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A generalization of the associated functional to the Lebesgue measure

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Abstract

In this paper we analyse a linear functional defined in the space of Laurent polynomials which can be considered as a generalization of the Lebesgue functional. We study the regularity and the semiclassical character of the functional and we construct the corresponding sequence of orthogonal polynomials.

Also, we obtain the differential equation that this family and the associated polynomials of first order satisfy.

Keywords: Orthogonal polynomials on the unit circle; Associated polynomials of first order; Semiclassical functional

In the linear space of Laurent polynomials Λ we define a linear and Hermitian functional $\mathfrak{L}:\Lambda \to \mathbb{C}$ which can be considered as a generalization of the Lebesgue functional.

We recall that $\mathfrak{L}$ is called regular (positive definite) if the principal minors Δ_n of the matrix of moments $M = (\mathfrak{L}(z^{i-j}))_{i,j\in\mathbb{N}} = (c_{i-j})_{i,j\in\mathbb{N}}$ are nonsingular (positive definite) and $\mathfrak{L}$ is Hermitian if $\mathfrak{L}(z^n) = \overline{\mathfrak{L}(z^{-n})}$.

Definition 1. For $p \in \mathbb{N}$ with $p \geqslant 1$ and $a \in \mathbb{C}$ with $|a| \neq 1$ we define the linear functional $\mathfrak{L}:\Lambda \to \mathbb{C}$ as follows: For $k \geqslant 0$,

$$\mathfrak{L}(z^k) = c_k = \begin{cases} 1, & k = 0, \\ a^j, & k = jp \text{ where } j \in \mathbb{N} - \{0\}, \\ 0, & \text{otherwise} \end{cases}$$

and for $k < 0$, $\mathfrak{L}(z^k) = \overline{\mathfrak{L}(z^{-k})}$.

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Since for $a = 0$, $\mathfrak{L}$ is the functional induced by the Lebesgue measure, we only analyse the case $a \neq 0$.

Theorem 2. *If we denote by* $\{\Delta_n\}_{n \geqslant 0}$ *the sequence of the principal minors of order* $n + 1$ *related to* $\mathfrak{L}$, *it holds*:

$$\Delta_0 = 1, \quad \Delta_1 = 1, \ldots, \Delta_{p-1} = 1 \quad \textit{and} \quad \Delta_{p+k} = (1 - |a|^2)^{k+1} \quad \forall k \geqslant 0.$$

Therefore $\mathfrak{L}$ *is regular, and for* $|a| < 1$, $\mathfrak{L}$ *is positive definite.*

Proof. First we prove that the principal minors verify the following relations:

$$\Delta_0 = 1, \quad \Delta_1 = 1, \ldots, \Delta_{p-1} = 1, \quad \Delta_p = 1 - |a|^2 \quad \text{and} \quad \Delta_{p+k} = \frac{\Delta_{p+k-1}^2}{\Delta_{p+k-2}} \quad \forall k \geqslant 1.$$

We distinguish two cases depending on $p = 1$ or $p > 1$.

Case I: For $p = 1$ the moment matrix associated to $\mathfrak{L}$ is

$$M = \begin{pmatrix} 1 & a & a^2 & \ldots & a^n & . \\ \bar{a} & 1 & a & \ldots & a^{n-1} & . \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \bar{a}^n & \bar{a}^{n-1} & \bar{a}^{n-2} & \ldots & 1 & . \\ . & . & . & & . & . \end{pmatrix}.$$

Then it is obvious that $\Delta_0 = 1$ and $\Delta_1 = (1 - |a|^2)$, and for $k \geqslant 1$ we have

$$\Delta_{1+k} = \left| \begin{array}{c:ccc:c} 1 & a & \ldots & a^k & a^{1+k} \\ \hdashline \bar{a} & 1 & & a^{k-1} & a^k \\ & \vdots & & & \vdots \\ \bar{a}^k & \bar{a}^{k-1} & & 1 & a \\ \hdashline \bar{a}^{1+k} & \bar{a}^k & \ldots & \bar{a} & 1 \end{array} \right|.$$

In order to evaluate Δ_{1+k} we apply a well-known property [5] of determinants:

$$\Delta_{1+k}\Delta_{1+k-2} = \operatorname{Adj}(1,1)\operatorname{Adj}(k+2,k+2) - \operatorname{Adj}(1,k+2)\operatorname{Adj}(k+2,1),$$

where $\operatorname{Adj}(i,l)$ represents the adjoint of the element in place (i,l).

Taking into account that

$$\operatorname{Adj}(1,1) = \operatorname{Adj}(k+2,k+2) = \Delta_k$$

and

$$\operatorname{Adj}(1,k+2) = 0 \tag{1}$$

we get

$$\Delta_{1+k}\Delta_{1+k-2} = \mathrm{Adj}(1,1)\mathrm{Adj}(k+2,k+2) = \Delta_k^2$$

from which the result follows.

Case II: For $p > 1$ the moment matrix associated to $\mathfrak{L}$ is

$$\begin{array}{rl} & \begin{array}{ccccccccccc} & & & & \text{column } p+1 & & & & \text{column } 2p+1 & & \\ & & & & \downarrow & & & & \downarrow & & \end{array} \\ \begin{array}{r} \\ \\ M = \\ \text{row } p+1 \rightarrow \\ \\ \end{array} & \begin{pmatrix} 1 & 0 & 0 & \ldots & a & 0 & \cdots & 0 & a^2 & \ldots & \\ 0 & 1 & 0 & & 0 & a & \ldots & 0 & 0 & a^2 & \ldots \\ \vdots & \vdots & \vdots & & \vdots & & & & & & \\ \bar{a} & 0 & 0 & & 1 & 0 & \cdots & 0 & a & 0 & \ldots \\ . & . & . & & . & . & & & . & . & \ldots \end{pmatrix} \end{array}$$

Then it is clear that

$$\Delta_0 = 1, \quad \Delta_1 = 1, \ldots, \Delta_{p-1} = 1.$$

Proceeding in the same way as in the previous case we obtain

$$\Delta_p . \det I = \mathrm{Adj}(1,1)\mathrm{Adj}(p+1,p+1) - \mathrm{Adj}(1,p+1)\mathrm{Adj}(p+1,1)$$

with $\det I = 1$ since I is the identity matrix.

Since

$$\mathrm{Adj}(1,1) = \mathrm{Adj}(p+1,p+1) = 1,$$

$$\mathrm{Adj}(1,p+1) = -\bar{a} = \overline{\mathrm{Adj}(p+1,1)} \tag{2}$$

then

$$\Delta_p = 1 - |a|^2.$$

Next we evaluate $\Delta_{\lambda p+k}$ with λ and k natural numbers greater than or equal to 1.

We distinguish between two cases depending on k. If k is not a multiple of p $(k \neq \mathring{p})$ the last element in the first row is 0, otherwise the last element in the first row is a power of a.

(i) If $k \neq \mathring{p}$ we may assume that $k \in \{1, 2, \dots, p-1\}$. We compute $\Delta_{\lambda p+k}$,

$$
\begin{array}{r|cccccccccccccc|}
\multicolumn{1}{r}{} & & & & & \text{column } p+1 & & & & & & & \text{column } \lambda p+k+1 & & \multicolumn{1}{c}{} \\
\multicolumn{1}{r}{} & & & & & \downarrow & & & & & & & \downarrow & & \multicolumn{1}{c}{} \\
 & 1 & \vdots & \cdots & 0 & a & 0 & \cdots 0 & \cdots & a^{\lambda-1} & 0 & \cdots 0 & a^{\lambda} & \cdots & \vdots\, 0 \\
\hline
 & 0 & 1 & & & & a & & & & & & & & \\
 & \vdots & \vdots & & & & & & & & & & & & \vdots \\
\text{row } p+1 \rightarrow & \bar{a} & & & & 1 & & & & & & & & & \vdots \\
 & 0 & \vdots & & & & & & & & & & & & \\
\Delta_{\lambda p+k} = & \vdots & \vdots & & & & & & & & & & & & \vdots \\
 & \bar{a}^{\lambda-1} & \vdots & & & & & & & 1 & & & & & \vdots \\
 & 0 & \vdots & & & & & & & & & & & & \vdots \\
 & \vdots & & & & & & & & & & & & & \\
 & 0 & & & & & & & & & & & & & \\
\text{row } \lambda p+k+1 \rightarrow & \bar{a}^{\lambda} & \vdots & & & & & & & & & & & & \vdots \\
 & \vdots & & & & & & & & & & & 1 & & \\
\hline
 & 0 & \vdots & & & & & & & & & & & & \vdots\, 1
\end{array}
$$

By applying the same property as in the previous cases we have

$$\Delta_{\lambda p+k}\Delta_{\lambda p+k-2} = \Delta^2_{\lambda p+k-1} - \operatorname{Adj}(1, \lambda p+k+1)\operatorname{Adj}(\lambda p+k+1, 1).$$

For computing $\operatorname{Adj}(1, \lambda p+k+1)$ it suffices to consider the rows $\lambda p+k+1$ and $(\lambda-1)p+k+1$ of $\Delta_{\lambda p+k}$.

Since row $\lambda p+k+1$ is

$$(0, \dots, 0, \underset{\substack{\uparrow \\ \text{column } k+1}}{\bar{a}^{\lambda}}, 0, \dots, 0, \underset{\substack{\uparrow \\ \text{column } p+k+1}}{\bar{a}^{\lambda-1}}, 0, \dots, \underset{\substack{\uparrow \\ \text{column } (\lambda-1)p+k+1}}{\bar{a}}, 0, \dots, 1)$$

and row $(\lambda-1)p+k+1$ is

$$(0, \dots, 0, \underset{\substack{\uparrow \\ \text{column } k+1}}{\bar{a}^{\lambda-1}}, 0, \dots, 0, \underset{\substack{\uparrow \\ \text{column } p+k+1}}{\bar{a}^{\lambda-2}}, 0, \dots, \underset{\substack{\uparrow \\ \text{column } (\lambda-1)p+k+1}}{1}, 0, \dots, a)$$

then

$$\operatorname{Adj}(1, \lambda p+k+1) = 0 \tag{3}$$

and therefore

$$\Delta_{\lambda p+k}\Delta_{\lambda p+k-2} = \Delta^2_{\lambda p+k-1}.$$

(ii) If $k = \mathring{p}$ it suffices to compute $\Delta_{\lambda p}$ where $\lambda \in \mathbb{N}$ with $\lambda > 1$.

$$\begin{array}{cc}
 & \begin{array}{ccc} & \text{column } p+1 & \qquad\qquad \text{column } (\lambda-1)p+k \\ & \downarrow & \qquad\qquad \downarrow \end{array} \\
\begin{array}{r} \\ \\ \\ \text{row } p+1 \rightarrow \\ \\ \Delta_{\lambda p} = \\ \\ \\ \\ \\ \\ \end{array} &
\left|\begin{array}{c|ccccccccccc|c}
1 & \cdots & 0 & a & 0 & \cdots & 0 & \cdots & a^{\lambda-1} & 0 & \cdots & 0 \cdots & a^{\lambda} \\
\hline
0 & 1 & & a & & & & & & & & & \vdots \\
\vdots & \vdots & & & & & & & & & & & \vdots \\
\bar{a} & & & 1 & & & & & & & & & \\
0 & \vdots & & & & & & & & & & & \vdots \\
\vdots & & & & & & & & & & & & \\
\bar{a}^{\lambda-1} & \vdots & & & & & & & 1 & & & & \vdots \\
0 & & & & & & & & & & & & \\
\vdots & \vdots & & & & & & & & & & & \vdots \\
0 & & & & & & & & & & & & \\
\vdots & \vdots & & & & & & & & & & 1 & \vdots \\
\hline
\bar{a}^{\lambda} & \vdots & & & & & & & & & & & 1
\end{array}\right|
\end{array}$$

Proceeding in the same way as in the preceding cases we conclude

$$\Delta_{\lambda p}\Delta_{\lambda p-2} = \Delta^2_{\lambda p-1} - \operatorname{Adj}(1,\lambda p+1)\operatorname{Adj}(\lambda p+1,1)$$

with

$$\operatorname{Adj}(1,\lambda p+1) = 0 \tag{4}$$

and as a consequence we have the result. □

Theorem 3. *For $p \in \mathbb{N}$ with $p \geqslant 1$ and $a \in \mathbb{C} - \{0\}$ with $|a| \neq 1$, the sequence $\{\phi_n(z)\}_0^\infty$ defined by*

$$\phi_k(z) = z^k, \quad 0 \leqslant k \leqslant p-1,$$

$$\phi_{p+k}(z) = z^k(z^p - a) \quad \forall k \geqslant 0$$

is the monic orthogonal polynomial sequence related to $\mathfrak{L}$ (MOPS ($\mathfrak{L}$)).

Proof. Since $\mathfrak{L}$ is regular, there exists $\{\phi_n(z)\}_0^\infty$ MOPS ($\mathfrak{L}$). To obtain this sequence we distinguish between two cases in which we use the following result [1]:

$$\phi_n(0)\Delta_{n-1} = \operatorname{Adj}(n+1,1) = \overline{\operatorname{Adj}(1,n+1)} \quad \text{in } \Delta_n. \tag{5}$$

Case I: For $p = 1$, we have

$$\phi_1(z) = \frac{\begin{vmatrix} 1 & a \\ 1 & z \end{vmatrix}}{\Delta_0} = z - a$$

and $\forall k \geqslant 1$

$$\phi_{1+k}(z) = \frac{\begin{vmatrix} 1 & a & \dots & a^{1+k} \\ \vdots & \vdots & & \vdots \\ \bar{a}^k & \bar{a}^{k-1} & \dots & a \\ 1 & z & \dots & z^{1+k} \end{vmatrix}}{\Delta_k}.$$

From (5) and (1),

$$\phi_{1+k}(0)\,\Delta_k = \mathrm{Adj}(k+2, 1) = 0.$$

By using Szegő's recurrence relation [7] we obtain the result.

Case II: For $p > 1$, proceeding in the same way as above we get

$$\phi_n(z) = z^n \quad \forall n \text{ such that } 0 \leqslant n \leqslant p - 1$$

and since

$$\phi_p(z) = \frac{\begin{vmatrix} 1 & 0 & \dots & a \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \\ 1 & z & \dots & z^p \end{vmatrix}}{\Delta_{p-1}}$$

by applying (5) and (2) we have $\phi_p(0) = -a$. Then we obtain $\phi_p(z)$ by using the recurrence relations.

To obtain the other polynomials we proceed in the way as for constructing $\phi_p(z)$ by using (3) and (4). □

Definition 4. Let $u : \Lambda \to \mathbb{C}$ be a regular and Hermitian linear functional.

We define the sequence of associated polynomials of first order, $\{\phi_n^{(1)}(y)\}_0^\infty$ by

$$\phi_n^{(1)}(y) = u_0^{-1} u\left(\frac{\phi_{n+1}(z) - \phi_{n+1}(y)}{z - y}\right) \quad \forall n \geqslant 0,$$

where $\{\phi_n(z)\}_0^\infty$ is the MOPS (u) and $u_0 = u(1)$ [2].

By using the above definition we obtain, in the following result, that the associated polynomials of first order related to our functional are the classical polynomials $\{z^n\}_0^\infty$.

Theorem 5. *The sequence of associated polynomials of first order corresponding to sequence* $\{\phi_n(z)\}_0^\infty$ *is* $\{z^n\}_0^\infty$.

Proof. By using the previous definition we have, for $1 \leqslant k < p$,

$$\phi_{k+1}^{(1)}(y) = \mathfrak{L}\left(\frac{\phi_k(z) - \phi_k(y)}{z - y}\right) = \mathfrak{L}(z^{k-1} + yz^{k-2} + \cdots + y^{k-1}) = y^{k-1}$$

and, for $k \geqslant 1$,

$$\phi_{p+k-1}^{(1)}(y) = \mathfrak{L}\left(\frac{\phi_{p+k}(z) - \phi_{p+k}(y)}{z - y}\right) = \mathfrak{L}\left(\sum_{j=0}^{p+k-1} z^j y^{p+k-1-j}\right) - a\mathfrak{L}\left(\sum_{j=0}^{k-1} z^j y^{k-1-j}\right)$$

$$= \sum_{j=0}^{p+k-1} c_j y^{p+k-1-j} - a\sum_{j=0}^{k-1} c_j y^{k-1-j}. \tag{6}$$

To prove that $\phi_{p+k-1}^{(1)}(y) = y^{p+k-1}$ we use $ac_{jp} = c_{(j+1)p}$ $\forall j \in \mathbb{N} - \{0\}$, which follows from Definition 1.

If $k \leqslant p$ then $0 \leqslant k - 1 \leqslant p - 1$ and $p \leqslant p + k - 1 \leqslant 2p - 1$. Therefore in the first term of the second member of (6) the only addends different from zero are those corresponding to c_0 and c_p. In the second term appear only the addends corresponding to c_0:

$$\phi_{p+k-1}^{(1)}(y) = c_0 y^{p+k-1} + c_p y^{k-1} - ac_0 y^{k-1} = y^{p+k-1}.$$

If $k \geqslant p + 1$ then $\exists r \in \mathbb{N}$, $r \geqslant 2$, $k \in [p + 1, rp]$. And thus $p \leqslant k - 1 \leqslant rp - 1$ and $2p \leqslant p + k - 1 \leqslant (r + 1)p - 1$.

Since in the first sum, the only terms different from zero are those corresponding to $c_0, c_p, c_{2p}, \ldots, c_{rp}$ and in the second those corresponding to $c_0, c_p, c_{2p}, \ldots, c_{(r-1)p}$, then like in the previous case we get the result. □

Next we analyse the semiclassical character of the functional $\mathfrak{L}$. First of all recall the following definition [6].

Definition 6. $\mathfrak{L}$ is a *semiclassical functional* if there exist polynomials $A(z)$ and $C(z)$ such that the series $G(z) = \sum_{-\infty}^{+\infty} c_k z^{-k}$ ($z = e^{i\theta}$) satisfies the following equation:

$$zA(z)G'(z) + iC(z)G(z) = 0.$$

Theorem 7. *$\mathfrak{L}$ is a semiclassical functional, verifying the above definition with*

$$A(z) = (1 - \bar{a}z^p)(z^p - a) \quad \text{and} \quad C(z) = ip(\bar{a}z^{2p} - a).$$

Proof. Since the series $G(z)$ associated with $\mathfrak{L}$ is

$$G(z) = \cdots + \bar{a}^n z^{np} + \cdots + \bar{a}^2 z^{2p} + \bar{a} z^p + 1 + a z^{-p} + a^2 z^{-2p} + \cdots + a^n z^{-np} + \cdots$$

then

$$G(z)(1 - \bar{a} z^p) = (1 - |a|^2) \sum_{j=0}^{+\infty} a^j z^{-jp}; \tag{7}$$

therefore for $|a| \neq 1$.

If we multiply (7) by z^p and by a, we conclude, from both expressions:

$$G(z)(1 - \bar{a} z^p)(z^p - a) = (1 - |a|^2) z^p. \tag{8}$$

By taking derivatives in (8) with respect to z:

$$G'(z)(1 - \bar{a} z^p)(z^p - a) + G(z)(-2\bar{a} p z^{2p-1} + (1 + |a|^2) p z^{p-1}) = p(1 - |a|^2) z^{p-1} \tag{9}$$

and by multiplying (9) by z and (8) by p we obtain

$$pG(z)(\bar{a} z^{2p} - a) - zG'(z)(1 - \bar{a} z^p)(z^p - a) = 0.$$

Then it is clear that this expression verifies the previous definition with $A(z) = (1 - \bar{a} z^p)(z^p - a)$ y $C(z) = ip(\bar{a} z^{2p} - a)$. □

Finally we obtain the differential equation satisfied by the MOPS ($\mathfrak{L}$).

Theorem 8. *The sequence* $\{\phi_n(z)\}_0^\infty$ *verifies the following differential equation*:

$$z^2 \phi_n''(z) + z(1 - 2n + p)\phi_n'(z) + n(n - p)\phi_n(z) = 0 \quad \forall n \geqslant 0.$$

Proof. From Theorem 3 follows

$$\phi_{p+k}(z) = z^{p+k} - a z^k \quad \forall n \geqslant 0. \tag{10}$$

If we take derivatives in the preceding expression and we take into account (10) we get

$$z\phi_{p+k}'(z) - k\phi_{p+k}(z) = p z^{p+k}. \tag{11}$$

Again, by taking derivatives in (11) and by multiplying by z we obtain

$$z^2 \phi_{p+k}''(z) + z(1 - k)\phi_{p+k}'(z) = p(p + k) z^{p+k}. \tag{12}$$

From (11) and (12) we deduce

$$z^2 \phi_{p+k}''(z) + z(1 - 2k - p)\phi_{p+k}'(z) + k(p + k)\phi_{p+k}(z) = 0 \quad \forall k \geqslant 0.$$

By putting $p + k = n$ we obtain the expression of the statement for all polynomials of degree greater than or equal to p. On the other hand, by taking into account that $\phi_n(z) = z^n, 0 \leqslant n < p$, it is easy to verify that these polynomials satisfy the equation too. □

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