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Doubly warped products

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Abstract: In this paper we study geodesic completeness of Riemannian doubly warped products and Lorentzian doubly warped products. We give necessary conditions for generalized Robertson–Walker space-times with doubly warped product spacial parts to be globally hyperbolic. We also state some results about Killing and conformal vector fields of doubly warped products.

Keywords: Geodesics, warped products, Lie derivative.

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1. Introduction

Singly warped products or simply warped products were first defined by O'Neill and Bishop in [9]. They used this concept to construct Riemannian manifolds with negative sectional curvature. Then Beem, Ehrlich and Powell pointed out that many exact solutions to Einstein's field equation can be expressed in terms of Lorentzian warped products in [6]. Furthermore, Beem and Ehrlich concluded that causality and completeness of warped products can be related to causality and completeness of components of warped products in [5]. O'Neill discussed warped products and explored curvature formulas of warped products in terms of curvatures of components of warped products in [15]. He also examined Robertson–Walker, static, Schwarschild and Kruskal space-times as warped products. Also, Besse considered warped products as Riemannian submersions and obtained some results for special cases in [8]. A Lorentzian warped product (M, g) of the form $M = (c, d) \times {}_f F$ with the metric $g = -dt^2 \oplus f^2 g_F$ where $f\colon (c, d) \to (0, \infty)$ is smooth and $-\infty \leqslant c < d \leqslant \infty$ is a *generalized Robertson–Walker* space-time. Generalized Robertson–Walker space-times are considered as model space-times in relativity theory (cf. [5, 14] and [15]). In [21], some results about stability, geodesic completeness and geodesic connectedness of generalized Robertson–Walker space-times were stated. Geodesic connectedness and conjugate points are studied [12] and geodesic completeness of these spaces are considered in [18]. In [22], curvature and Killing vector fields of generalized Robertson–Walker space-times are also considered.

In general, *doubly warped products* can be considered as a generalization of singly warped products. A doubly warped product (M, g) is a product manifold which is of the form $M = {}_f B \times {}_b F$ with the metric $g = f^2 g_B \oplus b^2 g_F$ where $b\colon B \to (0, \infty)$ and $f\colon F \to (0, \infty)$ are

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smooth maps. Beem and Powell considered these products for Lorentzian manifolds in [7]. Then Allison considered causality and global hyperbolicity of doubly warped products in [1] and null pseudocovexity of Lorentzian doubly warped products in [3]. Conformal properties of doubly warped products are studied by Gebarowski (cf. [13] or references therein).

One can also generalize singly warped products to multiwarped products. Briefly, a *multiwarped product* (M, g) is a product manifold of the form $M = B \times {}_{f_1}F_1 \times {}_{f_2}F_2 \times \cdots \times {}_{f_m}F_m$ with the metric $g = g_B \oplus f_1^2 g_{F_1} \oplus f_2^2 g_{F_2} \oplus \cdots \oplus f_m^2 g_{F_m}$, where for each $i \in \{1, \dots, m\}$, $f_i \colon B \to (0, \infty)$ is smooth and (F_i, g_{F_i}) is a pseudo-Riemannian manifold. Covariant derivatives and curvatures of multiwarped products are given in [4] for $m = 2$. In particular, when $B = (c, d)$ with the negative definite metric $g_B = -dt^2$, the corresponding multiwarped product $M = (c, d) \times {}_{f_1}F_1 \times {}_{f_2}F_2 \times \cdots \times {}_{f_m}F_m$ with the metric $g = -dt^2 \oplus f_1^2 g_{F_1} \oplus f_2^2 g_{F_2} \oplus \cdots \oplus f_m^2 g_{F_m}$ is called a *multiwarped space-time,* where for each $i \in \{1, \dots, m\}$, (F_i, g_{F_i}) is a Riemannian manifold and $-\infty \leqslant c < d \leqslant \infty$. Geodesic equations and geodesic connectedness of multiply warped space-times were studied by Flores and Sánchez in [11] and they also noted that the class of multiply warped space-times contains many well known relativistic space-times. In [20], Sánchez studied geodesic connectedness of generalized Reissner–Nordström space-times and noted that Reissner–Nordström space-times can be expressed as multiply warped products. Geodesic equations and geodesic connectedness of multiwarped space-times are studied in [11]. In [24], necessary and sufficient conditions are obtained about geodesic completeness of multiwarped space-times.

There are various types of warped products in addition to the ones considered above and some of these have proven useful in general relativity. In [19], it is shown that general relativistic solutions can always be locally embedded in Ricci-flat five-dimensional spaces. Furthermore, in [17] some physically motivated D-dimensional solutions studied by Wesson and Ponce de Leon were extended to $(D + 1)$ dimensions and all these extensions turn out to be various types of warped products.

In Section 3, we consider geodesic completeness of Riemannian and Lorentzian doubly warped products. If (B, g_B) and (F, g_F) are complete Riemannian manifolds and $0 < \inf(f)$ or $0 < \inf(b)$, then (M, g) is complete where $M = {}_fB \times_b F$ with the metric $g = -f^2 g_B \oplus b^2 g_F$. Also, we prove that (B, g_B) and (F, g_F) are complete Riemannian manifolds when (M, g) is complete.

We examine null geodesic completeness of Lorentzian doubly warped products (M, g) of the form $M = {}_f(c, d) \times {}_bF$ with the metric $g = -f^2 dt^2 \oplus b^2 g_F$ where $-\infty \leqslant c < d \leqslant \infty$, and we get relations between null geodesic completeness of (M, g) and the divergence of $\int_c^{w_0} b(s)\, ds$, $\int_{w_0}^d b(s)\, ds$ and we obtain relations between timelike geodesic incompleteness of (M, g) and the convergence of

$$\int_c^{w_0} \frac{b(s)}{\sqrt{1 + b^2(s)}}\, ds \qquad \text{or} \qquad \int_{w_0}^d \frac{b(s)}{\sqrt{1 + b^2(s)}}\, ds$$

for some $w_0 \in (c, d)$ when $0 < \inf(f) < \sup(f) < \infty$.

A *generalized Robertson–Walker* space-time with a doubly warped product fiber (M, g) is a warped product of the form $M = (c, d) \times_h ({}_fB \times_b F)$ with the metric $g = -dt^2 \oplus h^2(f^2 g_B + b^2 g_F)$, where $h \colon (c, d) \to (0, \infty)$ is smooth and (B, g_B), (F, g_F) are Riemannian manifolds.

We show that if (B, g_B) and (F, g_F) are complete Riemannian manifolds and $0 < \inf(b)$ or $0 < \inf(f)$ then the generalized Robertson–Walker space-time with the doubly warped product fiber is globally hyperbolic.

Finally, we state the formula for the Lie Derivative of the metrics of doubly warped products and then we give necessary and sufficient conditions for vector fields $\overline{X}$ on M of the form $\overline{X} = X + V$, where $X \in \mathfrak{L}(B)$ and $V \in \mathfrak{L}(F)$ to be Killing or conformal.

2. Preliminaries

Throughout this work any manifold M is assumed to be connected, Hausdorff, paracompact and smooth. A *pseudo-Riemannian manifold* (M, g) is a smooth manifold with a metric tensor g and a *Lorentzian manifold* (M, g) is a pseudo-Riemannian manifold with signature $(-, +, +, \ldots, +)$.

A smooth curve $\gamma\colon I \to M$ in an arbitrary pseudo-Riemannian manifold is said to be a *pre-geodesic* if it can be reparametrized so that the repametriziation is a geodesic. A parameter s for a pre-geodesic γ is called an affine parameter if $\gamma''(s) = 0$.

The Lorentzian manifold (M, g) is timelike (respectively null, spacelike) *complete* if all timelike (respectively null, spacelike) inextendible geodesics are complete (i.e., can be defined on all of $\mathbb{R}$).

Let (M, g) be a pseudo-Riemannian manifold and X be a nonzero vector field on M. X is *Killing* if $L_X g = 0$ and X is *conformal* with *conformal factor* $\Omega\colon M \to \mathbb{R}$ if $L_X g = 2\Omega g$, where L_X denotes the Lie derivative with respect to X.

Let (B, g_B) and (F, g_F) be r and s dimensional pseudo-Riemannian manifolds, respectively. Throughout this paper we use the *natural product coordinate system* on the product manifold $B \times F$. Let (p_0, q_0) be a point in M. Then there are coordinate charts (U, x) and (V, y) on B and F, respectively such that $p_0 \in B$ and $q_0 \in F$. Then we can define a coordinate chart (W, z) on M such that W is an open subset in M contained in $U \times V$ and $(p_0, q_0) \in W$ then for all (p, q) in W, $z(p, q) = (x(p), y(q))$, where $\pi\colon B \times F \to B$ and $\sigma\colon B \times F \to F$ are usual projection maps and $x = (x^1, \ldots, x^r)$ and $y = (y^{r+1}, \ldots, y^n)$. Here, for our convenience, we call the jth component of y as y^{r+j} for all $j \in \{1, \ldots, s\}$.

Let $\phi\colon B \to \mathbb{R} \in \mathfrak{D}(B)$ then the lift of ϕ to $B \times F$ is $\widetilde{\phi} = \phi \circ \pi \in \mathfrak{D}(B \times F)$, where $\mathfrak{D}(B)$ is the set of all smooth real-valued functions on B.

Moreover, one can define lifts of tangent vectors as: Let $X_p \in T_p(B)$ and $q \in F$ then the lift $\widetilde{X}_{(p,q)}$ of X_p is the unique tangent vector in $T_{(p,q)}(B \times \{q\})$ such that $d\pi_{(p,q)}(\widetilde{X}_{(p,q)}) = X_p$ and $d\sigma_{(p,q)}(\widetilde{X}_{(p,q)}) = 0$. We will denote the set of all lifts of all tangent vectors of B by $L_{(p,q)}(B)$.

Similarly, we can define lifts of vector fields. Let $X \in \mathfrak{X}(B)$ then the lift of X to $B \times F$ is the vector field $\widetilde{X} \in \mathfrak{X}(B \times F)$ whose value at each (p, q) is the lift of X_p to (p, q). We will denote the set of all lifts of all vector fields of B by $\mathfrak{L}(B)$.

Definition 2.1. Let (B, g_B) and (F, g_F) be pseudo-Riemannian manifolds and also let $b\colon B \to (0, \infty)$ and $f\colon F \to (0, \infty)$ be smooth functions. The doubly warped product is the product manifold $B \times F$ furnished with the metric tensor $g = f^2 g_B \oplus b^2 g_F$ defined by

$$g = (f \circ \sigma)^2 \pi^*(g_B) \oplus (b \circ \pi)^2 \sigma^*(g_F). \tag{2.1}$$

The functions $b\colon B \to (0,\infty)$ and $f\colon F \to (0,\infty)$ are called *warping* functions.

If (B, g_B) and (F, g_F) are both Riemannian manifolds, then $({}_fB \times {}_bF,\ f^2 g_B \oplus b^2 g_F)$ is also a Riemannian manifold. We call $({}_fB \times {}_bF,\ f^2 g_B \oplus b^2 g_F)$ a *Lorentzian doubly warped product* if (F, g_F) is Riemannian and either (B, g_B) is Lorentzian or else (B, g_B) is a one-dimensional manifold with a negative definite metric $-dt^2$. If neither b nor f is constant, then we have a *proper doubly warped product*.

By using the covariant derivative formulas for doubly warped products which can be found in [2], we can easily state the followings.

Proposition 2.2. *Let $M = {}_fB \times {}_bF$ be a pseudo-Riemannian doubly warped product with metric $g = f^2 g_B \oplus b^2 g_F$. Then*

(1) *The leaves $B \times \{q\}$ and the fibers $\{p\} \times F$ of the double warped product are totally umbilic.*

(2) *The leaf $B \times \{q\}$ is totally geodesic if* $\operatorname{grad}_F(f)|_q = 0$. *Similarly, the fiber $\{p\} \times F$ is totally geodesic if* $\operatorname{grad}_B(b)|_p = 0$.

Now, we will state the geodesic equations for doubly warped products. The version for singly warped products is well known, compare [15].

Proposition 2.3. *Let $M = {}_fB \times {}_bF$ be a pseudo-Riemannian doubly warped product with metric $g = f^2 g_B \oplus b^2 g_F$. Also let $\gamma = (\alpha, \beta)$ be a curve defined on some interval $I \subseteq \mathbb{R}$. Then $\gamma = (\alpha, \beta)$ is a geodesic if and only if for any $t \in I$,*

$$(1)\qquad \alpha'' = \frac{(b \circ \alpha)}{(f \circ \beta)^2}\, g_F(\beta', \beta')\, \operatorname{grad}_B(b) - \frac{2}{(f \circ \beta)} \frac{d(f \circ \beta)}{dt}\, \alpha'.$$

$$(2)\qquad \beta'' = \frac{(f \circ \beta)}{(b \circ \alpha)^2}\, g_B(\alpha', \alpha')\, \operatorname{grad}_F(f) - \frac{2}{(b \circ \alpha)} \frac{d(b \circ \alpha)}{dt}\, \beta'.$$

3. Completeness of doubly warped products

In this section, we obtain some results on *completeness* of *Lorentzian warped products* and *Riemannian warped products*.

3.1. The Riemannian case

In this subsection, we state some results about *completeness* of *Riemannian warped products*.

Proposition 3.1. *Let $M = {}_fB \times {}_bF$ be a Riemannian doubly warped product with the metric $g = f^2 g_B \oplus b^2 g_F$. If (B, g_B) and (F, g_F) are complete Riemannian manifolds and also* $\inf(b) > 0$ *or* $\inf(f) > 0$ *then (M, g) is a complete Riemannian manifold.*

Proof. Without loss of generality assume that $\inf(f) = \lambda > 0$. Note first that for any vector field X in M we have $\lambda^2 g_B(\pi(X), \pi(X)) + b^2 g_F(\sigma(X), \sigma(X)) \leqslant g(X, X)$. The first metric is a complete Riemannian (singly) warped metric by [15]. Clearly, this implies that the second metric is also complete.

We obtain the following result about global hyperbolicity of generalized Robertson–Walker space-times with doubly warped product fibers by using [5, Theorem 3.66] and the previous result.

Corollary 3.2. *Let $M = (c, d) \times_h ({}_f B \times_b F)$ be a Lorentzian singly warped product with the metric $g = -dt^2 \oplus h^2(f^2 g_B + b^2 g_F)$, where $h: (c, d) \to (0, \infty)$ is smooth and $-\infty \leqslant c < d \leqslant \infty$. If (B, g_B) and (F, g_F) are complete Riemannian manifolds and also if* $\inf(b) > 0$ *or* $\inf(f) > 0$, *then (M, g) is globally hyperbolic.*

Proposition 3.3. *Let $M = {}_f B \times_b F$ be a complete doubly Riemannian warped product with the metric $g = f^2 g_B \oplus b^2 g_F$. If (M, g) is a complete Riemannian manifold, then (B, g_B) and (F, g_F) are complete Riemannian manifolds.*

Proof. Let (p_n) be Cauchy in B. Then for a fixed $q \in F$ we have (p_n, q) is Cauchy in M. Because $d((p_n, q), (p_m, q)) = f(q) d_B(p_n, p_m)$. Thus there is a point $(p, q) \in M$ such that $\lim(p_n, q) = (p, q)$. Then since $d((p_n, q), (p, q)) = f(q) d_B(p_n, p)$ we have $\lim(p_n) = p$. Hence B is complete and so is F.

3.2. The Lorentzian Case

We now consider the nonspacelike geodesic completeness of *Lorentzian warped products* of the form $M = {}_f(c, d) \times_b F$ with the metric $g = -f^2 dt^2 \oplus b^2 g_F$ where $-\infty \leqslant c < d \leqslant \infty$. Here a space-time is said to be null (respectively, timelike) geodesically incomplete if some future directed null (respectively, timelike) geodesic can not be extended to be defined for arbitrary negative and positive values of an affine parameter. Since we are using the metric $-dt^2$ on (c, d), the curve $\gamma(t) = (t/f(q_0), q_0)$ with $q_0 \in F$ fixed and $\mathrm{grad}_F(f)(q_0) = 0$ is a unit speed timelike geodesic (M, g) independent of which warping function b is chosen and independent of f at other points of M.

Now, we will state the following fact to obtain some integral conditions for null geodesic completeness of Lorentzian doubly warped products.

Assume that (M, g) is a pseudo-Riemannian manifold and $\Omega: M \to (0, \infty)$ is a smooth map. Then we will call (M, g) is conformal to $(M, \Omega^2 g)$ with the conformal factor Ω. It is well known that any geodesic in (M, g) is also a pregeodesic in $(M, \Omega^2 g)$ so it is natural to have relations between null geodesic completeness of two conformal manifolds, i.e., (M, g) and $(M, \Omega^2 g)$.

Thus by using the affine parameter converting formula in [23, Problem 9.27] we state the followings:

(1) If (M, g) is null geodesically incomplete and $\sup(\Omega) < \infty$, then $(M, \Omega^2 g)$ is also null geodesically incomplete.

(2) If (M, g) is null geodesically complete and $\inf(\Omega) > 0$, then $(M, \Omega^2 g)$ is also null geodesically complete.

(3) If $0 < \inf(\Omega) < \sup(\Omega) < \infty$, then both (M, g) and $(M, \Omega^2 g)$ are null geodesically complete or incomplete.

Let $M = {}_f(c, d) \times {}_b F$ be a Lorentzian warped product with the metric $g = -f^2 dt^2 \oplus b^2 g_F$ where $-\infty \leqslant c < d \leqslant \infty$. Then the singly warped product, i.e., $((c, d) \times F, dt^2 \oplus b^2 f^{-2} g_F)$ is conformal to the original doubly warped product, i.e., $((c, d) \times F, -f^2 dt^2 \oplus b^2 g_F)$, with the conformal factor f. Hence one can easily obtain the following results by using the previous facts and [5, Theorem 3.70 and Remark 3.71].

Theorem 3.4. *Let $M = {}_f(c, d) \times {}_b F$ be a Lorentzian warped product with the metric $g = -f^2 dt^2 \oplus b^2 g_F$ where $-\infty \leqslant c < d \leqslant \infty$. Suppose that* $\sup(f) < \infty$ *and (F, g_F) is complete then*

(1) *if $\int_c^{w_0} b(s)\,ds < \infty$ for some $w_0 \in (c, d)$ then every future directed null geodesic is past incomplete.*

(2) *if $\int_{w_0}^d b(s)\,ds < \infty$ for some $w_0 \in (c, d)$ then every future directed null geodesic is future incomplete.*

Note that $(F, f^{-2} g_F)$ is complete when $\sup(f) < \infty$ and (F, g_F) is complete.

Theorem 3.5. *Let $M = {}_f(c, d) \times {}_b F$ be a Lorentzian warped product with the metric $g = -f^2 dt^2 \oplus b^2 g_F$ where $-\infty \leqslant c < d \leqslant \infty$. Suppose that* $\inf(f) > 0$ *and both (F, g_F) and $(F, f^{-2} g_F)$ are complete then*

(1) *if $\int_c^{w_0} b(s)\,ds = \infty$ for some $w_0 \in (c, d)$ then every future directed null geodesic is past complete.*

(2) *if $\int_{w_0}^d b(s)\,ds = \infty$ for some $w_0 \in (c, d)$ then every future directed null geodesic is future complete.*

Corollary 3.6. *Let $M = {}_f(c, d) \times {}_b F$ be a Lorentzian warped product with the metric $g = -f^2 dt^2 \oplus b^2 g_F$ where $-\infty \leqslant c < d \leqslant \infty$. Suppose that* $0 < \inf(f) < \sup(f) < \infty$ *and (F, g_F) is complete*

(1) *$\int_c^{w_0} b(s)\,ds = \infty$ for some $w_0 \in (c, d)$ if and only if every future directed null geodesic is past complete.*

(2) *$\int_{w_0}^d b(s)\,ds = \infty$ for some $w_0 \in (c, d)$ if and only if every future directed null geodesic is future complete.*

Here, we state a result concerning the null geodesic completeness of arbitrary pseudo-Riemannian doubly warped products.

Proposition 3.7. *Let $M = {}_f B \times {}_b F$ be a null complete pseudo-Riemannian doubly warped product with the metric $g = f^2 g_B \oplus b^2 g_F$. Then (B, g_B) and (F, g_F) are null complete pseudo-Riemannian manifolds.*

Proof. Let α be a null geodesic in B and let $\gamma = (\alpha, q)$ for $q \in F$. Then using $\alpha'' = 0$, $g_B(\alpha', \alpha') = 0$ and $\beta' = 0$ it follows from Proposition 2.3 that γ is a null geodesic in M.

In the following example, we will show that the converse of Proposition 3.7 is false, i.e., if (B, g_B) and (F, g_F) are null complete pseudo-Riemannian manifolds then (M, g) is not necessarily a null complete pseudo-Riemannian doubly warped product. Note that this example is a modification of an example of an incomplete warped product with a complete base and fiber (cf. [5, p. 108]).

Example 3.8. Take $B = F = \mathbf{L^2}$ where $\mathbf{L^2} = \mathbb{R}_1 \times \mathbb{R}$ is 2-dimensional *Minkowski* space with the metric $ds^2 = -dt^2 + dx^2$ and define $b\colon \mathbf{L^2} \to (\mathbf{0}, \infty)$, $f\colon \mathbf{L^2} \to (\mathbf{0}, \infty)$ as $b(x, y) = \exp(-x)$ and $f(u, v) = \exp(-u)$. We will show that $M = {}_f B \times {}_b F$ is not a null complete pseudo-Riemannian doubly warped product with the metric $g = f^2 g_B \oplus b^2 g_F$. To do that let's define $\alpha\colon (-\infty, \infty) \to B$ and $\beta\colon (-\infty, \infty) \to F$ as $\alpha(t) = (t, t)$ and $\beta(t) = (t, t)$. Clearly, α and β are complete null geodesics of B and F, respectively. Also, if $\gamma = (\alpha, \beta)$ then γ is a null pre-geodesic in M and $\gamma'' = -\gamma'$ by equations in Proposition 2.3. By using [23], one can compute the affine parameter as $p(t) = \int_0^t \exp(\int_0^u -ds)\,dt$, i.e., $p(t) = 1 - 1/\exp(t)$. Thus $\lim_{t\to\infty} p(t)$ is finite. Hence, γ is incomplete.

Here, we will call a timelike geodesic $\gamma = (\alpha, \beta)\colon I \to M$ as stationary if $\gamma(t) = (t, q)$ for any $t \in I$ and for some $q \in F$, where $\mathrm{grad}_F(f)(q) = 0$. In order to consider timelike geodesic completeness of doubly warped products, we first state the length of a unit speed non-stationary timelike geodesic in a Lorentzian doubly warped product of the form $M = {}_f(c, d) \times_b F$. Clearly, if the length is finite, then the geodesic will be incomplete.

Proposition 3.9. *Let $M = {}_f(c, d) \times {}_b F$ be a Lorentzian doubly warped product with the metric $g = -f^2 dt^2 \oplus b^2 g_F$ where $-\infty \leqslant c < d \leqslant \infty$ also let $\gamma = (\alpha, \beta)\colon I \to M$ be a unit speed non-stationary timelike geodesic in M. Assume that $0 \in I$ and $\alpha(0) = t_0 \in (c, d)$ also $\alpha'(0) = r > 0$ and if $L(\gamma)|_{t=t_1}^{t=t_2}$ denotes the length of γ between $t = t_1$ and $t = t_2$ and also $b_\alpha = b \circ \alpha$, $f_\beta = f \circ \beta$, then*

$$L(\gamma)|_{t=t_1}^{t=t_2} = \int_{t_1}^{t_2} \frac{(f_\beta^2 b_\alpha)(\alpha^{-1}(t))}{\sqrt{\int_0^{\alpha^{-1}(t)} f_\beta^2(x)(b_\alpha^2)'(x)dx + r^2(f_\beta^4 b_\alpha^2)(0)}}\, dt.$$

Proof. First, note that 0 may not be in I but without loss of generality we always suppose $0 \in I$. By Proposition 2.3(1) and $g(\gamma'(s), \gamma'(s)) = -1$, we have

$$-(f \circ \beta)^2(s)(\alpha'(s))^2 + (b \circ \alpha)^2(s) g_F(\beta'(s), \beta'(s)) = -1. \tag{3.2}$$

Note that $\alpha'(s) \neq 0$ for any $s \in I$. For simplicity, we will use the following notations: Let $\alpha(s) = t$ then $u = \alpha'(s) = dt/ds$ and $\alpha''(s) = d^2t/ds^2 = du/ds = u'$. Also $b \circ \alpha(s) = b_\alpha(s)$ and $f \circ \beta(s) = f_\beta(s)$. Then $d(f \circ \beta)(s)/ds = (f \circ \beta)'(s) = f_\beta'(s)$.

Now, $db/dt|_{\alpha(s)} = \dot{b}(\alpha(s))$ and by the change of variables formula for differentiation, we have

$$\dot{b}(\alpha(s)) = \frac{d(b \circ \alpha)(s)}{ds}\frac{ds}{dt} = b_\alpha'(s)\frac{1}{u}. \tag{3.3}$$

Then by using equation (3.3) and $g_B = -dt^2$, we have

$$\mathrm{grad}_B(b)|_{b(\alpha(s))} = -\dot{b}(\alpha(s)) = -b_\alpha'(s)\frac{1}{u}. \tag{3.4}$$

Also by using equation (3.2) and the notations introduced above we have

$$g_F(\beta', \beta') = \frac{-1 + f_\beta^2(s)u^2}{b_\alpha^2(s)}. \tag{3.5}$$

Thus equation (3.2) becomes

$$2uu' + u^2\left(4\frac{f_\beta'}{f_\beta} + 2\frac{b_{\alpha'}}{b_\alpha}\right) = 2\frac{b_{\alpha'}}{f_\beta^2 b_\alpha}. \tag{3.6}$$

This yields,

$$u = \frac{1}{f_\beta^2(s)b_\alpha(s)}\sqrt{\int_0^s f_\beta^2(x)(b_\alpha^2)'(x)\,dx + c}\,. \tag{3.7}$$

Recall that $\alpha(0) = t_0$, $\alpha(s) = t$ and $dt/ds|_{s=0} = \alpha'(0) = r$ also $ds/dt|_{t=t_0} = 1/r$. Thus, by equation (3.7)

$$c = \ r^2(f_\beta^4 b_\alpha^2)(0). \tag{3.8}$$

Hence, by equations (3.7) and (3.8)

$$s = \int_{w_0}^t \frac{(f_\beta^2 b_\alpha)(\alpha^{-1}(t))}{\sqrt{\int_0^{\alpha^{-1}(t)} f_\beta^2(x)(b_\alpha^2)'(x)\,dx + r^2(f_\beta^4 b_\alpha^2)(0)}}\,dt. \tag{3.9}$$

Finally, we obtain the result by equation (3.9) and $L(\gamma)|_{t=t_1}^{t=t_2} = s_2 - s_1$.

Theorem 3.10. *Let* $M = {}_f(c,d) \times {}_bF$ *be a Lorentzian warped product with the metric* $g = -f^2dt^2 \oplus b^2 g_F$ *where* $-\infty \leqslant c < d \leqslant \infty$. *Suppose that* $0 < \inf(f) < \sup(f) < \infty$ *then*

(1) *if* $\int_c^{w_0} b(s)/\sqrt{1+b^2(s)}\,ds < \infty$ *for some* $w_0 \in (c,d)$ *then every future directed timelike geodesic is past incomplete.*

(2) *if* $\int_{w_0}^d b(s)/\sqrt{1+b^2(s)}\,ds < \infty$ *for some* $w_0 \in (c,d)$ *then every future directed timelike geodesic is future incomplete.*

Proof. Let γ be an arbitrary future directed timelike geodesic in (M, g). Then we will use Proposition 3.9 to prove the conclusion. First of all by equation (3.5) we have that $u^2 f_\beta^2 > 1$. This implies that $u^2 > 1/(\sup(f))^2 > 0$. Also, equations (3.2) and (3.7) imply that $\int_0^s f_\beta^2(x)(b_\alpha^2)'(x)\,dx + c > f_\beta^2(s)(b_\alpha^2)(s)$.

We will define some notation for simplicity

$$I_1 = \int_{w_0}^{d} \frac{f_\beta^2 b_\alpha(s)}{\sqrt{\int_0^s f_\beta^2(x)(b_\alpha^2)'(x)\,dx + c}}\,ds,$$

$$I_2 = \int_{w_0}^{d} \frac{f_\beta^2 b_\alpha(s)}{\sqrt{\int_0^s f_\beta^2(x)(b_\alpha^2)'(x)\,dx + c + 1}}\,ds$$

and

$$I_3 = \int_{w_0}^{d} \frac{f_\beta^2 b_\alpha(s)}{\sqrt{f_\beta^2(s)(b_\alpha^2)(s) + 1}}\,ds.$$

Thus, by the limit comparison test for integrals we have that I_1 and I_2 are both convergent or divergent together because

$$1 < \lim_{t\to d^-} \left(1 + \frac{1}{\int_{w_0}^{t} f_\beta^2(x)(b_\alpha^2)'(x)dx + c}\right) < \infty.$$

Suppose that the above limit is divergent i.e., $\lim_{t\to d^-} \int_{w_0}^{t} f_\beta^2(x)(b_\alpha^2)'(x)\,dx + c = 0$. But then by equation (3.7) we have $\lim_{t\to d^-} u(t) = 0$ and this contradicts the facts that $u^2 > 1/(\sup(f))^2 > 0$ and $\int_0^s f_\beta^2(x)(b_\alpha^2)'(x)\,dx + c > f_\beta^2(s)(b_\alpha^2)(s)$. On the other hand by the comparison test for integrals we have that $I_2 < I_3$. Hence, we conclude that $\lim_{t_2\to d^-} L(\gamma)|_{t=t_1}^{t=t_2}$ is finite if I_3 is convergent and by using the assumption that $0 < \inf(f) < \sup(f) < \infty$ it is easy to see that I_3 is convergent when $\int_{w_0}^{d} b(s)/\sqrt{1+b^2(s)}\,ds$ is convergent.

Corollary 3.11. *Let* $M = {}_f(c,d) \times {}_bF$ *be a Lorentzian warped product with the metric* $g = -f^2dt^2 \oplus b^2 g_F$ *where* $-\infty \leqslant c < d \leqslant \infty$*. Suppose that* $0 < \inf(f) < \sup(f) < \infty$ *and* (F, g_F) *is complete. If* (M, g) *is future null incomplete, then* (M, g) *is future timelike incomplete.*

Proof. If $0 < \inf(f) < \sup(f) < \infty$, then $b/\sqrt{1+b^2} < b$. by Corollary 3.6, (M, g) is future null incomplete if and only if $\int_{w_0}^{d} b(s)\,ds < \infty$ for some $w_0 \in (c, d)$. Thus $\int_{w_0}^{d} b(s)/\sqrt{1+b^2(s)}\,ds < \infty$. Hence, the result follows from Theorem 3.10. Similar result holds for past incompleteness.

4. Killing vector fields

In this section, we give the formula for the Lie derivative of the metric of a doubly warped product. We then give some results about Killing and conformal vector fields of doubly warped products without proof.

Proposition 4.1. *Let $M = {}_fB \times {}_bF$ be a pseudo-Riemannian doubly warped product with metric $g = f^2 g_B \oplus b^2 g_F$ also let $X, Y, Z \in \mathfrak{L}(B)$ and $V, W, U \in \mathfrak{L}(F)$ then if $\overline{X} = X + V \in \mathfrak{X}(M)$, $\overline{Y} = Y + W \in \mathfrak{X}(M)$ and $\overline{Z} = Z + U \in \mathfrak{X}(M)$ then*

$$L_{\overline{X}} g(\overline{Y}, \overline{Z}) = f^2 L^B_X g_B(Y, Z) + 2 f V(f) g_B(Y, Z) \\ + b^2 L^F_V g_F(W, U) + 2bX(b) g_F(W, U).$$

Proof. In general, we know that (cf. [15]) $L_{\overline{X}} g(\overline{Y}, \overline{Z}) = g(\nabla_{\overline{Y}} \overline{X}, \overline{Z}) + g(\overline{Y}, \nabla_{\overline{Z}} \overline{X})$. The result follows from the covariant derivative formulas for doubly warped products and the definition of Lie derivative and the bilinearity of g.

Now, by making use of the above formula, we will state necessary and sufficient conditions for vector fields $\overline{X}$ on M of the form $\overline{X} = X + V$, where $X \in \mathfrak{L}(B)$ and $V \in \mathfrak{L}(F)$ to be Killing or conformal. Similiar problem was considered in [9] for singly warped products.

Theorem 4.2. *Let $M = {}_fB \times {}_bF$ be a pseudo-Riemannian doubly warped product with metric $g = f^2 g_B \oplus b^2 g_F$ also let $X \in \mathfrak{L}(B)$ and $V \in \mathfrak{L}(F)$, $\overline{X} = X + V \in \mathfrak{X}(M)$. Then $\overline{X}$ is a conformal vector field with conformal factor $\Omega\colon M \to \mathbb{R}$ if and only if*

(1) *Ω is constant.*

(2) *X and V are conformal vector fields with conformal factors Ω_B and Ω_F, respectively, where both Ω_B and Ω_F are constant such that $\Omega_B + \Omega_F = 0$.*

(3) *$X(b) = \Omega_B b$ and $V(f) = \Omega_F f$.*

By taking $\Omega \equiv 0$ in the above result, we obtain necessary and sufficient conditions for vector fields to be Killing.

Corollary 4.3. *Let $M = {}_fB \times {}_bF$ be a pseudo-Riemannian doubly warped product with metric $g = f^2 g_B \oplus b^2 g_F$ also let $X \in \mathfrak{L}(B)$ and $V \in \mathfrak{L}(F)$, $\overline{X} = X + V \in \mathfrak{X}(M)$. Then $\overline{X}$ is a Killing vector field if and only if*

(1) *X and V are conformal vector fields with conformal factors Ω_B and Ω_F, respectively, where both Ω_B and Ω_F are constant such that $\Omega_B + \Omega_F = 0$.*

(2) *$X(b) = \Omega_B b$ and $V(f) = \Omega_F f$.*

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