A class of irreducible representations of a Weyl group

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1. Let $V$ be a rational vector space of finite dimension, $R \subset \text{Hom}(V, \mathbb{Q})$ a (reduced) root system which generates $\text{Hom}(V, \mathbb{Q})$ and let $W \subset \text{Aut}(V)$ be the Weyl group of $R$. The purpose of this paper is to describe a class $\mathcal{F}_W$ of irreducible representations of $W$. This class arises naturally in the study of representations of a reductive group over a finite field (see [7, § 8]) and can be conjecturally related with the set of unipotent classes in that reductive group.

2. Let $E$ be an irreducible representation of $W$ (over $\mathbb{C}$). We shall associate with $E$ two polynomials $P_E(X), \bar{P}_E(X)$ with rational coefficients in an indeterminate $X$. Consider the graded $W$-module $\bar{S} = \sum_{t \geq 0} \bar{S}_t$, where $\bar{S}$ is the graded algebra of polynomial functions $V \otimes G \to \mathbb{C}$ modulo the ideal generated by $W$-invariant polynomials vanishing at 0. We set $P_E(X) = \sum_{t \geq 0} n_t X^t$ where $n_t$ is the multiplicity with which $E$ occurs in the $W$-module $\bar{S}_t$. Let $G$ be the adjoint Chevalley group over $k$ (an algebraic closure of the prime field $F_p$) with root system $R$. Let $G(q)$ be the group of $F_q$-rational points of $G$ (with respect to the standard $F_q$-rational structure of $G$), where $F_q$ is the subfield of $k$ with $q$ elements. Let us fix a homomorphism $\bar{h}: \mathbb{C}[X]^* \to \mathbb{C}$ ($\mathbb{C}[X]^* = \text{integral closure of } \mathbb{C}[X]$) such that $\bar{h}(X) = q$. It is known [2] that $\bar{h}$ gives rise to a 1 to 1 correspondence

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between the set of (isomorphism classes of) irreducible representations of \( W \) and the set of (isomorphism classes of) irreducible representations of \( G(q) \) occurring in \( \text{Ind}_{\mathbb{F}_q}^{G(q)}(1) \), where \( B \) is a Borel subgroup of \( G \) defined over \( \mathbb{F}_q \) and \( B(q) \) is its group of \( \mathbb{F}_q \)-rational points. The dimension of \( E_q \) is independent of the choice of \( h \), and equals \( \tilde{P}_E(q) \) where \( \tilde{P}_E(X) \) is a well-defined polynomial with rational coefficients, independent of \( q \).

The polynomials \( \tilde{P}_E(X) \) (also called “generic degrees”) have been computed in all cases (see [1] and the references there). The polynomials \( P_E(X) \) have been also computed in all cases (see [3] and the references there).

Let us write
\[
P_E(X) = c_{E} X^{a_E} + \ldots + d_{E} X^{b_E} \\
\tilde{P}_E(X) = c_{E} X^{\tilde{a}_E} + \ldots + d_{E} X^{\tilde{b}_E}
\]
where \( c_{E}, d_{E}, \tilde{c}_E, \tilde{d}_E \) are non-zero constants, \( a_E < b_E, \tilde{a}_E < \tilde{b}_E \) and the dots represent terms involving \( X^j \) with \( a_E < j < b_E \) (resp. \( \tilde{a}_E < j < \tilde{b}_E \)). It is an empirical observation that
\[
(2.1) \quad \tilde{a}_E < a_E < b_E < \tilde{b}_E.
\]

In the case where \( W \) is irreducible, we say that \( E \) is exceptional (cf. [2], [3]) if \( W \) is of type \( E_7 \) and \( \dim E = 512 \) or if \( W \) is of type \( E_8 \) and \( \dim E = 4096 \); if \( E \) is non-exceptional, we have
\[
(2.2) \quad \tilde{a}_E + \tilde{b}_E = a_E + b_E.
\]

This does not hold when \( E \) is exceptional: the sequences (2.1) corresponding to the two exceptional representations of a Weyl group of type \( E_7 \) are:
\[
11 < 11 < 51 < 52 \\
11 < 12 < 52 < 52;
\]
the sequences (2.1) corresponding to the four exceptional representations of a Weyl group of type \( E_8 \) are
\[
11 < 11 < 93 < 94 \\
11 < 12 < 94 < 94 \\
26 < 27 < 109 < 109 \\
26 < 26 < 108 < 109.
\]

**Definition.** In general, \( \mathcal{I}_W \) is the set of (isomorphism classes of) irreducible representation \( E \) of \( W \) which satisfy the equality \( \tilde{a}_E = a_E \).

In the case where \( W \) is a product of irreducible Weyl groups \( W_1, \ldots, W_m \) and \( E = E_1 \otimes \ldots \otimes E_m \) (with \( E_i \) irreducible \( W_i \)-modules) we have that \( E \in \mathcal{I}_W \) if and only if \( E_i \in \mathcal{I}_{W_i} \) for each \( i \). The identity representation
and the sign representation of \( W \) are always in \( \mathcal{S}_W \). In the case where \( W \) is irreducible, we have \( E \otimes \varepsilon_W \in \mathcal{S}_W \) whenever \( E \in \mathcal{S}_W \) is non-exceptional. (If \( E \in \mathcal{S}_W \) is exceptional, then \( E \otimes \varepsilon_W \notin \mathcal{S}_W \).) This follows immediately from (2.2) and from the known identities

\[
(2.3) \quad a_E \otimes \varepsilon_W + b_E = \bar{a}_E \otimes \varepsilon_W + \bar{b}_E = \nu(W)
\]

where \( \nu(W) \) is the number of reflections in \( W \).

In general, for \( E \in \mathcal{S}_W \), the constant \( \gamma_E \) (coefficient of \( X^{\alpha_E} \) in \( P_E(X) \)) is equal to 1. (This is an empirical observation.)

3. We now review a construction of [8] which generalizes a construction of I. G. Macdonald [9]. Let \( W' \) be a subgroup of \( W \). We can decompose uniquely \( V \) into a direct sum \( V = V' \oplus V'' \), where \( V'' \) is the set of \( W' \)-invariant vectors in \( V \) and \( V' \) is \( W' \)-stable. Let \( \mathcal{P}_i(V) \) be the space of homogeneous polynomials \( V \otimes \mathbb{C} \to \mathbb{C} \) of degree \( i \). We define similarly \( \mathcal{P}_i(V') \). The natural projection \( \pi: V \to V' \) induc es an injective linear map \( \pi^*: \mathcal{P}_i(V') \to \mathcal{P}_i(V) \). Let \( E_1 \) be an irreducible \( W' \)-submodule of \( \mathcal{P}_a(V') \) which occurs in \( \mathcal{P}_a(V) \) with multiplicity 1 and does not occur in \( \mathcal{P}_a(V') \) if \( i < a \). Then the \( W \)-submodule of \( \mathcal{P}_a(V) \) generated by \( \pi^*(E_1) \) is irreducible, it occurs with multiplicity 1 in \( \mathcal{P}_a(V) \) and it does not occur in \( \mathcal{P}_a(V') \) if \( i < a \) [8, (3.2)]; we denote it \( j^{W'}_W(E_1) \). (One could also characterize \( j^{W'}_W(E_1) \), up to isomorphism, as being the only irreducible submodule \( E \) of \( \text{Ind}^{W}_W(E_1) \) such that \( a_E = a \).)

4. Now let \( \Pi \) be a system of simple roots for the root system \( R \). For each subset \( I \) of \( \Pi \) we denote by \( R_I \) the root system in \( \text{Hom}(V, \mathbb{Q}) \) having \( I \) as set of simple roots and by \( W_I \subset \mathcal{W} \) the Weyl group of \( R_I \). We introduce some notations in \( G \). Let \( P_I \) be the parabolic subgroup of \( G \) containing \( B \), corresponding to \( I \). (Thus \( P_B = B \).) Let \( U_I \) be the unipotent radical of \( P_I \) and let \( L_I \) be a Levi subgroup of \( P_I \) defined over \( F_q \). We denote by \( P_I(q) \), \( U_I(q) \), \( L_I(q) \), the group of \( F_q \)-rational points of \( P_I \), \( U_I \), \( L_I \), respectively.

The following result gives a method of constructing representations in \( \mathcal{S}_W \):

**Proposition.** Let \( I \subset \Pi \) and let \( E_1 \in \mathcal{S}_{W_I} \). Then \( E = j^{W_I}_W(E_1) \in \mathcal{S}_W \) and \( a_E = a_{E_1} \).

We have \( \bar{a}_E = a_{E_1} = \bar{a}_{E_1} \). Hence it is enough to prove the following

**Lemma.** Let \( I \subset \Pi \) and let \( E_1 \) be an irreducible representation of \( W_I \). Let \( E \) be an irreducible \( W \)-submodule of \( \text{Ind}^{W_I}_W(E_1) \). Then \( \bar{a}_E < \bar{a}_{E_1} \).

Let \( E'_1 = E_1 \otimes \varepsilon_{W_I} \), \( E'_1 = E \otimes \varepsilon_W \). Using (2.3), we see that it is enough to prove the inequality

\[
(4.1) \quad \bar{b}_E < \bar{b}_{E'_1} + \nu(W) - \nu(W_I)
\]

where \( \nu(W_I) \) is the number of reflections in \( W_I \).
By assumption, the restriction of \( E \) to \( W_I \) contains \( E_1 \). Since \( \varepsilon_W|W_I = \varepsilon_{W_I} \), it follows that the restriction of \( E' \) to \( W_I \) contains \( E_1' \), hence \( E' \) is contained in \( \text{Ind}^W_{W_I}(E'_1) \). Let \( E'_q \) be the representation of \( G(q) \) corresponding to \( E' \) and let \( (E'_1)_q \) be the representation of \( L_I(q) \) corresponding to \( E'_1 \), as in 2. Then \( E'_q \) is contained in \( \text{Ind}^W_{W_I}((E'_1)_q) \), where \( (E'_1)_q \) is regarded as a \( P_I(q) \)-module with trivial action of \( U_I(q) \). It follows that

\[
\dim(E'_q) < \dim(\text{Ind}^W_{W_I}((E'_1)_q)) = \dim((E'_1)_q \cdot |G(q): P_I(q)|).
\]

We may regard the two sides of this inequality as polynomials in \( q \) with rational coefficients. Since this inequality is true for an arbitrary prime power \( q \), the polynomials in the left hand side must have a degree not bigger than that in the right hand side. This proves (4.1). A similar proof shows that, with the assumptions of the Proposition, we have \( \gamma_{E_1} > \gamma_E \). (Recall that \( \gamma_E \) is the coefficient of \( X^{\tilde{E}}_E \) in \( \tilde{P}_E(X) \).)

5. We shall now describe the set \( \mathcal{S}_W \) for each irreducible \( W \). (Here we make use of the results of [1], [3], see also the references there.)

Type \( A_{n-1} \). Any irreducible representation \( E \) of \( W \) is in \( \mathcal{S}_W \), since it is of the form \( j^W_{W_I}(\varepsilon_{W_I}) \) for some \( I \subset \Pi \). We have also \( P_B(X) = \tilde{P}_E(X) \).

Type \( B_n \) (or \( C_n \)). Let \( \lambda = (\lambda_1 < ... < \lambda_{m+1}) \), \( \beta = (\beta_1 < ... < \beta_m) \) be partitions such that \( \sum_{i=1}^{m+1} \lambda_i + \sum_{j=1}^{m} \beta_j = n \), \( \lambda_1 > 0 \), \( \beta_1 > 0 \). The dual partitions will be denoted by \( \lambda^* = (\lambda_1^* < ... < \lambda_{m+1}^*) \), \( \beta^* = (\beta_1^* < ... < \beta_m^*) \), respectively (with \( \lambda_1^* > 0 \), \( \beta_1^* > 0 \), if defined). Let \( R' \subset R \) be a root system of type \( A_{n-1} \) or of type \( B_n \). Define \( E_{\lambda, \beta} = j^W_{R'}(\varepsilon_{R'}) \). This gives a \( 1 \)-\( 1 \) correspondence between ordered pairs of partitions \( \lambda, \beta \) as above (with \( m \) arbitrary) and irreducible representations (up to isomorphism) of \( W \). (We regard the pair \( \lambda, \beta \) as being the same as the pair of partitions \( 0 < \lambda_1 < \lambda_2 < ... < \lambda_{m+1} \), \( 0 < \beta_1 < \beta_2 < ... < \beta_m \).)

It will be convenient to use a somewhat different parametrization for the representations of \( W \) (cf. [6, §§ 2, 3]).

Let \( W' \subset W \) be the Weyl group of \( R \). Define \( E_{\alpha, \beta} = j^W_{W'}(\varepsilon_{W'}) \). This gives a \( 1 \)-\( 1 \) correspondence between ordered pairs of partitions \( \alpha, \beta \) as above (with \( m \) arbitrary) and irreducible representations (up to isomorphism) of \( W \). (We regard the pair \( \alpha, \beta \) as being the same as the pair of partitions \( 0 < \alpha_1 < \alpha_2 < ... < \alpha_{m+1} \), \( 0 < \beta_1 < \beta_2 < ... < \beta_m \).)

It will be convenient to use a somewhat different parametrization for the representations of \( W \) (cf. [6, §§ 2, 3]).

Let \( \Phi_{n,1} \) be the set of arrays of integers \( \Lambda > 0 \)

\[
\Lambda = \left( \lambda_1 < \lambda_2 < ... < \lambda_{m+1} \right)
\]

such that \( \sum_{i=1}^{m+1} \lambda_i + \sum_{i=1}^{m} \mu_i = n+m^2 \) (\( m \) arbitrary), modulo the equiva-
lence relation given by

\[ \Lambda \sim \Lambda' = \left( 0 < \lambda_1 + 1 < \lambda_2 + 1 < \ldots < \lambda_{m+1} + 1 \right). \]

Then the set of ordered pairs of partitions \( \alpha, \beta \) as above can be put in \( 1-1 \) correspondence with the set \( \Phi_{n,1} \) by associating to \( \alpha, \beta \) the array \( \Lambda \) defined by \( \lambda_i = \alpha_i + i - 1 \) \((1 < i < m+1)\), \( \mu_j = \beta_j + j - 1 \) \((1 < j < m)\). If \( \alpha, \beta \) corresponds in this way to \( \Lambda \in \Phi_{n,1} \), we set \( E^\Lambda = E_{\alpha, \beta} \). From [6, § 2, 8.2] we have that

\[
\begin{align*}
\alpha_{B^\Lambda} &= 2 \sum_{1 \leq i < j \leq m+1} \inf (\lambda_i, \lambda_j) + 2 \sum_{1 \leq i < j \leq m} \inf (\mu_i, \mu_j) + \sum_{i=1}^{m} \mu_i \\
&\quad - \binom{2m-1}{2} - \binom{2m-3}{2} - \cdots \\
\tilde{\alpha}_{B^\Lambda} &= \sum_{1 \leq i < j \leq m+1} \inf (\lambda_i, \lambda_j) + \sum_{1 \leq i < j \leq m} \inf (\mu_i, \mu_j) + \sum_{1 \leq i \leq m+1} \inf (\lambda_i, \mu_i) \\
&\quad - \binom{2m-1}{2} - \binom{2m-3}{2} - \cdots 
\end{align*}
\]

hence

\[
\alpha_{B^\Lambda} - \tilde{\alpha}_{B^\Lambda} = \sum_{1 \leq i < j \leq m+1} \inf (\lambda_i, \lambda_j) + \sum_{1 \leq i < j \leq m} \inf (\mu_i, \mu_j) - \sum_{1 \leq i \leq m+1} \inf (\lambda_i, \mu_i) \\
\quad = \sum_{1 \leq i < j \leq m} (\lambda_i - \inf (\lambda_i, \mu_i)) + \sum_{1 \leq i < j \leq m+1} (\mu_i - \inf (\lambda_j, \mu_i)).
\]

This is, clearly, always \( > 0 \) and is equal to 0 precisely when

\[(5.1) \quad \lambda_1 < \mu_1 < \lambda_2 < \mu_2 < \ldots < \mu_m < \lambda_{m+1}.
\]

Thus, \( \mathcal{S}_W = \{ E^\Lambda | \Lambda \in \Phi_{n,1} \text{ satisfies (5.1)} \} \).

If \( E^\Lambda \notin \mathcal{S}_W \), \( \gamma_W \) equals \( 2 - c \) where \( c = m \) - number of equalities in (5.1).

**Type D** \((n > 2)\). Let \( \alpha = (\alpha_1 < \ldots < \alpha_m) \), \( \beta = (\beta_1 < \ldots < \beta_m) \) be partitions such that \( \sum_{i=1}^{m+1} \alpha_i + \sum_{j=1}^{m} \beta_j = n \), \( \alpha_1 > 0 \), \( \beta_1 > 0 \). Let \( \alpha' = (0 < \alpha_1 < \ldots < \alpha_m) \). Let \( E_{\alpha, \beta} \) be the representation of \( W \) obtained by restriction of the representation \( E_{\alpha', \beta} \) of the Weyl group of type \( B_n \) containing \( W \) as a subgroup of index 2. Then \( E_{\alpha, \beta} = E_{\beta, \alpha} \) is irreducible if \( \alpha \neq \beta \); on the other hand if \( \alpha = \beta \), \( E_{\alpha, \alpha} \) splits into two distinct irreducible \( W \)-modules \( E_{\alpha, \alpha}^I \), \( E_{\alpha, \alpha}^H \). All irreducible representations of \( W \) are obtained in this way. Let \( \Phi_{n,0} \) be the set of arrays of integers \( > 0 \)

\[
\Lambda = \left( \lambda_1 < \lambda_2 < \ldots < \lambda_m \right),
\]

\[
\mu_1 < \mu_2 < \ldots < \mu_m,
\]

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such that $\sum_{i=1}^{n} \lambda_i + \sum_{j=1}^{m} \mu_j = n + m(m-1)$ (m arbitrary), modulo the equivalence relation given by

$$\Lambda \sim \Lambda' = \left(0 < \lambda_1 + 1 < \lambda_2 + 1 < \ldots < \lambda_m + 1, 0 < \mu_1 + 1 < \mu_2 + 1 < \ldots < \mu_m + 1\right), \quad \Lambda \sim \left(\mu_1 < \mu_2 < \ldots < \mu_m\right) ;$$

we make the convention that each array such that $\lambda_i = \mu_i$ for all $i$ should be counted twice (i.e. it gives rise to two elements of $\Phi_{n,0}$). If $\alpha, \beta$ is a pair of partitions as above, with $\alpha \neq \beta$, define $\Lambda \in \Phi_{n,0}$ by $\lambda_i = \alpha_i + i - 1$ ($1 < i < m$), $\mu_j = \beta_j + j - 1$ ($1 < j < m$). We then set $e_{\alpha,\beta} = e^{\Lambda}$. If $\alpha = \beta$, the same formulae define two elements $\Lambda(I), \Lambda(II)$ of $\Phi_{n,0}$, and we set $e_{\alpha,\alpha}^{\Lambda(I)} = e_{\alpha,\alpha}^{\Lambda(II)}$.

Thus, we have a $1-1$ correspondence between $\Phi_{n,0}$ and the irreducible representations (up to isomorphism) of $W$.

From [6, § 2.8.2] we see that

$$a_{\beta} = 2 \sum_{1 \leq i < j \leq m} \inf (\lambda_i, \lambda_j) + 2 \sum_{1 \leq i < j \leq m} \inf (\mu_i, \mu_j) + \inf (\sum_{i=1}^{n} \lambda_i, \sum_{j=1}^{m} \mu_j) - \binom{2m-2}{2} - \binom{2m-4}{2} - \ldots$$

$$a_{\beta}^{\Lambda} = a_{\beta}^{\Lambda(I)} = a_{\beta}^{\Lambda(II)} = \sum_{1 \leq i < j \leq m} \inf (\lambda_i, \lambda_j) + \sum_{1 \leq i < j \leq m} \inf (\mu_i, \mu_j) + \sum_{1 \leq i < j \leq m} \inf (\lambda_i, \mu_j) - \binom{2m-2}{2} - \binom{2m-4}{2} - \ldots$$

when $\Lambda$ is such that $\lambda_i = \mu_i$ for some $i$ and

$$a_{\beta}^{\Lambda(I)} = a_{\beta}^{\Lambda(II)} = \tilde{a}_{\beta}^{\Lambda(I)} = \tilde{a}_{\beta}^{\Lambda(II)} =$$

$$4 \sum_{1 \leq i < j \leq m} \inf (\lambda_i, \lambda_j) + \sum_{i=1}^{n} \lambda_i - \binom{2m-2}{2} - \binom{2m-4}{2} - \ldots$$

when $\lambda_i = \mu_i$ for all $i$.

It follows easily that $\mathcal{S}_W$ consists of all $E^{\Lambda}$ ($\Lambda \in \Phi_{n,0}$) such that

$$(5.2) \quad \lambda_1 < \mu_1 < \lambda_2 < \mu_2 < \ldots < \lambda_m < \mu_m$$

or

$$\mu_1 < \lambda_1 < \mu_2 < \ldots < \mu_m < \lambda_m .$$

(In particular, when $\lambda_i = \mu_i$, for all $i$, both $E^{\Lambda(I)}$ and $E^{\Lambda(II)}$ are in $\mathcal{S}_W$.)

If $E^{\lambda_i}$ is in $\mathcal{S}_W$, where $\lambda_i \neq \mu_i$ for some $i$, we have $\tilde{\gamma}_{E^{\lambda_i}} = 2^{-c}$, where $c = (m-1)$ — number of equalities in (5.2). If $\Lambda$ satisfies $\lambda_i = \mu_i$ for all $i$, we have $\tilde{\gamma}_{E^{\Lambda(I)}} = \tilde{\gamma}_{E^{\Lambda(II)}} = 1$.

**Type $G_2$.** $\mathcal{S}_W$ consists of three representations: identity, reflection representation (on $V \otimes C$) and sign representation. The integers $a_{\beta}$ are given respectively by 0, 1, 6. The constants $\tilde{\gamma}_{E}$ are given respectively by $1, \frac{1}{6}, 1$. 328
**Type \(E_4\).** \(\mathcal{S}_W\) consists of 11 representations: identity (with \(a_E = 0, \widetilde{\gamma}_E = 1\)), sign (with \(a_E = 24, \widetilde{\gamma}_E = 1\)), reflection representation (with \(a_E = 1, \widetilde{\gamma}_E = \frac{1}{2}\)), reflection representation tensor with sign (with \(a_E = 13, \widetilde{\gamma}_E = \frac{1}{2}\)), the representation on \(\tilde{S}_2\) (with \(a_E = 2, \widetilde{\gamma}_E = 1\)), and its tensor product with sign (with \(a_E = 10, \widetilde{\gamma}_E = 1\)), the 4 eight dimensional representations (two of them with \(a_E = 3, \widetilde{\gamma}_E = 1\), the other two with \(a_E = 9, \widetilde{\gamma}_E = 1\)) and the twelve dimensional representation (with \(a_E = 4, \widetilde{\gamma}_E = \frac{1}{2}\)).

The representations of a Weyl group of type \(E_6, E_7\) or \(E_8\) will be denoted as in Frame [4, 5] and in [3].

**Type \(E_6\).** \(\mathcal{S}_W\) consists of:

<table>
<thead>
<tr>
<th>repres.</th>
<th>(a_E)</th>
<th>(\widetilde{\gamma}_E)</th>
<th>repres.</th>
<th>(a_E)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1_p)</td>
<td>0</td>
<td>1</td>
<td>(1_p')</td>
<td>38</td>
</tr>
<tr>
<td>(6_p)</td>
<td>1</td>
<td>1</td>
<td>(6_p')</td>
<td>25</td>
</tr>
<tr>
<td>(20_p)</td>
<td>2</td>
<td>1</td>
<td>(20_p')</td>
<td>20</td>
</tr>
<tr>
<td>(30_p)</td>
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<td>(\frac{1}{2})</td>
<td>(30_p')</td>
<td>15</td>
</tr>
<tr>
<td>(64_p)</td>
<td>4</td>
<td>1</td>
<td>(64_p')</td>
<td>13</td>
</tr>
<tr>
<td>(60_p)</td>
<td>5</td>
<td>1</td>
<td>(60_p')</td>
<td>11</td>
</tr>
<tr>
<td>(81_p)</td>
<td>6</td>
<td>1</td>
<td>(81_p')</td>
<td>10</td>
</tr>
<tr>
<td>(24_p)</td>
<td>6</td>
<td>(\frac{1}{2})</td>
<td>(24_p')</td>
<td>12</td>
</tr>
<tr>
<td>(80_p)</td>
<td>7</td>
<td>(\frac{1}{6})</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Whenever 2 representations are written in the same horizontal line, one equals the other tensored by sign; they have the same \(\widetilde{\gamma}_E\). (Similar conventions will be used for \(E_7\) and \(E_8\).)

**Type \(E_7\).** \(\mathcal{S}_W\) consists of:

<table>
<thead>
<tr>
<th>repres.</th>
<th>(a_E)</th>
<th>(\widetilde{\gamma}_E)</th>
<th>repres.</th>
<th>(a_E)</th>
</tr>
</thead>
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<td>1</td>
<td>(1_s')</td>
<td>63</td>
</tr>
<tr>
<td>(7_s)</td>
<td>1</td>
<td>1</td>
<td>(7_s')</td>
<td>46</td>
</tr>
<tr>
<td>(27_s)</td>
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<td>1</td>
<td>(27_s')</td>
<td>37</td>
</tr>
<tr>
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<td>(\frac{1}{2})</td>
<td>(56_s')</td>
<td>30</td>
</tr>
<tr>
<td>(21_s)</td>
<td>3</td>
<td>1</td>
<td>(21_s')</td>
<td>36</td>
</tr>
<tr>
<td>(120_s)</td>
<td>4</td>
<td>(\frac{1}{2})</td>
<td>(120_s')</td>
<td>25</td>
</tr>
<tr>
<td>(189_s)</td>
<td>5</td>
<td>1</td>
<td>(189_s')</td>
<td>22</td>
</tr>
<tr>
<td>(210_s)</td>
<td>6</td>
<td>1</td>
<td>(210_s')</td>
<td>21</td>
</tr>
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<td>(105_s)</td>
<td>6</td>
<td>1</td>
<td>(105_s')</td>
<td>21</td>
</tr>
<tr>
<td>(168_s)</td>
<td>6</td>
<td>1</td>
<td>(168_s')</td>
<td>21</td>
</tr>
<tr>
<td>(189_s)</td>
<td>7</td>
<td>1</td>
<td>(189_s')</td>
<td>20</td>
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Type $E_8$. $\mathcal{S}_W$ consists of:

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6. We now state a result which, in some sense, is a converse to the Proposition in 4.

**Theorem.** Assume that $W$ is irreducible $\neq \{1\}$, and let $E \in \mathcal{S}_W$. Then either $E$ or $E \otimes \epsilon_W$ is of the form $j_W^I(E_1)$, for some $I \subseteq \Pi$ and some $E_1 \in \mathcal{S}_W$.

When $W$ is of classical type the proof is elementary and will be omitted. When $W$ is an exceptional Weyl group, we have to appeal to the theory of Springer connecting representations of $W$ with unipotent classes in $G$ and we have to use also Dynkin's classification of unipotent classes. We shall first review the theory of Springer; after that we shall indicate a proof of the Theorem in the case where $W$ is of type $E_8$.

Assume that $p = \text{char } k$ is sufficiently large (with respect to the type of $G$). Let $u \in G$ be a unipotent element and let $A(u)$ be the group of components of the centralizer of $u$. To $u$ and to the identity representation of $A(u)$, Springer [11] associates an irreducible representation of $W$; the tensor product of this representation with the sign representation will be denoted $\rho_u$ (this agrees with the notation of [8]. For example, when $u = 1$, we have $\rho_u = \epsilon_W$. The map $u \to \rho_u$ defines an injective map from the set...
of unipotent conjugacy classes in $G$ to the set of irreducible representations of $W$.

Let $I \subset \Pi$ and let $v$ be an unipotent element in $L_I$. There exists a non-empty open subset of $U_I$ such that for $x$ in this subset, the $G$-conjugacy class of $vx$ is independent of $x$. Let $u = vx$ (with $x$ is this subset). Following [8] we say that the unipotent conjugacy class of $u$ in $G$ is "induced" by the class of $v$ in $L_I$. Let $\varphi_u$, $\varphi_v$ be the corresponding representations of $W$, $W_I$ respectively. Let $\beta(u)$ be the dimension of the variety of Borel subgroups of $G$ containing $u$. Define similarly $\beta(v)$ (with respect to $L_I$). If we assume that $P_{\varphi_u}(X) = X^{\beta(u)} + \text{higher order terms}$, we have

(6.1) \[ \varphi_u = j_{w_I}^{W}(\varphi_v) \]

(cf. [8, (3.5)] and $P_{\varphi_v}(X) = X^{\beta(v)} + \text{higher order terms}$ (since $\beta(u) = \beta(v)$, by [8, (1.3b)]).

7. Proof of the theorem for $W$ of type $E_8$. If $a$ is one of the numbers $0, 1, 2, 3, 4, 5, 7, 8, 9, 11, 16, 20, 23, 28, 30, 36, 42$, we can find $I_+ \subset \Pi$ such that $\nu(W_I) = a$. Also, the table shows that $\mathcal{S}_W$ contains a unique representation $E$ with $a_E = a$. This must be then equal to $j_{w_I}^{W}(e_{W_I})$. Next, $\mathcal{S}_W$ contains a unique representation $E$ with $a_E = 24$. This must be equal to $j_{w_I}^{W}(E_1)$ where $W_I$ is of type $D_7$ and $E_1$ is the representation of $W_I$ corresponding to

$$A = \begin{pmatrix} 0 & 1 & 2 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}. $$

(Note that $a_{E_1} = 24$.) $\mathcal{S}_W$ contains a unique representation $E$ with $a_E = 25$. This must be equal to $j_{w_I}^{W}(E_1)$ where $W_I$ is of type $E_6$ and $E_1$ is the reflection representation of $W_I$ tensored by the sign representation. (Note that $a_{E_1} = 25$.) $\mathcal{S}_W$ contains a unique representation $E$ with $a_E = 26$. This must be equal to $j_{w_I}^{W}(E_1)$ where $W_I$ is of type $E_6 \times A_1$ and $E_1$ is the reflection representation of $E_6$ tensored by the sign representation of $E_6 \times A_1$. (Note that $a_{E_1} = 25 + 1 = 26$.) $\mathcal{S}_W$ contains a unique representation $E$ with $a_E = 46$. This must be equal to $j_{w_I}^{W}(E_1)$ where $W_I$ is of type $E_7$ and $E_1$ is the reflection representation of $W_I$ tensored by the sign representation. (Note that $a_{E_1} = 46$.)

It is then enough to show that each of the two representations $\mathcal{S}_W$ with $a_E = 21$ and each of the two representations in $\mathcal{S}_W$ with $a_E = 14$ are of the form $j_{w_I}^{W}(E_1)$ for some $I_+ \subset \Pi$ and some $E_1 \in \mathcal{S}_{w_I}$.

Consider the unipotent elements $u, u'$ in $G$ with Dynkin diagrams

$$2 0 0 0 0 2 0 0 0 0 0 0 2$$

respectively. Then $u$ is induced by the identity element of $L_I \subset G$ with $W_I$ of type $D_5 \times A_1$ and $u'$ is induced by the identity element of $L_{I'} \subset G$ with $W_{I'}$ of type $A_6$. We have $\varphi_u = j_{w_I}^{W}(e_{w_I})$, 331
$\varphi_u = j_{W_J}^W(\varphi_{W_J})$. These are two distinct representations in $\mathcal{S}_W$ since $u, u'$ are not conjugate; they both have $a_B = 21$.

Consider now the unipotent elements $u_1, u_2$ in $G$ with Dynkin diagrams $\begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ respectively. Now $u_1$ is induced by a unipotent element $u_2$ in $L_J \subseteq G$ with $W_J$ of type $E_7$, whose Dynkin diagram is $\begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$. (Cf. [8, (1.9)].) Note that $\beta(u_2) = 14$. We shall show below that $\varphi_{u_2}$ is the representation in $\mathcal{S}_{W_J}$ denoted $378_a$. It can be also shown that $u_1$ is induced by the identity element of $L_{J'} \subseteq G$ with $W_{J'}$ of type $A_4 \times A_2 \times A_1$. (This is implicit in Mizuno's work [10, Lemma 45].) Thus

$$\varphi_{u_1} = j_{W_J}^W(\varphi_{u_2}) = j_{W_J}^W(378_a)$$

These are two distinct representations in $\mathcal{S}_W$ since $u_1, u_2$ are not conjugate; they both have $a_B = 14$. It remains to prove the following Lemma.

**LEMMA.** Assume that $W$ is of type $E_7$, and let $u \in G$ be a unipotent element with Dynkin diagram $\begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}$. Then $\varphi_u$ is the representation in $\mathcal{S}_W$ denoted $378_a$.

**PROOF OF THE LEMMA.** Let $C \subseteq PSO_{2n}(k)$ ($n > 3$) be the minimal unipotent class $\neq \{1\}$ and let $v \in C$. Then $\beta(v) = n^2 - 3n + 3$, $A(v) = \{1\}$. It is easy to see that the variety of Borel subgroups containing $v$ has exactly $n$ irreducible components. Hence $\dim(\varphi_v) = n$. Using, [11, (4.4)], one can show that the trace of a reflection on $\varphi_v$ equals $2 - n$. It follows that $\varphi_v$ is the reflection representation tensored with the sign representation of the Weyl group.

Now let $G$ be of type $E_7$ and let $I \subseteq II$ be such that $W_I$ is of type $D_5 \times A_1$. Consider a unipotent element $v_1 \in L_I$ whose projection to $PSO_{10}(k)$ is the $v$ considered above and whose projection to $PGL_3(k)$ is the identity. Let $\tilde{u}_1$ be a unipotent element in $G$ induced by $v_1$. We have $\beta(u_1) = \beta(\tilde{u}_1) = \beta(v_1) = 14$. But $G$ contains a unique unipotent class whose $\beta$ equals 14. Thus $u_1$ is conjugate to $\tilde{u}_1$. We have

$$\varphi_{u_1} = \varphi_{\tilde{u}_1} = j_{W_I}^W(\varphi_{v_1}) = j_{W_I}^W(E_1)$$

where $E_1$ is the reflection representation of $D_5$ tensored by the sign representation of $D_5 \times A_1$. Thus $\varphi_{u_1} \in \mathcal{S}_W$ and $a_{u_1} = 14$. The lemma follows.

**PROOF OF THE LEMMA.** Let $C \subseteq PSO_{2n}(k)$ ($n > 3$) be the minimal unipotent class $\neq \{1\}$ and let $v \in C$. Then $\beta(v) = n^2 - 3n + 3$, $A(v) = \{1\}$. It is easy to see that the variety of Borel subgroups containing $v$ has exactly $n$ irreducible components. Hence $\dim(\varphi_v) = n$. Using, [11, (4.4)], one can show that the trace of a reflection on $\varphi_v$ equals $2 - n$. It follows that $\varphi_v$ is the reflection representation tensored with the sign representation of the Weyl group.

Now let $G$ be of type $E_7$ and let $I \subseteq II$ be such that $W_I$ is of type $D_5 \times A_1$. Consider a unipotent element $v_1 \in L_I$ whose projection to $PSO_{10}(k)$ is the $v$ considered above and whose projection to $PGL_3(k)$ is the identity. Let $\tilde{u}_1$ be a unipotent element in $G$ induced by $v_1$. We have $\beta(u_1) = \beta(\tilde{u}_1) = \beta(v_1) = 14$. But $G$ contains a unique unipotent class whose $\beta$ equals 14. Thus $u_1$ is conjugate to $\tilde{u}_1$. We have

$$\varphi_{u_1} = \varphi_{\tilde{u}_1} = j_{W_I}^W(\varphi_{v_1}) = j_{W_I}^W(E_1)$$

where $E_1$ is the reflection representation of $D_5$ tensored by the sign representation of $D_5 \times A_1$. Thus $\varphi_{u_1} \in \mathcal{S}_W$ and $a_{u_1} = 14$. The lemma follows.

8. We now assume that $W$ is irreducible. Let $\kappa \in R$ be the root corresponding to the highest coroot. If $\tilde{I} \subseteq II \cup \{-\kappa\}$, we denote by $W_{\tilde{I}}$ the subgroup of $W$ generated by the reflections with respect to the roots in $\tilde{I}$. For each $E_1 \in \mathcal{S}_{W_{\tilde{I}}}$, it makes sense to consider $j_{W_{\tilde{I}}}^W(E_1)$. The set of all irreducible representations of $W$ (up to isomorphism) of the form $j_{W_{\tilde{I}}}^W(E_1)$

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for some \( I \subseteq \Pi \cup \{ -\alpha \} \) and some \( E_1 \in \mathcal{S}_{W_I} \) will be denoted \( \mathcal{F}_W \). For each \( E \in \mathcal{F}_W \) we define
\[
\alpha(E) = \sup \{ (\gamma - 1) I \subseteq \Pi \cup \{ -\alpha \}, \ E_1 \in \mathcal{S}_{W_I}, \ E = j_{W_I}^W(E_1) \}.
\]
We have always \( \mathcal{F}_W \supset \mathcal{S}_W \); the inclusion is strict unless \( W \) is of type \( A \).

Let us describe the set \( \mathcal{F}_W \) in the case of Weyl groups of classical type.

**Type \( B_n \).** \( \mathcal{F}_W \) consists of those representations \( E^A (A \in \Phi_{n,1}) \) for which
\[
\Lambda = (\lambda_1 < \lambda_2 < \ldots < \lambda_{m+1})
\]
satisfies
\[
(8.1) \quad \lambda_i < \mu_i (1 < i < m), \: \mu_i < \lambda_{i+1} + 1 (1 < i < m).
\]

**Type \( C_n \).** \( \mathcal{F}_W \) consists of those representations \( E^A (A \in \Phi_{n,1}) \) for which
\[
\Lambda = (\lambda_1 < \lambda_2 < \ldots < \lambda_{m+1})
\]
satisfies
\[
(8.2) \quad \lambda_i < \mu_i + 1 (1 < i < m), \: \mu_i < \lambda_{i+1} (1 < i < m).
\]

**Type \( D_n \).** \( \mathcal{F}_W \) consists of those representations \( E^A (A \in \Phi_{n,0}) \) for which
\[
\Lambda = (\lambda_1 < \lambda_2 < \ldots < \lambda_m)
\]
satisfies
\[
(8.3) \quad \lambda_i < \mu_i (1 < i < m), \: \mu_i < \lambda_{i+1} + 1 (1 < i < m-1).
\]

**Proposition.** Assume that \((W, R)\) is of type \( B_n, C_n \) or \( D_n \) and that \( p = \text{char}(k) \neq 2 \). Then there exists a 1–1 correspondence \( E \leftrightarrow \mathcal{E}_E \) between \( \mathcal{F}_W \) and the set of unipotent classes in \( G \) such that, if \( u_E \in \mathcal{E}_E \), we have \( \beta(u_E) = a_E \).

For type \( B_n \), we associate to \( E^A \in \mathcal{F}_W \) (see (8.1)) the partition of \( 2n+1 \) with parts
\[
2\lambda_1 + 1 + \delta_1 < 2\mu_1 - 1 + \delta'_1 < 2\lambda_2 - 1 + \delta_2 < \ldots < \\
< 2\lambda_i - 2i + 3 + \delta_i < 2\mu_i - 2i + 1 + \delta'_i < 2\lambda_{i+1} - 2i + 1 + \delta_{i+1} < \ldots < \\
< 2\lambda_{m+1} - 2m + 1 + \delta_{m+1}
\]
where
\[
\delta_i = \begin{cases} 
-1 & \text{if } \lambda_i = \mu_i \text{ and } i < m \\
1 & \text{if } \lambda_i = \mu_{i-1} - 1 \text{ and } i > 2 \\
0 & \text{otherwise}
\end{cases} \quad (1 < i < m+1)
\]
Using (8.1) we see that this partition is well defined and it has the property that every even part \( \neq 0 \) occurs an even number of times; hence there is a well defined unipotent element \( u_A \in \text{SO}_{2n+1}(k) \) (up to conjugacy) whose Jordan cells have sizes given by the parts of this partition.

For type \( C_n \), we associate to \( E^A \in \mathcal{T}_w \) (see (8.2); we assume as we may, that \( \lambda_1 = 0 \)) the partition of \( 2n \) with parts

\[
2\mu_1 + \delta_1 < 2\mu_2 - 2 + \delta'_1 < 2\mu_2 - 2 + \delta_2 < \ldots < \\
< 2\mu_i - 2i + \delta_i < 2\mu_{i+1} - 2i + \delta'_i < 2\mu_{i+1} - 2i + \delta_{i+1} < \ldots < \\
< 2\lambda_{m+1} - 2m + \delta'_m
\]

where

\[
\delta'_i = \begin{cases} 
-1 & \text{if } \lambda_{i+1} = \mu_i \\
1 & \text{if } \lambda_i = \mu_i + 1 \quad (1 \leq i < m) \\
0 & \text{otherwise} 
\end{cases}
\]

Using (8.2) we see that this partition is well defined and it has the property that every odd part occurs an even number of times; hence there is a well defined unipotent element \( u_A \in \text{Sp}_{2n}(k) \) (up to conjugacy) whose Jordan cells have sizes given by the parts of this partition.

For type \( D_n \), we associate to \( E^A \in \mathcal{T}_w \) (see (8.3)) the partition of \( 2n \) with parts

\[
2\lambda_1 + 1 + \delta_1 < 2\mu_1 - 1 + \delta'_1 < 2\mu_2 - 1 + \delta_2 < \ldots < \\
< 2\lambda_i - 2i + 3 + \delta_i < 2\mu_i - 2i + 1 + \delta'_i < 2\mu_{i+1} - 2i + 1 + \delta_{i+1} < \ldots < \\
< 2\lambda_{m+1} - 2m + 1 + \delta'_m
\]

where

\[
\delta_i = \begin{cases} 
-1 & \text{if } \lambda_{i+1} = \mu_i \\
1 & \text{if } \lambda_i = \mu_i - 1 \text{ and } i > 2 \quad (1 \leq i < m) \\
0 & \text{otherwise} 
\end{cases}
\]

and

\[
\delta'_i = \begin{cases} 
-1 & \text{if } \lambda_{i+1} = \mu_i - 1 \text{ and } i < m - 1 \\
1 & \text{if } \lambda_i = \mu_i \quad (1 \leq i < m) \\
0 & \text{otherwise} 
\end{cases}
\]

Using (8.3) we see that this partition is well defined and it has the
property that every even part ≠ 0 occurs an even number of times; hence in the case where \( \lambda_i \neq \mu_i \) for some \( i \) there is a well defined unipotent element \( u_A \in SO_{2n}(k) \) (up to conjugacy) whose Jordan cells have sizes given by the parts of this partition. In the case where \( \lambda_i = \mu_i \) for all \( i \), we get a partition of \( 2n \) all of whose parts are even and there are exactly two unipotent elements \( u_{A(1)}, u_{A(2)} \) whose Jordan cells have sizes given by the parts of this partition. We make them correspond to \( E_{A(1)}, E_{A(2)} \) respectively.

One checks that the correspondence we have defined has, in each case, the required properties.

9. We now state a conjecture in the general case. We assume that \( p \) is sufficiently large.

**CONJECTURE.** *The Springer construction \( u \to q_u \) defines a \( 1-1 \) correspondence between the set of unipotent classes in \( G \) and the set \( \mathcal{F}_W \). We have \( a_{q_u} = \beta(u) \) and \( \alpha(q_u) = |\lambda(u)|.\)

This conjecture suggests the existence of a remarkable set \( X \) of unipotent classes in \( G \): \( X \) consists of those unipotent classes for which the corresponding representation in \( \mathcal{F}_W \) is actually in \( \mathcal{F}_W \).

**REFERENCES**

10. Mizuno, K. – The conjugate classes of unipotent elements of the Chevalley groups \( E_n \) (\( n = 7 \) or 8). Preprint, University of Tokyo.

*Added in proof.* It can be verified, using recent results of T. Shoji, that our conjecture is true (except possibly for the formula for \( \alpha(q_u) \)) in the case where \( G \) is a classical group.