\Delta p_i^{(k)}(t) < 0$. Note that $k > 1$ since the first column of t contains all letters $1, 2, \ldots, c_1$. Let $k' \leqslant k$ be minimal such that $\Delta p_i^{(a)}(t) = 0$ for all $k' \leqslant a < k$. Then inductively for $a = k-1, k-2, \ldots, k'$ it follows from (3.7) that $m_i^{(a-1)} = 0$ and column a of t contains $i+1$ but no i. Construct a new t' from t by replacing all letters $i+1$ in columns $k', k'+1, \ldots, k$ by i. This accomplishes that $\Delta p_j^{(a)}(t') \geqslant 0$ for all j and $1 \leqslant a < k$, $\Delta p_i^{(k)}(t') \geqslant 0$, and $\Delta p_j^{(a)}(t') \geqslant 0$ for all $a \geqslant k$ such that $m_j^{(a)} > 0$. Repeating the above construction, if necessary, eventually yields a new tableau t'' such that finally $\Delta p_j^{(a)}(t'') \geqslant 0$ for all j and a. $\quad\square$

3.2. Fermionic formula

The following statistics can be defined on the set of unrestricted rigged configurations. For $(\nu, J) \in \mathrm{RC}(L, \lambda)$ let

$$cc(\nu, J) = cc(\nu) + \sum_{(a,i) \in \mathcal{H}} \big|J^{(a,i)}\big|,$$

where $|J^{(a,i)}|$ is the sum of all parts of the quasipartition $J^{(a,i)}$ and

$$cc(\nu) = \frac{1}{2} \sum_{a,b \in I} \sum_{j,k \geqslant 1} (\alpha_a \mid \alpha_b) \min(j,k) m_j^{(a)} m_k^{(b)}.$$

Definition 3.7. The RC polynomial is defined as

$$M(L, \lambda) = \sum_{(\nu, J) \in \mathrm{RC}(L, \lambda)} q^{cc(\nu, J)}.$$

The RC polynomial is in fact S_n-symmetric in the weight λ, that is, $M(L, \lambda) = M(L, \sigma(\lambda))$ for $\sigma \in S_n$, where $\sigma(\lambda) = (\lambda_{\sigma(1)}, \lambda_{\sigma(2)}, \ldots, \lambda_{\sigma(n)})$. This is not obvious from its definition as both (3.1) and the lower bounds are not symmetric with respect to λ, but follows from Corollary 4.2 and the fact that $X(B, \lambda)$ is symmetric in λ [7,26].

Let $\mathcal{SA}(\lambda^{\mathrm{part}})$ be the set of all nonempty subsets of $\mathcal{A}(\lambda^{\mathrm{part}})$ and set

$$M_i^{(a)}(S) = \max\big\{M_i^{(a)}(t) \mid t \in S\big\} \quad \text{for } S \in \mathcal{SA}(\lambda^{\mathrm{part}}).$$

By inclusion–exclusion arguments analogous to [23], the set of all allowed riggings for a given $\nu \in \mathrm{C}(L, \lambda)$ is

$$\bigcup_{S \in \mathcal{SA}(\lambda^{\mathrm{part}})} (-1)^{|S|+1}\big\{J \mid J^{(a,i)} \text{ is a } \big(M_i^{(a)}(S), p_i^{(a)}, m_i^{(a)}\big)\text{-quasipartition}\big\}.$$

The q-binomial coefficient $\begin{bmatrix} m+p \\ m \end{bmatrix}$, defined as

$$\begin{bmatrix} m + p \\ m \end{bmatrix} = \frac{(q)_{m+p}}{(q)_m (q)_p}$$

where $(q)_n = (1 - q)(1 - q^2) \cdots (1 - q^n)$, is the generating function of partitions with at most m parts each not exceeding p. Hence the polynomial $M(L, \lambda)$ may be rewritten as

$$M(L, \lambda) = \sum_{S \in \mathcal{S}\mathcal{A}(\lambda^{\mathrm{part}})} (-1)^{|S|+1} \sum_{\nu \in \mathrm{C}(L, \lambda)} q^{cc(\nu) + \sum_{(a,i) \in \mathcal{H}} m_i^{(a)} M_i^{(a)}(S)}$$
$$\times \prod_{(a,i) \in \mathcal{H}} \begin{bmatrix} m_i^{(a)} + p_i^{(a)} - M_i^{(a)}(S) \\ m_i^{(a)} \end{bmatrix}$$

called *fermionic formula*. In [7,11] a fermionic formula for the same polynomial was given in the special case when L is the multiplicity array of $B = B^{1,s_k} \otimes \cdots \otimes B^{1,s_1}$ or $B = B^{r_k,1} \otimes \cdots \otimes B^{r_1,1}$. However, the form of the fermionic formulas are different and a direct link between the rigged configurations of this paper and those of [20] is not yet known.

4. Bijection

In this section we define the bijection $\Phi : \mathcal{P}(B, \lambda) \to \mathrm{RC}(L, \lambda)$ from paths to unrestricted rigged configurations algorithmically. The algorithm generalizes the bijection of [14] to the unrestricted case. The main result is summarized in the following theorem.

Theorem 4.1. *Let* $B = B^{r_k,s_k} \otimes \cdots \otimes B^{r_1,s_1}$, *$L$ the corresponding multiplicity array and* $\lambda = (\lambda_1, \ldots, \lambda_n)$ *a sequence of nonnegative integers. There exists a bijection* $\Phi : \mathcal{P}(B, \lambda) \to$ $\mathrm{RC}(L, \lambda)$ *which preserves the statistics, that is,* $\overleftarrow{D}(b) = cc(\Phi(b))$ *for all* $b \in \mathcal{P}(B, \lambda)$.

A different proof of Theorem 4.1 is given in [22] by proving directly that the crystal structure on rigged configurations and paths coincide. The results in [22] hold for all simply-laced types, not just type $A_{n-1}^{(1)}$. Hence Theorem 4.1 holds whenever there is a corresponding bijection for the highest weight elements (for example, for type $D_n^{(1)}$ for symmetric powers [24] and antisymmetric powers [21]). Using virtual crystals and the method of folding Dynkin diagrams, these results can be extended to other affine root systems. In this paper we use the crystal structure to prove that the statistics is preserved. It follows from Theorem 4.13 that the algorithmic definition for Φ of this paper and the definition of [22] agree.

An immediate corollary of Theorem 4.1 is the relation between the fermionic formula for the RC polynomial of Section 3 and the unrestricted Kostka polynomials of Section 2.

Corollary 4.2. *With the same assumptions as in Theorem* 4.1, $X(B, \lambda) = M(L, \lambda)$.

4.1. Operations on crystals

To define Φ we first need to introduce certain maps on paths and rigged configurations. These maps correspond to the following operations on crystals:

(1) If $B = B^{1,1} \otimes B'$, let $\mathrm{lh}(B) = B'$. This operation is called *left-hat*.
(2) If $B = B^{r,s} \otimes B'$ with $s \geqslant 2$, let $\mathrm{ls}(B) = B^{r,1} \otimes B^{r,s-1} \otimes B'$. This operation is called *left-split*.
(3) If $B = B^{r,1} \otimes B'$ with $r \geqslant 2$, let $\mathrm{lb}(B) = B^{1,1} \otimes B^{r-1,1} \otimes B'$. This operation is called *box-split*.

In analogy we define $\mathrm{lh}(L)$ (respectively $\mathrm{ls}(L)$, $\mathrm{lb}(L)$) to be the multiplicity array of $\mathrm{lh}(B)$ (respectively $\mathrm{ls}(B)$, $\mathrm{lb}(B)$), if L is the multiplicity array of B. The corresponding maps on crystal elements are given by:

(1) Let $b = c \otimes b' \in B^{1,1} \otimes B'$. Then $\mathrm{lh}(b) = b'$.
(2) Let $b = c \otimes b' \in B^{r,s} \otimes B'$, where $c = c_1 c_2 \cdots c_s$ and c_i denotes the ith column of c. Then
$\mathrm{ls}(b) = c_1 \otimes c_2 \cdots c_s \otimes b'$.
(3) Let

$$
b = \boxed{\begin{array}{c} b_1 \\ b_2 \\ \vdots \\ b_r \end{array}} \otimes b' \in B^{r,1} \otimes B',
$$

where $b_1 < \cdots < b_r$. Then

$$
\mathrm{lb}(b) = \boxed{b_r} \otimes \boxed{\begin{array}{c} b_1 \\ \vdots \\ b_{r-1} \end{array}} \otimes b'.
$$

In the next subsection we define the corresponding maps on rigged configurations, and give the bijection in Section 4.3.

4.2. Operations on rigged configurations

Suppose $L_1^{(1)} > 0$. The main algorithm on rigged configurations as defined in [13,14] for admissible rigged configurations can be extended to our setting. For a tuple of nonnegative integers $\lambda = (\lambda_1, \ldots, \lambda_n)$, let λ^- be the set of all nonnegative tuples $\mu = (\mu_1, \ldots, \mu_n)$ such that $\lambda - \mu = \epsilon_r$ for some $1 \leqslant r \leqslant n$ where ϵ_r is the canonical rth unit vector in $\mathbb{Z}^n$. Define $\delta : \mathrm{RC}(L, \lambda) \to \bigcup_{\mu \in \lambda^-} \mathrm{RC}(\mathrm{lh}(L), \mu)$ by the following algorithm. Let $(\nu, J) \in \mathrm{RC}(L, \lambda)$. Set $\ell^{(0)} = 1$ and repeat the following process for $a = 1, 2, \ldots, n-1$ or until stopped. Find the smallest index $i \geqslant \ell^{(a-1)}$ such that $J^{(a,i)}$ is singular. If no such i exists, set $\mathrm{rk}(\nu, J) = a$ and stop. Otherwise set $\ell^{(a)} = i$ and continue with $a + 1$. Set all undefined $\ell^{(a)}$ to ∞.

The new rigged configuration $(\tilde{\nu}, \tilde{J}) = \delta(\nu, J)$ is obtained by removing a box from the selected strings and making the new strings singular again. Explicitly

$$
m_i^{(a)}(\tilde{\nu}) = m_i^{(a)}(\nu) + \begin{cases} 1 & \text{if } i = \ell^{(a)} - 1, \\ -1 & \text{if } i = \ell^{(a)}, \\ 0 & \text{otherwise.} \end{cases}
$$

The partition $\tilde{J}^{(a,i)}$ is obtained from $J^{(a,i)}$ by removing a part of size $p_i^{(a)}(\nu)$ for $i = \ell^{(a)}$, adding a part of size $p_i^{(a)}(\tilde{\nu})$ for $i = \ell^{(a)} - 1$, and leaving it unchanged otherwise. Then $\delta(\nu, J) \in \mathrm{RC}(\mathrm{lh}(L), \mu)$ where $\mu = \lambda - \epsilon_{\mathrm{rk}(\nu, J)}$.

Proposition 4.3. δ *is well defined.*

The proof is given in Appendix A.

Example 4.4. Let L be the multiplicity array of $B = B^{1,1} \otimes B^{2,1} \otimes B^{2,3}$ and $\lambda = (2, 2, 2, 1, 1, 1)$. Then

$$(\nu, J) = \boxed{} \; -1 \quad \boxed{} \; 0 \quad \boxed{} \; 0 \quad \boxed{} \; -1 \quad \boxed{} \; -1 \in \mathrm{RC}(L, \lambda).$$

Writing the vacancy numbers next to each part instead of the riggings we get

$$\boxed{} \; -1 \quad \boxed{} \; 0 \quad \boxed{} \; 1 \quad \boxed{} \; -1 \quad \boxed{} \; -1$$

Hence $\ell^{(1)} = \ell^{(2)} = 1$ and all other $\ell^{(a)} = \infty$, so that

$$\delta(\nu, J) = \boxed{} \; -1 \quad \boxed{} \; 0 \quad \boxed{} \; 0 \quad \boxed{} \; -1 \quad \boxed{} \; -1.$$

Also $cc(\nu, J) = 2$.

The inverse algorithm of δ denoted by δ^{-1} is defined as follows. Let $L_1^{(1)} = \bar{L}_1^{(1)} + 1$, $L_i^{(k)} = \bar{L}_i^{(k)}$ for all $i, k \neq 1$. Let $\bar{\lambda}$ be a weight and $\lambda = \bar{\lambda} + \epsilon_r$ for some $1 \leqslant r \leqslant n$. Define $\delta^{-1} : \mathrm{RC}(\bar{L}, \bar{\lambda}) \to \mathrm{RC}(L, \lambda)$ by the following algorithm. Let $(\bar{\nu}, \bar{J}) \in \mathrm{RC}(\bar{L}, \bar{\lambda})$. Let $s^{(r)} = \infty$. For $k = r - 1$ down to 1, select the longest singular string in $(\bar{\nu}, \bar{J})^{(k)}$ of length $s^{(k)}$ (possibly of zero length) such that $s^{(k)} \leqslant s^{(k+1)}$. With the convention $s^{(0)} = 0$ we have $s^{(0)} \leqslant s^{(1)}$ as well. $\delta^{-1}(\bar{\nu}, \bar{J}) = (\nu, J)$ is obtained from $(\bar{\nu}, \bar{J})$ by adding a box to each of the selected strings, and resetting their labels to make them singular with respect to the new vacancy number for $\mathrm{RC}(L, \lambda)$, and leaving all other strings unchanged.

Proposition 4.5. δ^{-1} *is well defined.*

This proposition will also be proved in Appendix A.

Let $s \geqslant 2$. Suppose $B = B^{r,s} \otimes B'$ and L the corresponding multiplicity array. Note that $\mathrm{C}(L, \lambda) \subset \mathrm{C}(\mathrm{ls}(L), \lambda)$. Under this inclusion map, the vacancy number $p_i^{(a)}$ for ν increases by $\delta_{a,r} \chi(i < s)$. Hence there is a well-defined injective map $\mathrm{ls}_{rc} : \mathrm{RC}(L, \lambda) \to \mathrm{RC}(\mathrm{ls}(L), \lambda)$ given by the identity map $\mathrm{ls}_{rc}(\nu, J) = (\nu, J)$.

Suppose $r \geqslant 2$ and $B = B^{r,1} \otimes B'$ with multiplicity array L. Then there is an injection $\mathrm{lb}_{rc} : \mathrm{RC}(L, \lambda) \to \mathrm{RC}(\mathrm{lb}(L), \lambda)$ defined by adding singular strings of length 1 to $(\nu, J)^{(a)}$ for $1 \leqslant a < r$. Note that the vacancy numbers remain unchanged under lb_{rc}.

4.3. Bijection

The map $\Phi : \mathcal{P}(B, \lambda) \to \mathrm{RC}(L, \lambda)$ is defined recursively by various commutative diagrams. Note that it is possible to go from $B = B^{r_k, s_k} \otimes B^{r_{k-1}, s_{k-1}} \otimes \cdots \otimes B^{r_1, s_1}$ to the empty crystal via successive application of lh, ls and lb.

Definition 4.6. Define that map $\Phi : \mathcal{P}(B, \lambda) \to \mathrm{RC}(L, \lambda)$ such that the empty path maps to the empty rigged configuration and such that the following conditions hold:

(1) Suppose $B = B^{1,1} \otimes B'$. Then the following diagram commutes:

$$
\begin{array}{ccc}
\mathcal{P}(B, \lambda) & \xrightarrow{\ \Phi\ } & \mathrm{RC}(L, \lambda) \\
{\scriptstyle \mathrm{lh}}\downarrow & & \downarrow{\scriptstyle \delta} \\
\displaystyle\bigcup_{\mu \in \lambda^-} \mathcal{P}(\mathrm{lh}(B), \mu) & \xrightarrow[\ \Phi\ ]{} & \displaystyle\bigcup_{\mu \in \lambda^-} \mathrm{RC}(\mathrm{lh}(L), \mu)
\end{array}
$$

(2) Suppose $B = B^{r,s} \otimes B'$ with $s \geqslant 2$. Then the following diagram commutes:

$$
\begin{array}{ccc}
\mathcal{P}(B, \lambda) & \xrightarrow{\ \Phi\ } & \mathrm{RC}(L, \lambda) \\
{\scriptstyle \mathrm{ls}}\downarrow & & \downarrow{\scriptstyle \mathrm{ls}_{rc}} \\
\mathcal{P}(\mathrm{ls}(B), \lambda) & \xrightarrow[\ \Phi\ ]{} & \mathrm{RC}(\mathrm{ls}(L), \lambda)
\end{array}
$$

(3) Suppose $B = B^{r,1} \otimes B'$ with $r \geqslant 2$. Then the following diagram commutes:

$$
\begin{array}{ccc}
\mathcal{P}(B, \lambda) & \xrightarrow{\ \Phi\ } & \mathrm{RC}(L, \lambda) \\
{\scriptstyle \mathrm{lb}}\downarrow & & \downarrow{\scriptstyle \mathrm{lb}_{rc}} \\
\mathcal{P}(\mathrm{lb}(B), \lambda) & \xrightarrow[\ \Phi\ ]{} & \mathrm{RC}(\mathrm{lb}(L), \lambda)
\end{array}
$$

Proposition 4.7. *The map Φ of Definition* 4.6 *is a well-defined bijection.*

The proof is given in Appendix B.

Example 4.8. Let $B = B^{1,1} \otimes B^{2,1} \otimes B^{2,3}$ and $\lambda = (2, 2, 2, 1, 1, 1)$. Then

$$
b = \boxed{3} \otimes \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \end{array} \otimes \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline \end{array} \in \mathcal{P}(B, \lambda)
$$

and $\Phi(b)$ is the rigged configuration (ν, J) of Example 4.4. We have $\overleftarrow{D}(b) = cc(\nu, J) = 2$.

Example 4.9. Let $n = 4$, $B = B^{2,2} \otimes B^{2,1}$ and $\lambda = (2, 2, 1, 1)$. Then the multiplicity array is $L_1^{(2)} = 1, L_2^{(2)} = 1$ and $L_i^{(a)} = 0$ for all other (a, i). There are 7 possible unrestricted paths in $\mathcal{P}(B, \lambda)$. For each path $b \in \mathcal{P}(B, \lambda)$ the corresponding rigged configuration $(\nu, J) = \Phi(b)$ together with the tail energy and cocharge is summarized below:

$$
b = \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & 2 \\ \hline \end{array} \otimes \begin{array}{|c|} \hline 3 \\ \hline 4 \\ \hline \end{array} \qquad (\nu, J) = \square \ 0 \qquad \begin{array}{c} \square \\ \square \end{array} \begin{array}{c} -1 \\ -1 \end{array} \quad \square \ 0 \qquad \overleftarrow{D}(b) = 0 = cc(\nu, J),
$$

$$
b = \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & 4 \\ \hline \end{array} \otimes \begin{array}{|c|} \hline 2 \\ \hline 3 \\ \hline \end{array} \qquad (\nu, J) = \square \ {-1} \qquad \begin{array}{c} \square \\ \square \end{array} \begin{array}{c} 0 \\ 0 \end{array} \quad \square \ 0 \qquad \overleftarrow{D}(b) = 1 = cc(\nu, J),
$$

$$
b = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 & 3 \\ \hline \end{array} \otimes \begin{array}{|c|} \hline 1 \\ \hline 4 \\ \hline \end{array} \qquad (\nu, J) = \square \ 0 \qquad \begin{array}{c} \square \\ \square \end{array} \begin{array}{c} 0 \\ 0 \end{array} \quad \square \ {-1} \qquad \overleftarrow{D}(b) = 1 = cc(\nu, J),
$$

$$
b = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 & 4 \\ \hline \end{array} \otimes \begin{array}{|c|} \hline 1 \\ \hline 3 \\ \hline \end{array} \qquad (\nu, J) = \square \ 0 \qquad \begin{array}{c} \square \\ \square \end{array} \begin{array}{c} 0 \\ -1 \end{array} \quad \square \ 0 \qquad \overleftarrow{D}(b) = 1 = cc(\nu, J),
$$

$$b = \boxed{\begin{array}{cc} 1 & 3 \\ 2 & 4 \end{array}} \otimes \boxed{\begin{array}{c} 1 \\ 2 \end{array}} \qquad (\nu, J) = \square \; 0 \qquad \begin{array}{c} \square \; 0 \\ \; 0 \end{array} \qquad \square \; 0 \qquad \overleftarrow{D}(b) = 2 = cc(\nu, J),$$

$$b = \boxed{\begin{array}{cc} 1 & 1 \\ 2 & 3 \end{array}} \otimes \boxed{\begin{array}{c} 2 \\ 4 \end{array}} \qquad (\nu, J) = \square \; -1 \qquad \boxed{} \; 0 \qquad \square \; -1 \qquad \overleftarrow{D}(b) = 0 = cc(\nu, J),$$

$$b = \boxed{\begin{array}{cc} 1 & 2 \\ 3 & 4 \end{array}} \otimes \boxed{\begin{array}{c} 1 \\ 2 \end{array}} \qquad (\nu, J) = \square \; -1 \qquad \boxed{} \; 1 \qquad \square \; -1 \qquad \overleftarrow{D}(b) = 1 = cc(\nu, J).$$

The unrestricted Kostka polynomial in this case is $M(L, \lambda) = 2 + 4q + q^2 = X(B, \lambda)$.

4.4. Crystal operators on unrestricted rigged configurations

Let $B = B^{r_k, s_s} \otimes \cdots \otimes B^{r_1, s_1}$ and L be the multiplicity array of B. Let $\mathcal{P}(B) = \bigcup_\lambda \mathcal{P}(B, \lambda)$ and $\mathrm{RC}(L) = \bigcup_\lambda \mathrm{RC}(L, \lambda)$. Note that the bijection Φ of Definition 4.6 extends to a bijection from $\mathcal{P}(B)$ to $\mathrm{RC}(L)$. Let f_a and e_a for $1 \leqslant a < n$ be the crystal operators acting on the paths in $\mathcal{P}(B)$. In [22] analogous operators $\tilde{f}_a$ and $\tilde{e}_a$ for $1 \leqslant a < n$ acting on rigged configurations in $\mathrm{RC}(L)$ were defined.

Definition 4.10. (See [22, Definition 3.3].)

(1) Define $\tilde{e}_a(\nu, J)$ by removing a box from a string of length k in $(\nu, J)^{(a)}$ leaving all colabels fixed and increasing the new label by one. Here k is the length of the string with the smallest negative rigging of smallest length. If no such string exists, $\tilde{e}_a(\nu, J)$ is undefined.

(2) Define $\tilde{f}_a(\nu, J)$ by adding a box to a string of length k in $(\nu, J)^{(a)}$ leaving all colabels fixed and decreasing the new label by one. Here k is the length of the string with the smallest nonpositive rigging of largest length. If no such string exists, add a new string of length one and label -1. If the result is not a valid unrestricted rigged configuration $\tilde{f}_a(\nu, J)$ is undefined.

Example 4.11. Let L be the multiplicity array of $B = B^{1,3} \otimes B^{3,2} \otimes B^{2,1}$ and let

$$(\nu, J) = \boxed{} \; -3 \quad \boxed{} \; 0 \quad \boxed{} \; -1 \in \mathrm{RC}(L).$$
$$ \square \; -1 \qquad\qquad \square \; 1 \qquad\qquad \square \; -1$$

Then

$$\tilde{f}_3(\nu, J) = \boxed{} \; -3 \quad \boxed{} \; 1 \quad \boxed{} \; -2$$
$$\phantom{\tilde{f}_3(\nu, J) =} \square \; -1 \qquad\qquad \square \; 1 \qquad\qquad \square \; -1$$

and

$$\tilde{e}_3(\nu, J) = \boxed{} \; -3 \quad \boxed{} \; -1 \quad \boxed{} \; 1 \, .$$
$$\phantom{\tilde{e}_3(\nu, J) =} \square \; -1 \qquad\qquad \square \; 0$$

Define $\tilde{\varphi}_a(\nu, J) = \max\{k \geqslant 0 \mid \tilde{f}_a(\nu, J) \neq 0\}$ and $\tilde{\varepsilon}_a(\nu, J) = \max\{k \geqslant 0 \mid \tilde{e}_a(\nu, J) \neq 0\}$. The following lemma is proven in [22].

Lemma 4.12. (See [22, Lemma 3.6].) *Let $(\nu, J) \in \mathrm{RC}(L)$. For fixed $a \in \{1, 2, \ldots, n - 1\}$, let $p = p_i^{(a)}$ be the vacancy number for large i and let $s \leqslant 0$ be the smallest nonpositive label in $(\nu, J)^{(a)}$; if no such label exists set $s = 0$. Then $\tilde{\varphi}_a(\nu, J) = p - s$.*

Theorem 4.13. *Let* $B = B^{r_k,s_k} \otimes \cdots \otimes B^{r_1,s_1}$ *and L the multiplicity array of B. Then the following diagrams commute*:

$$
\begin{array}{ccc}
\mathcal{P}(B) & \xrightarrow{\ \Phi\ } & \mathrm{RC}(L) \\
{\scriptstyle f_a}\downarrow & & \downarrow{\scriptstyle \tilde{f}_a} \\
\mathcal{P}(B) & \xrightarrow[\ \Phi\ ]{} & \mathrm{RC}(L)
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{P}(B) & \xrightarrow{\ \Phi\ } & \mathrm{RC}(L) \\
{\scriptstyle e_a}\downarrow & & \downarrow{\scriptstyle \tilde{e}_a} \\
\mathcal{P}(B) & \xrightarrow[\ \Phi\ ]{} & \mathrm{RC}(L).
\end{array}
\tag{4.1}
$$

The proof of Theorem 4.13 is given in Appendix C. Note that Proposition 4.7 and Theorem 4.13 imply that the operators $\tilde{f}_a, \tilde{e}_a$ give a crystal structure on $\mathrm{RC}(L)$. In [22] it is shown directly that $\tilde{f}_a$ and $\tilde{e}_a$ define a crystal structure on $\mathrm{RC}(L)$.

4.5. Proof of Theorem 4.1

By Proposition 4.7 Φ is a bijection which proves the first part of Theorem 4.1. By Theorem 4.13 the operators $\tilde{f}_a$ and $\tilde{e}_a$ give a crystal structure on $\mathrm{RC}(L)$ induced by the crystal structure on $\mathcal{P}(B)$ under Φ. The highest weight elements are given by the usual rigged configurations and highest weight paths, respectively, for which Theorem 4.1 is known to hold by [14]. The energy function $\overleftarrow{D}$ is constant on classical components. By [22, Theorem 3.9] the statistics cc on rigged configurations is also constant on classical components. Hence Φ preserves the statistic.

4.6. Implementation

The bijection Φ and its inverse have been implemented as a C++ program. The code is available in [4]. In early stages of this project these programs have been invaluable to produce data and check conjectures regarding the unrestricted rigged configurations. The programs have also been incorporated into MuPAD-Combinat as a dynamic module by Francois Descouens [18]. For example, the command

```
riggedConfigurations::RcPathsEnergy::
    fromOnePath([[[3]],[[2],[1]],[[4,5,6],[1,2,3]]])
```

calculates $\Phi(b)$ with b as in Example 4.8.

Appendix A. Proof of Propositions 4.3 and 4.5

Propositions 4.3 and 4.5 state that δ is a well-defined bijection. Since the proofs are similar we only give the proof of Proposition 4.3. The proof of Proposition 4.5 is available in the electronic version of this paper math.CO/0509194 or Chapter 2 of [4].

To prove that δ is well defined it needs to be shown that $(\bar{\nu}, \bar{J}) = \delta(\nu, J) \in \mathrm{RC}(\bar{L}, \bar{\lambda})$. Here $\bar{L}$ is given by $\bar{L}_1^{(1)} = L_1^{(1)} - 1$, $\bar{L}_i^{(a)} = L_i^{(a)}$ for all other i, a, and $\bar{\lambda} = \lambda - \epsilon_r$ where $r = \mathrm{rk}(\nu, J)$.

Let us first show that $\bar{\lambda}$ indeed has nonnegative entries. Assume the contrary that $\bar{\lambda}_r < 0$. This can happen only if $\lambda_r = 0$. Suppose $t \in \mathcal{A}(\lambda^{\mathrm{part}})$ is such that $M_j^{(k)}(t) \leqslant p_j^{(k)}(\nu)$ for all j, k. By (3.3), $p_i^{(r)}(\nu) = -\lambda_{r+1}$ for large i. Let ℓ be the size of the largest part in $\nu^{(r)}$, so that $m_j^{(r)}(\nu) = 0$ for $j > \ell$. By definition of vacancy numbers, $p_i^{(r)}(\nu) \geqslant p_j^{(r)}(\nu)$ for $i \geqslant j \geqslant \ell$. Also

we have $M_j^{(r)}(t) \geqslant -\lambda_{r+1}$ for all j. Hence, $-\lambda_{r+1} \leqslant M_j^{(r)}(t) \leqslant p_j^{(r)}(v) \leqslant p_i^{(r)}(v) = -\lambda_{r+1}$ implies

$$M_i^{(r)}(t) = M_j^{(r)}(t) = p_j^{(r)}(v) = p_i^{(r)}(v) \quad \text{for all } \ell \leqslant j \leqslant i. \tag{A.1}$$

This means that the string of length ℓ in $(v, J)^{(r)}$ is singular and $\Delta p_j^{(r)}(t) = 0$ for all $j \geqslant \ell$. We claim that $m_j^{(r-1)}(v) = 0$ for $j > \ell$. By (3.6) we get

$$\begin{aligned} S &:= -\chi(j \in t_{.,r}) + \chi(j \in t_{.,r+1}) + \chi(j+1 \in t_{.,r}) - \chi(j+1 \in t_{.,r+1}) \\ &\geqslant m_j^{(r-1)}(v) + m_j^{(r+1)}(v) \end{aligned}$$

for $j > \ell$. Clearly, $m_j^{(r-1)}(v) = 0$ unless $1 \leqslant S \leqslant 2$. If $S = 2$ we have $j+1 \in t_{.,r}$ and $j \in t_{.,r+1}$ which implies $M_j^{(r)}(t) = M_{j+1}^{(r)}(t) + 1$, a contradiction to (A.1). Hence $S = 2$ is not possible. Similarly, we can show that $S = 1$ is not possible. This proves that $m_j^{(r-1)}(v) = 0$ for $j > \ell$. Hence $\ell^{(r-1)} \leqslant \ell$ which contradicts the assumption that $r = \mathrm{rk}(v, J)$ since $(v, J)^{(r)}$ has a singular string of length ℓ. Therefore $\lambda_r > 0$.

It remains to show that $(\bar{v}, \bar{J})$ is admissible, which means that the parts of $\bar{J}$ lie between the corresponding lower bound for some $\bar{t} \in \mathcal{A}(\bar{\lambda}^{\mathrm{part}})$ and the vacancy number. To prove this we need the following preliminary results.

Remark A.1. Let (v, J) be admissible with respect to $t \in \mathcal{A}(\lambda^{\mathrm{part}})$. Suppose that $\Delta p_{i-1}^{(k)}(t) + \Delta p_{i+1}^{(k)}(t) \geqslant 1$ and $\Delta p_i^{(k)}(t) = m_i^{(k)}(v) = 0$. Then by (3.6) there are five choices for the letters i and $i+1$ in columns k and $k+1$ of t:

(1) $i+1$ in column k;
(2) $i+1$ in column k and $k+1$, i in column $k+1$;
(3) i in column $k+1$;
(4) i in column k and $k+1$, $i+1$ in column k;
(5) $i+1$ in column k, i in column $k+1$.

In cases (1) and (2) we have $m_i^{(k-1)}(v) = 0$. Changing letter $i+1$ to i in column k to form a new tableau t' has the effect $M_i^{(k)}(t') = M_i^{(k)}(t) - 1$, $M_i^{(k-1)}(t') = M_i^{(k-1)}(t) + 1$ and all other lower bounds remain unchanged. In cases (3) and (4) we have $m_i^{(k+1)}(v) = 0$. Changing letter i to $i+1$ in column $k+1$ to form a new tableau t' has the effect $M_i^{(k)}(t') = M_i^{(k)}(t) - 1$, $M_i^{(k+1)}(t') = M_i^{(k+1)}(t) + 1$ and all other lower bounds remain unchanged. Finally in case (5) either $m_i^{(k-1)}(v) = 0$ or $m_i^{(k+1)}(v) = 0$. Changing $i+1$ to i in column k (respectively i to $i+1$ in column $k+1$) has the same effect as in case (1) (respectively case (3)).

This shows that under the replacement $t \mapsto t'$ we have $\Delta p_i^{(k)}(t') > 0$ and by Lemma 3.6 (v, J) is admissible with respect to some tableau t''.

Let λ be a weight such that $\lambda_r > 0$ for a given $1 \leqslant r \leqslant n$. Set $\bar{\lambda} = \lambda - \epsilon_r$. Recall that $c_k = \lambda_{k+1} + \lambda_{k+2} + \cdots + \lambda_n$ is the height of the kth column of $t \in \mathcal{A}(\lambda^{\mathrm{part}})$. Let us define the map $\mathcal{D}_r : \mathcal{A}(\lambda^{\mathrm{part}}) \to \mathcal{A}(\bar{\lambda}^{\mathrm{part}})$ with $\bar{t} = \mathcal{D}_r(t)$ as follows. If $t_{1,r} < c_{r-1}$ then

$$\bar{t}_{i,k} = \begin{cases} t_{i+1,k} & \text{for } 1 \leqslant k \leqslant r-1 \text{ and } 1 \leqslant i < c_k, \\ t_{i,k} & \text{for } r \leqslant k \leqslant n \text{ and } 1 \leqslant i \leqslant c_k. \end{cases} \tag{A.2}$$

If $t_{1,r} = c_{r-1}$ then there exists $1 \leqslant j \leqslant c_r$ such that $t_{i,r} = t_{i-1,r} - 1$ for $2 \leqslant i \leqslant j$ and $t_{j+1,r} < t_{j,r} - 1$ if $j < c_r$. In this case

$$\bar{t}_{i,k} = \begin{cases} t_{i+1,k} & \text{for } 1 \leqslant k \leqslant r-1 \text{ and } 1 \leqslant i < c_k, \\ t_{i,r} - 1 & \text{for } k = r \text{ and } 1 \leqslant i \leqslant j, \\ t_{i,r} & \text{for } k = r \text{ and } j < i \leqslant c_r, \\ t_{i,k} & \text{for } r < k \leqslant n \text{ and } 1 \leqslant i \leqslant c_k. \end{cases} \tag{A.3}$$

Note that by definition the entries of $\mathcal{D}_r(t)$ are strictly decreasing along columns. Let $\bar{c}_k = \bar{\lambda}_{k+1} + \cdots + \bar{\lambda}_n$. Then we have $\bar{c}_k = c_k - 1$ for $1 \leqslant k \leqslant r-1$ and $\bar{c}_k = c_k$ for $r \leqslant k \leqslant n$. Again by definition $\bar{t}_{j,1} \in \{1, 2, \ldots, \bar{c}_1\}$ for all $1 \leqslant j \leqslant \bar{c}_1$ and $\bar{t}_{j,k} \in \{1, 2, \ldots, \bar{c}_{k-1}\}$ for all $2 \leqslant j \leqslant \bar{c}_k$ and $1 \leqslant k \leqslant n$. Therefore, $\mathcal{D}_r(t) \in \mathcal{A}(\bar{\lambda}^{\text{part}})$.

Example A.2. Let $t = \begin{array}{|c|c|c|} \hline 3 & 3 & 2 \\ \hline 2 & 1 \\ \cline{1-2} 1 \\ \cline{1-1} \end{array}$ and $r = 3$. Then $\mathcal{D}_r(t) = \begin{array}{|c|c|c|} \hline 2 & 1 & 1 \\ \hline 1 \\ \cline{1-1} \end{array}$.

We will use the following lemma and remark in the proofs.

Lemma A.3. *Let* $B = B^{r_l, s_l} \otimes \cdots \otimes B^{r_1, s_1}$ *with* $r_l = 1 = s_l$. *Let* $(\bar{\nu}, \bar{J}) = \delta(\nu, J)$ *and let* $\mathrm{rk}(\nu, J) = r$. *For* $1 < k < r$ *let* $i = t_{1,k}$. *Then one of the following conditions hold*:

(1) $m_i^{(k)}(\nu) = 0$, *or*

(2) $m_i^{(k)}(\nu) = 1$, *in which case* δ *selects the part of length* i *in* $\nu^{(k)}$.

Proof. Note that $i = t_{1,k} \geqslant c_k$. By (3.2) we have $|\nu^{(k)}| \leqslant c_k$, so that either $m_i^{(k)}(\nu) = 0$ or $i = c_k$ and $\nu^{(k)}$ consists of just one part of size i. In this case $m_i^{(k)}(\nu) = 1$ and δ has to select this single part. $\quad\square$

Remark A.4. By (3.2) we have

$$\left| \nu^{(r)} \right| = \left| \nu^{(r-1)} \right| - \lambda_r + \sum_{i \geqslant 1} s_i \chi(r_i \geqslant r),$$

$$\left| \nu^{(r+1)} \right| = \left| \nu^{(r-1)} \right| - \lambda_r - \lambda_{r+1} + 2 \sum_{i \geqslant 1} s_i \chi(r_i \geqslant r) - \sum_{i \geqslant 1} s_i \delta_{r_i, r}.$$

Note that for $a > 0$

$$\sum_{i \geqslant 1} \min(a, i) L_i^{(r)} = \sum_{i \geqslant 1} s_i \chi(s_i \leqslant a) \delta_{r_i, r} + \sum_{i \geqslant 1} a \chi(s_i > a) \delta_{r_i, r}.$$

Then if $|\nu^{(r-1)}| = c_{r-1} - k$ for some $k \geqslant 0$ it follows that

$$-2 \left| \nu^{(r)} \right| + \left| \nu^{(r+1)} \right| + \sum_{i \geqslant 1} \min(a, i) L_i^{(r)} = -2\lambda_{r+1} - c_{r+1} + k - \sum_{i \geqslant 1} \max(s_i - a, 0) \delta_{r_i, r}.$$

Now we are ready to show that $(\bar{\nu}, \bar{J})$ is admissible, which means that the parts of $\bar{J}$ lie between the corresponding lower bound for some $\bar{t} \in \mathcal{A}(\bar{\lambda}^{\text{part}})$ and the vacancy number. Let

$t \in \mathcal{A}(\lambda^{\mathrm{part}})$ be such that (v, J) is admissible with respect to t. By the same arguments as in the proof of Proposition 3.12 of [14] the only problematic case is when

$$m_{\ell-1}^{(k)}(v) = 0, \qquad \Delta p_{\ell-1}^{(k)}(t) = 0, \quad \ell^{(k-1)} < \ell \text{ and } \ell \text{ finite}, \tag{A.4}$$

where $\ell = \ell^{(k)}$.

Assume that $\Delta p_{\ell-2}^{(k)}(t) + \Delta p_{\ell}^{(k)}(t) \geqslant 1$ and (A.4) holds. By Remark A.1 with $i = \ell - 1$, there exists a new tableau t' such that $\Delta p_{\ell-1}^{(k)}(t') > 0$ so that the problematic case is avoided.

Hence assume that $\Delta p_{\ell-2}^{(k)}(t) + \Delta p_{\ell}^{(k)}(t) = 0$ and (A.4) holds. Let $\ell' < \ell$ be maximal such that $m_{\ell'}^{(k)}(v) > 0$. If no such ℓ' exists, set $\ell' = 0$.

Suppose that there exists $\ell' < j < \ell$ such that $\Delta p_{j-1}^{(k)}(t) > 0$. Let i be the maximal such j. Then by Remark A.1 we can find a new tableau t' such that $\Delta p_{i}^{(k)}(t') > 0$ and (v, J) is admissible with respect to t'. Repeating the argument we can achieve $\Delta p_{\ell-1}^{(k)}(t'') > 0$ for some new tableau t'', so that the problematic case does not occur.

Hence we are left to consider the case $\Delta p_{i}^{(k)}(t) = 0$ for all $\ell' \leqslant i \leqslant \ell$. If $m_{i}^{(k-1)}(v) = 0$ for all $\ell' < i < \ell$, then by the same arguments as in the proof of Proposition 3.12 of [14] we arrive at a contradiction since $\ell^{(k-1)} \leqslant \ell'$, but the string of length ℓ' in $(v, J)^{(k)}$ is singular which implies that $\ell^{(k)} \leqslant \ell' < \ell$. Hence there must exist $\ell' < i < \ell$ such that $m_{i}^{(k-1)}(v) > 0$ and $\ell^{(k-1)} = i$. By (3.6) the same five cases as in Remark A.1 occur as possibilities for the letters i and $i + 1$ in columns k and $k + 1$ of t. In cases (3), (4) and case (5) if $m_{i}^{(k-1)}(v) = 2$, we have $m_{i}^{(k+1)}(v) = 0$. Replace i in column $k + 1$ by $i + 1$ in t to get a new tableau t'. In all other cases $m_{i}^{(k-1)}(v) = 1$; replace the letter $i + 1$ in column k by i to obtain t'. The replacement $t \mapsto t'$ yields $\Delta p_{i}^{(k)}(t') > 0$ in all cases. The change of lower bound $M_{i}^{(k-1)}(t') = M_{i}^{(k-1)}(t) + 1$ in cases (1), (2) and (5) when $m_{i}^{(k-1)}(v) \neq 2$ will not cause any problems since $m_{i}^{(k-1)}(v) = 1$ so that after the application of δ there is no part of length i in the $(k-1)$th rigged partition. Then again repeated application of Remark A.1 achieves $\Delta p_{\ell-1}^{(k)}(t'') > 0$ for some tableau t'', so that the problematic case does not occur.

Let t'' be the tableau we constructed so far. Note that in all constructions above, either a letter $i + 1$ in column k is changed to i, or a letter i in column $k + 1$ is changed to $i + 1$. In the latter case $i + 1 \leqslant \ell \leqslant |v^{(k)}| \leqslant c_k$. Hence t'' satisfies the constraint that $t''_{i,k} \in \{1, 2, \dots, c_{k-1}\}$ for all i, k.

Now let $\bar{t} = \mathcal{D}_r(t'')$. We know $\bar{t} \in \mathcal{A}(\bar{\lambda}^{\mathrm{part}})$. We will show that the parts of $\bar{J}$ lie between the corresponding lower bound with respect to $\bar{t} \in \mathcal{A}(\bar{\lambda}^{\mathrm{part}})$ and the vacancy number.

If $t''_{1,r} < c_{r-1}$ then by Lemma A.3 $M_{i}^{(k)}(\bar{t}) \leqslant M_{i}^{(k)}(t'')$ for all k and i such that $m_{i}^{(k)}(\bar{v}) > 0$. Hence by Lemma 3.6 we have that $(\bar{v}, \bar{J})$ is admissible with respect to $\bar{t}$.

Let $t''_{1,r} = c_{r-1}$. Then there exists j as in the definition of $\mathcal{D}_r$. We claim that

(i) $m_{i}^{(r-1)}(v) = 0$ for $i > c_{r-1} - j$ and $m_{c_{r-1}-j}^{(r-1)}(v) \leqslant 1$,
(ii) if $m_{c_{r-1}-j}^{(r-1)}(v) = 1$, then $\ell^{(r-1)} = c_{r-1} - j$.

Note that $M_{i}^{(r-1)}(\bar{t}) = M_{i}^{(r-1)}(t'') + 1$ for $c_{r-1} - j \leqslant i < c_{r-1}$ and $M_{i}^{(k)}(\bar{t}) \leqslant M_{i}^{(k)}(t'')$ for all other k and i such that $m_{i}^{(k)}(\bar{v}) > 0$. Hence if the claim is true using Lemma A.3 we have

$M_i^{(k)}(\bar{t}) \leqslant M_i^{(k)}(t'')$ for all k and i such that $m_i^{(k)}(\bar{v}) > 0$. Therefore by Lemma 3.6 we have that $(\bar{v}, \bar{J})$ is admissible with respect to $\bar{t}$.

It remains to prove the claim. Note that if $|v^{(r-1)}| < c_{r-1} - j$ then our claim is trivially true. Let $|v^{(r-1)}| = c_{r-1} - k$ for some $0 \leqslant k \leqslant j$. If all parts of $v^{(r-1)}$ are strictly less than $c_{r-1} - j$, again our claim is trivially true. Let the largest part in $v^{(r-1)}$ be $c_{r-1} - p \geqslant c_{r-1} - j$ for some $k \leqslant p \leqslant j$. Let a be the largest part in $v^{(r)}$.

First suppose $a > c_{r-1} - p$ and $a = c_r - q$ for some $0 \leqslant q < c_r$. Then $a = c_{r-1} - (\lambda_r + q)$ which implies that

$$M_a^{(r)}(t'') \geqslant -(c_r - \lambda_r - q) + (c_{r+1} - q) = \lambda_r - \lambda_{r+1}.$$

This means $p_a^{(r)}(v) \leqslant M_a^{(r)}(t'')$ since $p_b^{(r)}(v) \geqslant p_a^{(r)}(v)$ for all $b \geqslant a$ and $p_b^{(r)} = \lambda_r - \lambda_{r+1}$ for large b. If $p_a^{(r)}(v) < M_a^{(r)}(t'')$, it contradicts that $p_a^{(r)}(v) \geqslant M_a^{(r)}(t'')$. If $p_a^{(r)}(v) = M_a^{(r)}(t'')$, it contradicts the fact that $r = \mathrm{rk}(v, J)$ since we get a singular part of length a in $v^{(r)}$ which is larger than the largest part in $v^{(r-1)}$. Therefore $a > c_{r-1} - p$ is not possible.

Hence $a \leqslant c_{r-1} - p$. Using Remark A.4 we get,

$$p_a^{(r)}(v) = Q_a\big(v^{(r-1)}\big) - 2|v^{(r)}| + Q_a\big(v^{(r+1)}\big) + \sum_{i \geqslant 1} \min(a, i) L_i^{(r)}$$

$$\leqslant a + p - k - 2|v^{(r)}| + |v^{(r+1)}| + \sum_{i \geqslant 1} \min(a, i) L_i^{(r)}$$

$$= a + p - 2\lambda_{r+1} - c_{r+1} - \sum_{i \geqslant 1} \max(s_i - a, 0)\delta_{r_i, r}. \tag{A.5}$$

Since $p_a^{(r)}(v) \geqslant M_a^{(r)}(t'') \geqslant -\lambda_{r+1}$ we get

$$c_r - \left(p - \sum_{i \geqslant 1} \max(s_i - a, 0)\delta_{r_i, r} \right) \leqslant a \leqslant c_r.$$

Hence $a = c_r - q$ for $0 \leqslant q \leqslant p - \sum_{i \geqslant 1} \max(s_i - a, 0)\delta_{r_i, r}$. Then from (A.5) with $a = c_r - q$ we get

$$p_a^{(r)}(v) \leqslant p - q - \lambda_{r+1} - \sum_{i \geqslant 1} \max(s_i - a, 0)\delta_{r_i, r} \leqslant \lambda_r - \lambda_{r+1}, \tag{A.6}$$

where we used that $0 \leqslant p - q \leqslant \lambda_r$ which follows from $a = c_r - q \leqslant c_{r-1} - p$.

If $a > c_{r-1} - j$, as in the case $a > c_{r-1} - p$ we have

$$M_a^{(r)}(t'') \geqslant -(c_r - \lambda_r - q) + (c_{r+1} - q) = \lambda_r - \lambda_{r+1} \geqslant p_a^{(r)}(v).$$

Hence we get a contradiction unless $p_a^{(r)}(v) = M_a^{(r)}(t'')$. By (A.6) and the fact that $0 \leqslant p - q \leqslant \lambda_r$ we know $p_a^{(r)}(v) = \lambda_r - \lambda_{r+1}$ happens only when $p - q = \lambda_r$ and $\sum_{i \geqslant 1} \max(s_i - a, 0)\delta_{r_i, r} = 0$. This means the largest part in $v^{(r-1)}$ is of length $c_{r-1} - p = c_r - q = a$. Since we have a singular string of length a in $v^{(r)}$ this contradicts the fact that $r = \mathrm{rk}(v, J)$.

If $a \leqslant c_{r-1} - j$ then $M_a^{(r)}(t'') \geqslant -(c_r - j) + (c_{r+1} - q) = j - q - \lambda_{r+1} \geqslant p_a^{(r)}(v)$ because of (A.6) and the fact that $j \geqslant p$. Again we get a contradiction unless $p_a^{(r)}(v) = M_a^{(r)}(t'')$. But this happens only when $p_a^{(r)}(v) = j - q - \lambda_{r+1}$ which gives $p = j$ because $p_a^{(r)}(v)$ attains the right-hand side of (A.6). This means the largest part in $v^{(r-1)}$ is $c_{r-1} - j$. Furthermore, for large i we have $p_i^{(r)} = \lambda_r - \lambda_{r+1} \geqslant j - q - \lambda_{r+1} + (c_{r-1} - j - a) = \lambda_r - \lambda_{r+1}$ which shows that

besides $c_{r-1} - j$ all parts in $\nu^{(r-1)}$ have to be less than or equal to a. But the part of length a in $\nu^{(r)}$ is singular, so we have to have $c_{r-1} - j > a$ and $\ell^{(r-1)} = c_{r-1} - j$ else it will contradict the fact that $r = \text{rk}(\nu, J)$. This proves our claim.

Hence $(\bar{\nu}, \bar{J})$ is admissible with respect to $\bar{t} \in \mathcal{A}(\bar{\lambda}^{\text{part}})$ and therefore δ is well defined.

Example A.5. Let L be the multiplicity array of $B = (B^{1,1})^{\otimes 4}$ and $\lambda = (0, 1, 0, 1, 2)$. Let

$$(\nu, J) = \boxed{} -1 \quad \boxed{} \; 0 \quad \boxed{} -1 \quad \boxed{} -1 \in \text{RC}(L, \lambda).$$

Let

$$t = \begin{array}{|c|c|c|c|} \hline 4 & 4 & 3 & 3 \\ \hline 3 & 2 & 2 & 2 \\ \hline 2 & 1 & 1 \\ \cline{1-3} 1 \\ \cline{1-1} \end{array}$$

be the corresponding lower bound tableau. Then

$$\delta(\nu, J) = \boxed{} -1 \quad \boxed{} \; 0 \quad \boxed{} -1 \quad \boxed{} -1 \cdot$$

Note that in this example $\ell = \ell^{(4)} = 2$ and it satisfies (A.4) with $k = 4$. Also $\Delta p_{\ell-2}^{(4)}(t) + \Delta p_{\ell}^{(4)}(t) = 0$ with $\Delta p_i^{(4)}(t) = 0$ for all $0 \leqslant i \leqslant \ell$. Since $m_1^{(3)}(\nu) = 1$ and $2 \in t_{\cdot,4}$ this is an example where we get the new tableau t' by replacing the $2 \in t_{\cdot,4}$ by 1 and then the corresponding lower bound tableau for $\delta(\nu, J)$ is

$$\mathcal{D}_5(t') = \begin{array}{|c|c|c|c|} \hline 3 & 2 & 2 & 1 \\ \hline 2 & 1 & 1 \\ \cline{1-3} 1 \\ \cline{1-1} \end{array} \; .$$

Appendix B. Proof of Proposition 4.7

In this section a proof of Proposition 4.7 is given stating that the map Φ of Definition 4.6 is a well-defined bijection.

The proof proceeds by induction on B using the fact that it is possible to go from $B = B^{r_k, s_k} \otimes B^{r_{k-1}, s_{k-1}} \otimes \cdots \otimes B^{r_1, s_1}$ to the empty crystal via successive application of lh, ls and lb. Suppose that B is the empty crystal. Then both sets $\mathcal{P}(B, \lambda)$ and $\text{RC}(L, \lambda)$ are empty unless λ is the empty partition, in which case $\mathcal{P}(B, \lambda)$ consists of the empty partition and $\text{RC}(L, \lambda)$ consists of the empty rigged configuration. In this case Φ is the unique bijection mapping the empty partition to the empty rigged configuration.

Consider the commutative diagram (4.6) of Definition 4.6. By induction

$$\Phi: \bigcup_{\mu \in \lambda^-} \mathcal{P}\big(\text{lh}(B), \mu\big) \to \bigcup_{\mu \in \lambda^-} \text{RC}\big(\text{lh}(L), \mu\big)$$

is a bijection. By Propositions 4.3 and 4.5 δ is a bijection, and by definition it is clear that lh is a bijection as well. Hence $\Phi = \delta^{-1} \circ \Phi \circ \text{lh}$ is a well-defined bijection.

Suppose that $B = B^{r,1} \otimes B'$ with $r \geqslant 2$. By induction Φ is a bijection for $\text{lb}(B) = B^{1,1} \otimes B^{r-1,1} \otimes B'$. Hence to prove that (4.6) uniquely determines Φ for B it suffices to show that

Φ restricts to a bijection between the image of $\mathrm{lb}\colon \mathcal{P}(B, \lambda) \to \mathcal{P}(\mathrm{lb}(B), \lambda)$ and the image of $\mathrm{lb}_{rc}\colon \mathrm{RC}(L, \lambda) \to \mathrm{RC}(\mathrm{lb}(L), \lambda)$. Let

$$b = \boxed{b_r} \otimes \boxed{\begin{array}{c} b_1 \\ \vdots \\ b_{r-1} \end{array}} \otimes b' \in \mathcal{P}\big(\mathrm{lb}(B), \lambda\big)$$

with $b_{r-1} < b_r$. Let $(\nu, J) = \Phi(b)$ which is in $\mathrm{RC}(\mathrm{lb}(L), \lambda)$. We will show that $(\nu, J)^{(a)}$ has a singular string of length one for $1 \leqslant a \leqslant r - 1$.

By induction we know for $(\bar{\nu}, \bar{J}) = \Phi(\bar{b})$ where

$$\bar{b} = \boxed{b_{r-1}} \otimes \boxed{\begin{array}{c} b_1 \\ \vdots \\ b_{r-2} \end{array}} \otimes b' \in \mathrm{lb}\big(B^{r-1,1} \otimes B'\big)$$

with $b_{r-2} < b_{r-1}$, $(\bar{\nu}, \bar{J})^{(a)}$ has a singular string of length one for $1 \leqslant a \leqslant r - 2$. Let

$$\bar{b}' = \boxed{\begin{array}{c} b_1 \\ \vdots \\ b_{r-1} \end{array}} \otimes b' \quad \text{and} \quad (\bar{\nu}', \bar{J}') = \Phi(\bar{b}').$$

This "unsplitting" on the rigged configuration side removes the singular string of length one from $(\bar{\nu}, \bar{J})^{(a)}$ for $1 \leqslant a \leqslant r - 2$ yielding $(\bar{\nu}', \bar{J}')$.

Let $\bar{s}^{(a)}$ be the length of the selected strings by δ^{-1} associated with b_{r-1}. Note that $\bar{s}^{(a)} = 0$ for $1 \leqslant a \leqslant r - 2$. Now let $s^{(a)}$ be the selected strings by δ^{-1} associated with b_r. Since $b_{r-1} < b_r$ we have by construction that $s^{(a+1)} \leqslant \bar{s}^{(a)}$. In particular $s^{(r-1)} \leqslant \bar{s}^{(r-2)} = 0$ and therefore, $s^{(r-1)} = 0$. This implies that $s^{(a)} = 0$ for $1 \leqslant a \leqslant r - 1$. Hence $(\nu, J)^{(a)}$ has a singular string of length one for $1 \leqslant a \leqslant r - 1$.

Conversely, let $(\nu, J) \in \mathrm{lb}_{rc}(\mathrm{RC}(L, \lambda))$, that is, $(\nu, J)^{(a)}$ has singular string of length one for $1 \leqslant a \leqslant r - 1$. Let

$$b = \Phi^{-1}(\nu, J) = \boxed{b_r} \otimes \boxed{\begin{array}{c} b_1 \\ \vdots \\ b_{r-1} \end{array}} \otimes b' \in \mathcal{P}\big(\mathrm{lb}(B), \lambda\big).$$

We want to show that $b_{r-1} < b_r$. Let $(\bar{\nu}, \bar{J}) = \delta(\nu, J)$ and $\ell^{(a)}$ be the length of the selected string in $(\nu, J)^{(a)}$ by δ. Then $\ell^{(a)} = 1$ for $1 \leqslant a \leqslant r - 1$ and the change of vacancy numbers from (ν, J) to $(\bar{\nu}, \bar{J})$ is given by

$$p_i^{(a)}(\bar{\nu}) = p_i^{(a)}(\nu) - \chi\big(\ell^{(a-1)} \leqslant i < \ell^{(a)}\big) + \chi\big(\ell^{(a)} \leqslant i < \ell^{(a+1)}\big). \tag{B.1}$$

This implies that $(\bar{\nu}, \bar{J})^{(r-1)}$ has no singular string of length less than $\ell^{(r)}$ since $\ell^{(r-1)} = 1$. Let $(\bar{\nu}', \bar{J}') = \mathrm{lb}_{rc}(\bar{\nu}, \bar{J})$. Denote by $\bar{\ell}^{(a)}$ the length of the singular string selected by δ in $(\bar{\nu}', \bar{J}')^{(a)}$. Then by induction $\bar{\ell}^{(a)} = 1$ for $1 \leqslant a \leqslant r - 2$ and by (B.1) we get $\bar{\ell}^{(a)} \geqslant \ell^{(a+1)}$ for $a \geqslant r - 1$. Therefore $\bar{\ell}^{(a)} \geqslant \ell^{(a+1)}$ for all $1 \leqslant a \leqslant n$. Hence $b_{r-1} < b_r$. This proves that Φ in (3) is uniquely determined.

Let us now consider the case $B = B^{r,s} \otimes B'$ where $s \geqslant 2$. Any map Φ satisfying (2) is injective by definition and unique by induction. To prove the existence and surjectivity it suffices to prove that the bijection Φ maps the image of $\mathrm{ls}\colon \mathcal{P}(B, \lambda) \to \mathcal{P}(\mathrm{ls}(B), \lambda)$ to the image of $\mathrm{ls}_{rc}\colon \mathrm{RC}(L, \lambda) \to \mathrm{RC}(\mathrm{ls}(L), \lambda)$. This can be done by similar arguments as in [14]. Details are available in the electronic version of this paper math.CO/0509194 or Chapter 2 of [4].

Appendix C. Proof of Theorem 4.13

In this section we prove that the crystal operators on paths and rigged configurations commute with the bijection Φ.

The following lemma is a result of [14, Lemma 3.11] about the convexity of the vacancy numbers.

Lemma C.1 *(Convexity). Let* $(v, J) \in \mathrm{RC}(L)$:

(1) *For all* $i, k \geqslant 1$ *we have* $-p_{k-1}^{(i)}(v) + 2p_k^{(i)}(v) - p_{k+1}^{(i)}(v) \geqslant m_k^{(i-1)}(v) - 2m_k^{(i)}(v) + m_k^{(i+1)}(v)$.

(2) *Let* $m_k^{(i)}(v) = 0$ *for* $a < k < b$. *Then* $p_k^{(i)}(v) \geqslant \min(p_a^{(i)}(v), p_b^{(i)}(v))$.

(3) *Let* $m_k^{(i)}(v) = 0$ *for* $a < k < b$. *If* $p_a^{(i)}(v) = p_{a+1}^{(i)}(v)$ *and* $p_{a+1}^{(i)}(v) \leqslant p_b^{(i)}(v)$ *then* $p_{a+1}^{(i)}(v) = p_k^{(i)}(v)$ *for all* $a \leqslant k \leqslant b$.

(4) *Let* $m_k^{(i)}(v) = 0$ *for* $a < k < b$. *If* $p_b^{(i)}(v) = p_{b-1}^{(i)}(v)$ *and* $p_{b-1}^{(i)}(v) \leqslant p_a^{(i)}(v)$ *then* $p_{b-1}^{(i)}(v) = p_k^{(i)}(v)$ *for all* $a \leqslant k \leqslant b$.

Proof. The proof of (1) is given in [15, Appendix] (see also (3.5)), (2) follows from repeated use of (1), and the proof of (3) and (4) follow from (1) and (2). $\quad\square$

Lemma C.2. *Let* $B = B^{1,1} \otimes B'$ *and let* L *and* L' *be the multiplicity arrays of* B *and* B'. *For* $1 \leqslant i < n$ *the following diagrams commute if* $\tilde{f}_i$ *is always defined*:

$$\begin{array}{ccc} \mathrm{RC}(L) \xrightarrow{\;\delta\;} \mathrm{RC}(L') & \quad & \mathrm{RC}(L) \xrightarrow{\;\delta\;} \mathrm{RC}(L') \\ \tilde{f}_i \downarrow \qquad\qquad \downarrow \tilde{f}_i & \quad & \tilde{e}_i \downarrow \qquad\qquad \downarrow \tilde{e}_i \\ \mathrm{RC}(L) \xrightarrow[\;\delta\;]{} \mathrm{RC}(L') & \quad & \mathrm{RC}(L) \xrightarrow[\;\delta\;]{} \mathrm{RC}(L') \end{array} \qquad (\mathrm{C.1})$$

Proof. We prove (C.1) for $\tilde{f}_i$ here; the proof for $\tilde{e}_i$ is similar. Let us introduce some notation. Let $(v, J) \in \mathrm{RC}(L)$ and let $\ell^{(a)}$ be the length of the singular string selected by δ in $(v, J)^{(a)}$ for $1 \leqslant a < n$. Let $(\bar{v}, \bar{J}) = \delta(v, J)$ and $(\tilde{v}, \tilde{J}) = \tilde{f}_i(v, J)$. Let $\tilde{\ell}^{(a)}$ be the length of the singular string selected by δ in $(\tilde{v}, \tilde{J})^{(a)}$ for $1 \leqslant a < n$ and ℓ (respectively $\bar{\ell}$) be the length of the string selected by $\tilde{f}_i$ in $(v, J)^{(i)}$ (respectively in $(\bar{v}, \bar{J})^{(i)}$). A string of length k and label x_k in $(v, J)^{(a)}$ is denoted by (k, x_k).

Using the definition of $\tilde{f}_i$ it is easy to see that the diagram (C.1) commutes trivially except when $\ell^{(i-1)} - 1 \leqslant \ell \leqslant \ell^{(i)}$. We list the nontrivial cases as follows:

(a) $\ell^{(i-1)} < \infty, \ell^{(i)} = \infty, \ell + 1 \geqslant \ell^{(i-1)}$.

(b) $\ell^{(i)} < \infty, \ell^{(i-1)} \leqslant \ell + 1 \leqslant \ell^{(i)}$.

(c) $\ell^{(i)} < \infty$ *and* $\ell^{(i)} = \ell$.

Note that since $\tilde{f}_i$ fixes all the colabels, the singular strings (except the new string of length $\ell + 1$) remain singular under the action of $\tilde{f}_i$. Let (ℓ, x_ℓ) be the string selected by $\tilde{f}_i$ in $(v, J)^{(i)}$. The new string of length $\ell + 1$ can be singular in $(\tilde{v}, \tilde{J})^{(i)}$ only if $p_{\ell+1}^{(i)}(v) = x_\ell + 1$. Also note that by the definition of $\tilde{f}_i$ if $m_k^{(i)}(v) > 0$ and (k, x_k) is a string in $(v, J)^{(i)}$ then

$$x_\ell < x_k \leqslant p_k^{(i)}(v), \quad \text{if } k > \ell,$$
$$x_\ell \leqslant x_k \leqslant p_k^{(i)}(v), \quad \text{if } k < \ell. \tag{C.2}$$

Let us now consider all the nontrivial cases.

Case (a): If the new string of length $\ell + 1$ in $(\tilde{v}, \tilde{J})^{(i)}$ is nonsingular, then (C.1) commutes trivially. Let us consider the case when the new string of length $\ell + 1$ in $(\tilde{v}, \tilde{J})^{(i)}$ is singular. We have $p_{\ell+1}^{(i)}(v) = x_\ell + 1$ and since $\ell^{(i-1)} < \infty, \ell^{(i)} = \infty$ we have $p_j^{(i)}(\bar{v}) = p_j^{(i)}(v) - 1$ for $j \geqslant \ell^{(i-1)}$. In particular $p_{\ell+1}^{(i)}(\bar{v}) = p_{\ell+1}^{(i)}(v) - 1 = x_\ell$. The labels in $(\bar{v}, \bar{J})^{(i)}$ are the same as in $(v, J)^{(i)}$. Hence $\bar{\ell} = \ell$, but the result is not a valid rigged configuration since $p_{\ell+1}^{(i)}(\bar{v}) - 2 < x_\ell - 1$. So, $\tilde{f}_i(\bar{v}, \bar{J})$ is undefined, which contradicts the assumptions of Lemma C.2.

Cases (b) and (c) can be proved in a similar fashion to [22, Lemma 4.10]. Details are available in the electronic version of this paper math.CO/0509194 or Chapter 2 of [4]. □

Lemma C.3. *Let $B = B^{r,1} \otimes B'$, $r \geqslant 2$, and let L be the multiplicity array of B. For $1 \leqslant i < n$ the following diagrams commute*:

$$
\begin{array}{ccc}
\mathrm{RC}(L) \xrightarrow{\mathrm{lb}_{rc}} \mathrm{RC}(\mathrm{lb}(L)) & \qquad & \mathrm{RC}(L) \xrightarrow{\mathrm{lb}_{rc}} \mathrm{RC}(\mathrm{lb}(L)) \\
\tilde{f}_i \downarrow \qquad\qquad \downarrow \tilde{f}_i & & \tilde{e}_i \downarrow \qquad\qquad \downarrow \tilde{e}_i \\
\mathrm{RC}(L) \xrightarrow[\mathrm{lb}_{rc}]{} \mathrm{RC}(\mathrm{lb}(L)) & & \mathrm{RC}(L) \xrightarrow[\mathrm{lb}_{rc}]{} \mathrm{RC}(\mathrm{lb}(L))
\end{array} \tag{C.3}
$$

Proof. Note that if $i > r - 1$ then the proof of (C.3) is trivial. Suppose $1 \leqslant i \leqslant r - 1$. The proof for $\tilde{e}_i$ is very similar to the proof for $\tilde{f}_i$, so here we only prove (C.3) for $\tilde{f}_i$. Let $(v, J) \in \mathrm{RC}(L)$. Let (ℓ, x_ℓ) be the string selected by $\tilde{f}_i$ in $(v, J)^{(i)}$. Let $(\bar{v}, \bar{J}) = \mathrm{lb}_{rc}(v, J)$. By definition of lb_{rc} we get $(\bar{v}, \bar{J})^{(k)}$ by adding a singular string of length one to $(v, J)^{(k)}$ for $1 \leqslant k \leqslant r - 1$. Hence to show that the diagram (C.3) commutes it suffices to show that the label for the new singular string of length one in $(\bar{v}, \bar{J})^{(i)}$ satisfies $p_1^{(i)}(\bar{v}) \geqslant x_\ell$. Note that $p_1^{(i)}(\bar{v}) = p_1^{(i)}(v)$ for all $1 \leqslant i \leqslant r - 1$.

If $m_1^{(i)}(v) > 0$ then $x_1^{(i)} \geqslant x_\ell$ by (C.2). So, $p_1^{(i)}(\bar{v}) = p_1^{(i)}(v) \geqslant x_1^{(i)} \geqslant x_\ell$. If $m_1^{(i)}(v) = 0$ let j be smallest such that $m_j^{(i)}(v) > 0$ and (j, x_j) be a string in $(v, J)^{(i)}$. By Lemma C.1(2) we get $p_1^{(i)}(v) \geqslant \min(p_0^{(i)}(v), p_j^{(i)}(v))$. Recall that $p_0^{(i)}(v) = 0$ and $x_\ell \leqslant 0$ by the definition of $\tilde{f}_i$. So, if $p_j^{(i)}(v) \geqslant 0$ then $p_j^{(i)}(\bar{v}) = p_1^{(i)}(v) \geqslant 0 \geqslant x_\ell$. If $p_j^{(i)}(v) < 0$ then $p_1^{(i)}(v) \geqslant p_j^{(i)}(v)$. But $p_j^{(i)}(v) \geqslant x_j \geqslant x_\ell$. Hence $p_1^{(i)}(\bar{v}) = p_1^{(i)}(v) \geqslant p_j^{(i)}(v) \geqslant x_\ell$ and we are done. □

Lemma C.4. *Let $B = B^{r,s} \otimes B'$, $r \geqslant 1, s \geqslant 2$, and let L be the multiplicity array of B. For $1 \leqslant i < n$ the following diagrams commute*:

$$
\begin{array}{ccc}
\mathrm{RC}(L) \xrightarrow{\mathrm{ls}_{rc}} \mathrm{RC}(\mathrm{ls}(L)) & \qquad & \mathrm{RC}(L) \xrightarrow{\mathrm{ls}_{rc}} \mathrm{RC}(\mathrm{ls}(L)) \\
\tilde{f}_i \downarrow \qquad\qquad \downarrow \tilde{f}_i & & \tilde{e}_i \downarrow \qquad\qquad \downarrow \tilde{e}_i \\
\mathrm{RC}(L) \xrightarrow[\mathrm{ls}_{rc}]{} \mathrm{RC}(\mathrm{ls}(L)) & & \mathrm{RC}(L) \xrightarrow[\mathrm{ls}_{rc}]{} \mathrm{RC}(\mathrm{ls}(L))
\end{array} \tag{C.4}
$$

Proof. Let $(v, J) \in \mathrm{RC}(L)$. By definition ls_{rc} only changes the vacancy numbers in $(v, J)^{(r)}$. Hence the proof of this lemma is trivial. □

Now we will prove Theorem 4.13.

Proof of Theorem 4.13. To prove this theorem we will use a diagram of the form
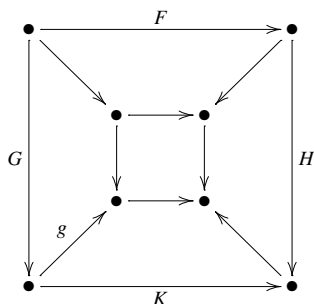
We view this diagram as a cube with front face given by the large square. By [14, Lemma 5.3] if the squares given by all the faces of the cube except the front commute and the map g is injective then the front face also commutes.

We will prove Theorem 4.13 by using induction on B as we did in the proof of the bijection of Proposition 4.7. First let $B = B^{1,1} \otimes B'$. We prove Theorem 4.13 for $\tilde{f}_i$ by using Lemma C.2 and the following diagram when f_i and $\tilde{f}_i$ are defined:
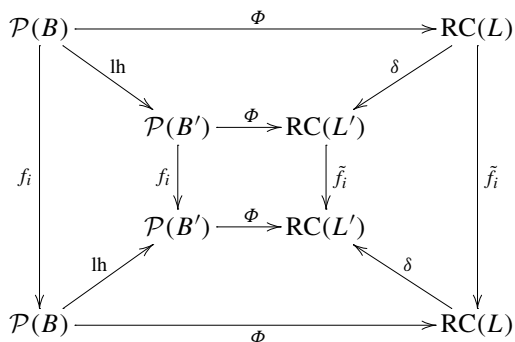
Note the top and the bottom faces commute by Definition 4.6(1). The right face commutes by Lemma C.2. The left face commutes by definition of f_i on the paths and we know lh is injective. By induction hypothesis the back face commutes. Hence the front face must commute.

Let us now prove Theorem 4.13 when not all f_i (respectively $\tilde{f}_i$) in the above diagram are defined. Let $(v, J) \in \mathrm{RC}(L)$, $(\bar{v}, \bar{J}) = \delta(v, J)$, $b = \Phi^{-1}(v, J)$ and $b' = \Phi^{-1}(\bar{v}, \bar{J})$. We need to show the following cases:

(1) $f_i(b)$ is defined and $f_i(b')$ is undefined if and only if $\tilde{f}_i(v, J)$ is defined and $\tilde{f}_i(\bar{v}, \bar{J})$ is undefined. In addition $\Phi(f_i(b)) = \tilde{f}_i(v, J)$.
(2) $f_i(b)$ is undefined and $f_i(b')$ is defined if and only if $\tilde{f}_i(v, J)$ is undefined and $\tilde{f}_i(\bar{v}, \bar{J})$ is defined.
(3) $f_i(b)$ and $f_i(b')$ are both undefined if and only if $\tilde{f}_i(v, J)$ and $\tilde{f}_i(\bar{v}, \bar{J})$ are both undefined.

For case (1) suppose that $\tilde{f}_i(v, J) = (\tilde{v}, \tilde{J})$ is defined, but $\tilde{f}_i(\bar{v}, \bar{J})$ is undefined. Then we are in the situation described in case (a) of Lemma C.2. That is $\ell^{(i-1)} < \infty$, $\ell^{(i)} = \infty$, $\ell + 1 \geqslant \ell^{(i-1)}$ and the new string of length $\ell + 1$ is singular in $(\tilde{v}, \tilde{J})^{(i)}$. In this situation note that $m^{(i)}_{\ell+1}(\bar{v}) = 0$,

else $p_{\ell+1}^{(i)}(\bar{v}) \geqslant x_{\ell+1} > x_\ell$ by (C.2), which is a contradiction to $p_{\ell+1}^{(i)}(\bar{v}) = x_\ell$ as discussed in case (a) of Lemma C.2. Suppose $j > \ell$ be smallest such that $m_j^{(i)}(\bar{v}) > 0$. Then

$$p_j^{(i)}(\bar{v}) \geqslant x_j > x_\ell = p_{\ell+1}^{(i)}(\bar{v}). \tag{C.5}$$

By Lemma C.1(2), $p_{\ell+1}^{(i)}(\bar{v}) \geqslant \min(p_\ell^{(i)}(\bar{v}), p_j^{(i)}(\bar{v}))$. By (C.5) this implies $p_{\ell+1}^{(i)}(\bar{v}) \geqslant p_\ell^{(i)}(\bar{v})$. But $x_\ell = p_{\ell+1}^{(i)}(\bar{v}) \geqslant p_\ell^{(i)}(\bar{v}) \geqslant x_\ell$, hence we get $p_{\ell+1}^{(i)}(\bar{v}) = p_\ell^{(i)}(\bar{v})$. Again by Lemma C.1(3) since $m_k^{(i)}(\bar{v}) = 0$ for $\ell < k < j$ we get $p_{\ell+1}^{(i)}(\bar{v}) = p_j^{(i)}(\bar{v})$ which contradicts (C.5). Hence $m_j^{(i)}(\bar{v}) = 0$ for $j > \ell$. Also by Lemma C.1(1) $p_{\ell+1}^{(i)}(\bar{v}) = p_\ell^{(i)}(\bar{v})$ with $m_j^{(i)}(\bar{v}) = 0$ for $j > \ell$ implies that $m_j^{(i+1)}(\bar{v}) = 0$ for $j > \ell$. Since $\bar{v}^{(i+1)}$ and $\tilde{v}^{(i+1)}$ have the same shape we get $m_j^{(i+1)}(\tilde{v}) = 0$ for $j > \ell$. Hence $\tilde{\ell}^{(a)} = \ell^{(a)}$ for $1 \leqslant a \leqslant i-1$, $\tilde{\ell}^{(i)} = \ell+1$ and $\tilde{\ell}^{(i+1)} = \infty$. There-fore we proved that if $\Phi^{-1}(\bar{v}, \bar{J}) = b' \in B'$ then $\Phi^{-1}(v, J) = i \otimes b'$ and $\Phi^{-1}(\tilde{v}, \tilde{J}) = i+1 \otimes b'$. But $\tilde{f}_i(\bar{v}, \bar{J}) = 0$ implies $f_i(\Phi^{-1}(\bar{v}, \bar{J})) = 0$ since by induction we have that $\Phi^{-1} \circ \tilde{f}_i = f_i \circ \Phi^{-1}$ for B'. Hence $f_i(\Phi^{-1}(v, J)) = \Phi^{-1}(\tilde{v}, \tilde{J}) = \Phi^{-1}(\tilde{f}_i(v, J))$, so that indeed $f_i(b)$ is defined, $f_i(b')$ and $\Phi(f_i(b)) = \tilde{f}_i(v, J)$.

Now suppose that $f_i(b)$ is defined and $f_i(b')$ is undefined. This implies that $b = i \otimes b'$. By induction $\tilde{f}_i(\bar{v}, \bar{J})$ is undefined so that by Lemma 4.12 we have $\bar{p} = \bar{s}$ where $\bar{p} = p_j^{(i)}(\bar{v})$ for large j and $\bar{s}$ is the smallest label occurring in $(\bar{v}, \bar{J})^{(i)}$. Since b is obtained from b' by adding i it follows that the vacancy numbers change as $p := p_j^{(i)}(v) = \bar{p} + 1$ for large j under δ^{-1} and the new smallest label occurring in $(v, J)^{(i)}$ is $s = \bar{s}$. Hence $\tilde{\varphi}_i(v, J) = p - s = 1$, so that $\tilde{f}_i(v, J)$ is defined. It remains to prove that $\Phi(f_i(b)) = \tilde{f}_i(v, J)$. Note that $f_i(b) = i+1 \otimes b'$. Let ℓ be the length of the largest part in $(\bar{v}, \bar{J})^{(i)}$. Suppose that $\bar{v}^{(i-1)}$ or $\bar{v}^{(i+1)}$ has a part strictly bigger than ℓ. In this case $p_\ell^{(i)}(\bar{v}) < \bar{p} = \bar{s}$ contradicting the fact that $\bar{s} \leqslant p_\ell^{(i)}(\bar{v})$ is the smallest label occurring in $(\bar{v}, \bar{J})^{(i)}$. Hence both $\bar{v}^{(i-1)}$ and $\bar{v}^{(i+1)}$ have only parts of length less or equal to ℓ. Also by Lemma 3.6 we have $p_\ell^{(i)}(\bar{v}) = \bar{s} = s$ which shows that both δ^{-1} adding $i+1$ and $\tilde{f}_i$ pick the string of length ℓ in $(\bar{v}, \bar{J})^{(i)}$. Hence $\Phi(f_i(b)) = \tilde{f}_i(v, J)$.

Let us now consider case (2). Suppose that $\tilde{f}_i(v, J)$ is undefined and $\tilde{f}_i(\bar{v}, \bar{J})$ is defined. Again by Lemma 4.12 we have that $p = s$ where $p = p_j^{(i)}(v)$ for large j and s is the smallest label in $(v, J)^{(i)}$. If $\mathrm{rk}(v, J) < i+1$, then s is still the smallest label in $(\bar{v}, \bar{J})$ and by the change in vacancy numbers $\bar{p} \leqslant p$. Hence by Lemma 4.12 $\tilde{\varphi}_i(\bar{v}, \bar{J}) = \bar{p} - s \leqslant 0$ contradicting that $\tilde{f}_i(\bar{v}, \bar{J})$ is defined. Hence we must have $\mathrm{rk}(v, J) \geqslant i+1$. In fact we want to show that $\mathrm{rk}(v, J) = i+1$. Suppose $\mathrm{rk}(v, J) > i+1$. Then by the change in vacancy numbers by δ we have $\bar{p} = p = s$, so that $\tilde{\varphi}_i(\bar{v}, \bar{J}) = s - \bar{s}$. So to achieve $\tilde{\varphi}_i(\bar{v}, \bar{J}) > 0$ we need $\bar{s} < s$. This can only happen if $p_{\ell^{(i)}-1}^{(i)}(v) = s$ and $\ell^{(i-1)} < \ell^{(i)}$. If $m_{\ell^{(i)}-1}^{(i)}(v) > 0$, then the string of length $\ell^{(i)} - 1$ is singular. Since $\ell^{(i-1)} < \ell^{(i)}$ this contradicts the fact that δ picks the string of length $\ell^{(i)}$ in $(v, J)^{(i)}$. If $m_{\ell^{(i)}-1}^{(i)}(v) = 0$, by convexity Lemma C.1, we get a similar contradiction. Hence we have that $b = i+1 \otimes b$. Note that the above arguments also shows that $\tilde{\varphi}_i(\bar{v}, \bar{J}) = 1$ since $\bar{s} \geqslant s$ and $\bar{p} = p - 1$ if $\mathrm{rk}(v, J) = i+1$. Hence $f_i(b)$ is undefined since $\varphi_i(b') = \tilde{\varphi}_i(\bar{v}, \bar{J}) = 1$.

Consider Case (2) where $f_i(b)$ is undefined and $f_i(b')$ is defined. This implies that $b = i+1 \otimes b'$. By induction $\tilde{\varphi}_i(\bar{v}, \bar{J}) = \varphi_i(b') = 1$ so that by Lemma 4.12 we have $\bar{p} = \bar{s} + 1$. Hence $\tilde{\varphi}_i(v, J) = p - s = \bar{p} - 1 - s = \bar{s} - s$ by the change of vacancy numbers. Therefore $\tilde{\varphi}_i(v, J) = 0$ if $\bar{s} = s$. It remains to show that $p_{\ell+1}^{(i)}(v) \geqslant \bar{s}$ where $\ell := s^{(i)}$ is the length of the string in $(\bar{v}, \bar{J})^{(i)}$ selected by δ^{-1}. Hence the only problem occurs if $p_{\ell+1}^{(i)}(\bar{v}) = \bar{s}$ and $s^{(i-1)} < \ell$. If $m_{\ell+1}^{(i)}(\bar{v}) > 0$,

this means that there is a singular string of length $\ell + 1 > s^{(i)}$ in $(\bar{v}, \bar{J})^{(i)}$ contradicting the maximality of $s^{(i)}$. If $m_{\ell+1}^{(i)}(\bar{v}) = 0$ one can again use convexity to arrive at similar contradiction.

By exclusion case (3) follows from all the previous cases where at least one f_i or $\tilde{f}_i$ is defined. Now let $B = B^{r,1} \otimes B'$ where $r \geqslant 2$. Consider the following diagram:
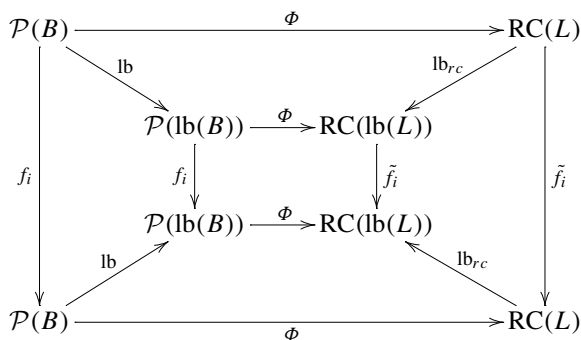
Again the top and the bottom faces commute because of Definition 4.6(3). The right face commutes by Lemma C.3. The left face commutes by definition of f_i on the paths and we know lb is injective. By induction hypothesis the back face commutes too. Hence the front face commutes.

Finally let $B = B^{r,s} \otimes B'$ where $s \geqslant 2$. Consider the following diagram:
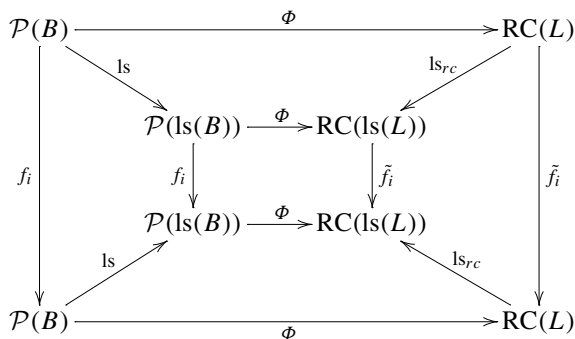
As in the previous cases by Definition 4.6(2), Lemma C.4 and induction hypothesis all the faces commute except the front. Since the map ls is injective the front face of the above diagram commutes. This completes the proof of Theorem 4.13. □

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