AGGREGATION OF NONNEGATIVE INTEGER-VALUED EQUATIONS

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Results are given for aggregating a system of nonnegative integer equations of the form

\[ x_j = b_j, \quad j \in N, \]

where all \( b_j \) are nonnegative integers. We seek integer weights \( w_j, j \in N, \) so that the single equation

\[ \sum_{j \in N} w_j x_j = \sum_{j \in N} w_j b_j \]

is uniquely solved by (1) when the \( x_j \) are constrained to nonnegative integers. The specification of such weights permits the aggregation of a system of general linear or nonlinear integer-valued equations of the form

\[ f_j(y) = b_j, \quad j \in N, \quad y \in Y \]

where the range of each \( f_j(y) \) is a subset of the nonnegative integers for \( y \in Y \). For this, \( x_j \) is simply replaced by \( f_j(y), y \in Y, \) in (1) and (2).

A number of results have been given for aggregating integer-valued equations (see, e.g., [1–9]). Babayev and Mamadov [2] have provided a noteworthy aggregation rule for the case where all \( b_j = 1 \), yielding weights that are significantly smaller than those obtained from the rules for aggregating equations with general (nonnegative integer) right hand sides. Smallest possible weights are desirable not only from a theoretical standpoint, but also from a practical (numerical analysis) standpoint, in order to keep the coefficients of the aggregated equation within a manageable range. Accordingly, it is of special interest to identify a rule for the case of general right hand sides that yields the same weights as [2] when all \( b_j = 1 \).
Such a rule is the contribution of this paper. Our result yields different outcomes for different permutations of the indexes \( j \in N \), based on the magnitudes of the \( b_j \) values. (When all \( b_j \) have the same value, every permutation yields the same outcome.) In addition, we identify a particular ordering of the \( b_j \) values that completely dominates all others, yielding weights which, when arranged in nondecreasing sequence, are smaller than the weights obtained by any non-equivalent ordering of \( b_j \) values.

**Notation and results**

Define

\[
a_k = 1 + \sum_{j=1}^{k-1} a_j b_j \quad \text{for } k \geq 2,
\]

or equivalently:

\[
a_k = \prod_{j=1}^{k-1} (b_j + 1) \quad \text{for } k \geq 1
\]

(understanding by convention \( \prod_{j=1}^{0} \) yields the value 1).

Note that \( a_k \) is precisely the number of integer vectors \((x_1, \ldots, x_{k-1})\) such that \( 0 \leq x_j \leq b_j \) for \( j = 1, \ldots, k-1 \).

Our main result is the following.

**Theorem.** Assume the \( b_j \) values are arranged so that \( b_p = 0 \) and \( b_q \neq 0 \) imply \( p > q \). Let

\[
w_j^* = \begin{cases} 
    a_{n+1} - a_j & \text{if } b_n \neq 0, \\
    2a_{n+1} - a_j & \text{if } b_n = 0.
\end{cases}
\]

Then, for \( w_j = w_j^* \), \( j \in N \), the solution to (2) in nonnegative integers is uniquely given by (1).

We require two intermediate results in order to prove this theorem. (We will also subsequently indicate alternative weights for the case \( b_n = 0 \), implied by the theorem itself.)

**Lemma 1.** Let \( u \) be any nonnegative integer, and let the \( b_j \) values be ordered arbitrarily. Then, an optimal solution \( x_j = x_j' \), \( j \in N \), to the problem

\[
\begin{align*}
\text{minimize} & \quad \sum_{j \in N} x_j \\
\text{subject to} & \quad \sum_{j \in N} a_j x_j = u, \\
& \quad x_j \geq 0 \text{ and integer, } \ j \in N
\end{align*}
\]
may be obtained by successively maximizing each \( x_j \), in reverse order of indexing, subject to \( \sum_{j \in N} a_j x_j \leq u \), i.e.

\[
x'_n = \left[ \frac{u}{a_n} \right] \quad \text{and} \quad x'_k = \left[ \frac{u - \sum_{j=k+1}^n a_j x'_j}{a_k} \right] \quad \text{for } k < n
\]

(where \([ \cdot ]\) denotes the largest integer not exceeding the quantity inside).

Moreover, this optimal solution is unique if

(a) all coefficients \( a_j \) have different values, or

(b) \( x'_j = 0 \) for all coefficients \( a_j \) that do not have unique values.

Proof. Note first the problem has a solution in nonnegative integers since \( a_1 = 1 \). Let \( x_j = x''_j \) denote any optimal solution and let \( p \) be the largest index such that \( x_p' < x_p'' \). If \( p \) exists, then

\[
\sum_{j=1}^{p-1} a_j x''_j = \sum_{j=1}^{p-1} a_j x'_j + a_p (x_p'' - x_p'),
\]

Thus, the solution given by \( x_j = x''_j - u_j \) for \( j \leq p - 1 \), \( x_p = x_p'' + 1 \), and \( x_j - x''_j \) for \( j > p \), must also be optimal. Further, granting the optimality of \( x'' \), there is exactly one index of such that \( u_a = 1 \), and \( u_j = 0 \) for all other \( j \), establishing \( a_a = a_p \). Hence, if \( x_j' = 0 \), for all \( a_j \) that are not unique, this situation cannot occur and \( x'' \) must be uniquely optimal. Otherwise, repetition of the foregoing process transforms \( x'' \) into \( x' \), establishing the optimality of \( x' \). This completes the proof.

It may be noted that Lemma 1 can be sharpened by specifying that the objective is to

\[
\text{minimize } \sum_{j \in N} c_j x_j
\]

where the coefficients \( c_j \) are any positive values satisfying \( c_j \leq c_j - 1 (b_j - 1 + 1) \) (for \( j \geq 2 \)). The uniqueness of the optimal solution than occurs when \( x_j' = 0 \), \( x_{j+1}' = 0 \) for all \( j \) such that \( c_j / a_j = c_{j+1} / a_{j+1} \). Also, the result applies to any positive coefficients \( a_j \) that divide all \( a_k \) for \( k > j \), since we allow each \( b_j \) to be any nonnegative integer.

Lemma 2. Assume that the indexing of the \( b_j \) values is restricted as in Theorem 1 so that 0 valued \( b_i \) are indexed last. Define \( a_0 = a_{n+1} - 1 \). Then, the following system

\[
\begin{align*}
\sum_{j \in N} x_j & \leq \sum_{j \in N} b_j, \\
\sum_{j \in N} a_j x_j & = a_0, \quad x_j \geq 0 \text{ and integer, } j \in N
\end{align*}
\]

has the unique solution \( x_j = b_j \), \( j \in N \).

Proof. Consider minimizing \( \sum_{j \in N} x_j \) subject to (3). From the definition of \( a_k \), for \( k = n + 1 \) we have \( a_0 = a_n (b_n + 1) - 1 \). Thus, by Lemma 1, \( x_n' = b_n \). This leaves a
residual right hand side of $a_n - 1$. But $a_n = a_{n-1}(b_{n-1} + 1)$. Hence, again by Lemma 1, $x'_n = b_{n-1}$. Repeating the process establishes that $x_j = b_j$, $j \in N$, minimizes $\sum_{j \in N} x_j$ subject to (3). It remains to show this minimizing solution is unique. Note the condition $a_j = a_{j+1}$ corresponds to $b_j = 0$. But by the assumption concerning the indexes $j$ for which $b_j = 0$, it follows that $b_{j-1} = 0$ also. This satisfies the stipulation of Lemma 1 for uniqueness, thereby completing the proof.

We are now ready to establish the validity of the main result.

Proof of the Theorem. Noting that $a_0 = \sum_{j \in N} a_j b_j$ (by Lemma 2) and rewriting equation (2) for $w_j = w_j^*$, as specified by the theorem, we obtain

$$Ma_{n+1} \left( \sum_{j \in N} x_j - \sum_{j \in N} b_j \right) = \sum_{j \in N} a_j x_j - a_0$$

where $M = 1$ if $b_n \neq 0$ and $M = 2$ if $b_n = 0$. Lemma 2 immediately implies that if $\sum_{j \in N} x_j = \sum_{j \in N} b_j$, then (1) provides the unique solution to (2).

Suppose the contrary. From $a_{n+1} = a_n(b_n + 1)$, we have

$$\sum_{j \in N} a_j x_j = a_0 + qM(a_n(b_n + 1)) \quad (4)$$

where $q$ is the nonzero integer identified by

$$\sum_{j \in N} x_j = \sum_{j \in N} b_j + q.$$

First, suppose $q < 0$, and let $x_j = x_j^*$, $j \in N$ be a nonnegative integer solution to (2) that satisfies (4). Then, letting $x_j = x_j^*$ for $j < n$ and $x_n = x_n^* + M(b_n + 1)(-q)$, we have a solution to (3) with $x_n \geq b_{n-1} + q$. But since $x_n \leq b_n$ for any solution to (3), this yields a contradiction.

Next suppose $q > 0$. By Lemma 1, a minimizing solution to $\sum_{j \in N} x_j$ subject to (4) may be obtained by setting

$$x_n = b_n + qM(b_n + 1)$$

and

$$x_j = b_j \quad \text{for } j < n.$$ 

But this yields $\sum_{j \in N} x_j = \sum_{j \in N} b_j + qM(b_n + 1)$, which exceeds $\sum_{j \in N} b_j + q$. Again, a contradiction is obtained, thus completing the proof.

It may be noted by the foregoing proof that if $b_n = 0$, then $M = 2$ can be replaced by $M = 1$, with $a_n(b_n + 1)$ replaced by $a_n(b_n + 2)$. However, this yields the same result as simply specifying $M = 2$, since when $b_n = 0$ we have $a_{n+1} = a_n$ and $a_n(b_n + 2) = 2a_n = 2a_{n+1}$.

Nevertheless, the theorem directly implies alternative weights for the case $b_n = 0$, as expressed by the following.
Corollary 1. For the ordering of the $b_j$ indicated in the theorem, let $q$ be the largest index such that $b_q \neq 0$. Weights $w_j^*$, $j \in \mathbb{N}$, for which (1) is the unique nonnegative integer solution to (2) may then be given by

$$w_j^* = a_{q+1} - a_j \quad \text{for } j \leq q,$$

$$w_j^* = \sum_{k=1}^{q} w_k^* b_k^* + 1 \quad \text{for } j > q.$$  

Proof. By the theorem, the form of the aggregated equation for $x_j = b_j$, $j = 1, \ldots, q$, is obtained for $w_j^*$, $j \leq q$, as specified in Corollary 1. Moreover, the right-hand side for this equation is precisely one less than the value specified for each $w_j^*$, $j > q$. Since $b_j = 0$ for $j > q$, this will also be the right-hand side for the aggregated equation over all $j \in \mathbb{N}$, and hence $x_j = 0$ must result for $j > q$. The corollary follows at once.

Our final concern is to arrange the $b_j$ values to produce the ‘best’ weights $w_j^*$, $j \in \mathbb{N}$, for both the Theorem and Corollary 1.

Corollary 2. The best (smallest) vector of weights $(w_1^*, w_2^*, \ldots, w_n^*)$ is given by indexing the $b_j$ so that

$$b_1 \geq b_2 \geq \cdots \geq b_n$$

Proof. Corollary 1’s prescription for $j \leq q$ is equivalent to defining

$$w_1^* = \prod_{k=1}^{q} (b_k + 1) - 1$$

and

$$w_j^* = \prod_{k=1}^{q} (b_k + 1) - \prod_{k=1}^{q} (b_k + 1) \quad \text{for } 1 < j \leq q.$$  

The indexing doesn’t affect $w_1^*$, but each remaining $w_j^*$, for $1 < j \leq q$, is clearly minimized by the ordering $b_1 \geq b_2 \geq \cdots \geq b_q$. Also, if $q < n$ ($b_n = 0$), then Corollary 1 yields

$$w_{q+1}^* = a_{q+1} \left( \sum_{j=1}^{q} b_j \right) - \sum_{j=1}^{q} a_j b_j + 1 = 2 + a_{q+1} \left( \sum_{j=1}^{q} b_j - 1 \right).$$

This is a constant value independent of the indexing. Thus, the 0-valued $b_j$ are also appropriately indexed.

The foregoing results raise some interesting issues. The aggregating equation for all $b_j = 1$ corresponds to that of (2), which has been conjectured to dominate all
other aggregating equations for this case. However, it is not true in the general case that the new equation yields uniformly better coefficients than other aggregating equations. This prompts the question: Is the generalization we have developed the only possible generalization of (2)? If so, this would seem to indicate that no one form of aggregation can be dominant in all situations. A second question is whether a specific range of $b_j$ values (other than all $b_j=1$) can be identified for which the new result is in fact 'best'. The number-theoretic flavor of the present development gives these questions intriguing overtones, but also suggests that they may be difficult to answer.

References