On selfinjective artin algebras having generalized standard quasitubes

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MSC: 16D50; 16G10; 16G70

ABSTRACT

We give a complete description of the Morita equivalence classes of all connected selfinjective artin algebras for which the Auslander–Reiten quiver admits a family of quasitubes having common composition factors, closed under composition factors, and consisting of modules not lying on infinite short cycles.

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1. Introduction and the main result

Throughout the paper, by an algebra we mean a basic, connected artin algebra over a commutative artinian ring k. For an algebra A we denote by modA the category of finitely generated right A-modules. Given a module M in modA, we denote by [M] the image of M in the Grothendieck group K0(A) of A. Thus [M] = [N] if and only if the modules M and N have the same composition factors, including the multiplicities. An algebra A is called selfinjective if A is an injective module, or, equivalently, the projective and injective modules in modA coincide.

An important combinatorial and homological invariant of the module category modA of an algebra A is its Auslander–Reiten quiver ΓA. The Auslander–Reiten quiver ΓA describes the structure of the quotient category modA/ rad∞(modA), where rad∞(modA) is the infinite Jacobson radical of A. In particular, by a result due to Auslander [6], A is of finite representation type if and only if rad∞(modA) = 0. In general, it is important to study the behavior of the components of ΓA in the category modA. Following [45], a component C of ΓA is called generalized standard if rad∞(X, Y) = 0 for all modules X and Y in C. It has been proved in [45] that every generalized standard component C of ΓA is almost periodic; that is, all but finitely many DTr-orbits in C are periodic. Moreover, by a result of [59], the additive closure add(C) of a generalized standard component C of ΓA is closed under extensions in modA. We note that, for a selfinjective, every infinite generalized standard component C of ΓA is either acyclic with finitely many DTr-orbits or is a quasitube (the stable part C of C is a stable tube).

In the representation theory of selfinjective algebras, a prominent role is played by the selfinjective algebras of quasitilted type, that is, the orbit algebras B/G, where B is the repetitive algebra of a quasitilted algebra and G is an admissible group of automorphisms of B, which is in fact an infinite cyclic group generated by a strictly positive automorphism of B. Recall that the quasitilted algebras are those of the form EndH(T), where T is a tilting object in a hereditary Ext-finite abelian category H, or, equivalently, the algebras A of global dimension at most two and with every indecomposable module in...
Let $A$ be a basic, connected, selfinjective artin algebra. The following statements are equivalent.

Theorem 1.1. Let $A$ be a basic, connected, selfinjective artin algebra. The following statements are equivalent.

(i) $\Gamma_A$ admits a nonempty family $\mathcal{C} = (C_i)_{i \in I}$ of quasitubes having common composition factors, closed on composition factors, and consisting of modules which do not lie on infinite short cycles in mod $A$.

(ii) $A$ is isomorphic to an orbit algebra $B/G$, where $B$ is an almost concealed canonical algebra and $G$ is an infinite cyclic group of automorphisms of $B$ of one of the following forms:

(a) $G = (\varphi v^2_B)$, for a strictly positive automorphism $\varphi$ of $B$,

(b) $G = (\varphi v^2_B)$, for $B$ a tubular algebra and $\varphi$ a rigid automorphism of $B$,

(c) $G = (\varphi v^2_B)$, for $B$ of Euclidean or wild type and $\varphi$ a rigid automorphism of $B$ acting freely on the nonstable tubes of the unique separating family $\mathcal{T}$ of ray tubes of $\Gamma_B$.

where $v^2_B$ is the Nakayama automorphism of $B$.

Following [48], a family $\mathcal{C} = (C_i)_{i \in I}$ of components of $\Gamma_A$ is said to have common composition factors if, for each pair $i$ and $j$ in $I$, there exist modules $X_i \in C_i$ and $X_j \in C_j$ with $[X_i] = [X_j]$. Moreover, $\mathcal{C}$ is closed under composition factors if, for every indecomposable module $M$ and $N$ in mod $A$ with $[M] = [N]$, $M \in \mathcal{C}$ forces $N \in \mathcal{C}$. Further, by a short cycle in mod $A$ we mean a sequence $M \xrightarrow{f} N \xrightarrow{g} M$ of nonzero nonisomorphisms between indecomposable modules in mod $A$ [38], and such a cycle is said to be infinite if at least one of the homomorphisms $f$ or $g$ belongs to rad$^\infty$(mod $A$). We also mention that, by a result proved in [38], every indecomposable module $M$ in mod $A$ which does not lie on a short cycle is uniquely determined by $[M]$ (up to isomorphism).

As a direct consequence of Theorem 1.1 and results on selfinjective algebras of canonical type, established in Section 6 (Propositions 6.4 and 6.5), we obtain the following fact.

Corollary 1.2. Let $A$ be a basic, connected, selfinjective artin algebra. The following statements are equivalent.

(i) $\Gamma_A$ admits a family $\mathcal{T} = (T_i)_{i \in I}$ of stable tubes with common composition factors, closed under composition factors, and consisting of modules which do not lie on infinite short cycles in mod $A$.

(ii) $A$ is isomorphic to an orbit algebra $B/G$, where $B$ is a concealed canonical algebra and $G$ is an infinite cyclic group of automorphisms of $B$ of the form $(\varphi v^2_B)$ for a positive automorphism $\varphi$ of $B$.

We refer to [11,22,27–30,41,42,48] for constructions and basic properties of concealed canonical algebras.

By general theory (see [33,61]), an infinite component $\mathcal{C}$ of the Auslander–Reiten quiver $\Gamma_A$ of a selfinjective algebra $A$ is cyclic (every module in $\mathcal{C}$ lies on an oriented cycle in $\Gamma_A$) if and only if $\mathcal{C}$ is a quasitube. Moreover, $\Gamma_A$ is said to be cyclic if every component of $\Gamma_A$ is cyclic. Then we obtain the following consequence of Theorem 1.1 and the known structure of the Auslander–Reiten quivers of selfinjective algebras of canonical type (see Theorems 6.3 and Proposition 6.4).

Corollary 1.3. Let $A$ be a basic, connected, selfinjective artin algebra. The following statements are equivalent.

(i) $\Gamma_A$ is cyclic and admits a family $\mathcal{C} = (C_i)_{i \in I}$ of quasitubes having common composition factors, closed under composition factors, and consisting of modules which do not lie on infinite short cycles in mod $A$.
(ii) $Γ_A$ is cyclic and admits a family $T = (γ_i)_{i∈I}$ of stable tubes having common composition factors, closed under composition factors, and consisting of modules which do not lie on infinite short cycles in mod $A$.

(iii) $A$ is isomorphic to an orbit algebra $\widetilde{B}/G$, where $B$ is a tubular algebra and $G$ is an infinite cyclic group of $\widetilde{B}$ of the form $(ϕ^2)_{B}$ for a positive automorphism $ϕ$ of $B$.

We refer to [23–27,40,41] for constructions and basic properties of tubular algebras.

As an immediate consequence of Theorem 1.1 and the fact that the ordinary valued quivers of quasitilted algebras are acyclic [17], we obtain the following fact.

**Corollary 1.4.** Let $A$ be a basic, connected, selfinjective artin algebra whose Auslander–Reiten quiver $Γ_A$ admits a family $C$ of quasitubes with common composition factors, closed under composition factors, and consisting of modules which do not lie on infinite short cycles. Then the center of $A$ is a field, and hence $A$ is a finite-dimensional algebra over a field.

The paper is organized as follows. In Section 2, we describe basic properties of quasitubes which are fundamental for the proofs of the main results. In Section 3, we recall known characterizations of quasitilted algebras of canonical type, playing a prominent role in the proof of Theorem 1.1. Section 4 is devoted to quasitube enlargements of concealed canonical algebras, essential for further considerations. In Section 5, we recall criteria for selfinjective algebras to be orbit algebras of repetitive algebras established by the second and third named authors, applied in the proof of Theorem 1.1. In Section 6, we describe the module categories of selfinjective algebras of canonical type and prove the implication (ii)$⇒$(i) of Theorem 1.1. Section 7 is devoted to the proof of the implication (i)$⇒$(ii) of Theorem 1.1.

For basic background on the representation theory of algebras applied in the paper, we refer to the books [2,7,15,40,42,43] and to the survey articles [51,58,60].

The main results of the paper have been presented by the first named author during the Fourteenth International Conference on Representations of Algebras (ICRAXIV) held in Tokyo in August 2010.

2. Quasitubes

The purpose of this section is to present results on quasitubes of Auslander–Reiten quivers of algebras, playing a prominent role in the proof of Theorem 1.1.

Recall that, if $A_∞$ is the quiver $0 → 1 → 2 → \cdots$, then $ZA_∞$ is the translation quiver of the form

\[
\begin{array}{cccccc}
(i+1,0) & (0) & (i+1,0) & (i+2,0) \\
(i+1,1) & (i,1) & (i+1,1) & \cdot & \cdot & \cdot \\
(i+1,2) & (i,2) & \cdot & \cdot & \cdot & \cdot \\
(i+1,3) & \cdot & \cdot & \cdot & \cdot & \cdot \\
(i+1,4) & \cdot & \cdot & \cdot & \cdot & \cdot \\
(i+1,5) & \cdot & \cdot & \cdot & \cdot & \cdot \\
\end{array}
\]

with $τ(i,j) = (i - 1,j)$ for $i ∈ \mathbb{Z}, j ∈ \mathbb{N}$. For $r ≥ 1$, denote by $ZA_∞/(τ^r)$ the translation quiver obtained from $ZA_∞$ by identifying each vertex $((i,j)$ of $ZA_∞$ with the vertex $τ^r(i,j)$ and each arrow $x → y$ in $ZA_∞$ with the arrow $τ^r x → τ^r y$, and call it the stable tube of rank $r$. The $τ$-orbit of a stable tube $Γ'$ formed by all vertices having exactly one immediate predecessor (equivalently, successor) is called the mouth of $Γ'$.

Let $(Γ', τ)$ be a translation quiver (with trivial valuations). For some vertices $x$ in $Γ'$, called pivots, we shall define two admissible operations [4] modifying $(Γ', τ)$ to a new translation quiver $(Γ', τ')$, depending on the shape of paths in $Γ'$ starting from $x$.

(ad 1) Suppose that $Γ'$ admits an infinite sectional path

\[x = x_0 → x_1 → x_2 → \cdots\]

starting at $x$, and assume that every sectional path in $Γ'$ starting at $x$ is a subpath of the above path. For $t ≥ 1$, let $Γ'_t$ be the following translation quiver, isomorphic to the Auslander–Reiten quiver of the full $t × t$ upper triangular matrix algebra over a field:
We then let \( \Gamma'' \) be the translation quiver having as vertices those of \( \Gamma' \), those of \( \Gamma_t \), additional vertices \( z_j \) and \( x'_i \) (where \( i \geq 0, 1 \leq j \leq t \)), and having arrows as in the figure below:

The translation \( \tau' \) of \( \Gamma'' \) is defined as follows: \( \tau'z_{ij} = z_{i-1,j-1} \) if \( i \geq 1, j \geq 2 \), \( \tau'z_{1j} = x_{i-1} \) if \( i \geq 1 \), \( \tau'z_{0j} = y_{j-1} \) if \( j \geq 2 \), \( \tau'z_{01} = x_0 \) is projective, \( \tau'x'_0 = y_t \), \( \tau'x'_i = z_{i-1,t} \) if \( i \geq 1 \), \( \tau'X_i = X'_i \) provided \( x_i \) is not injective in \( \Gamma' \); otherwise \( x'_i \) is injective in \( \Gamma'' \). For the remaining vertices of \( \Gamma'' \), \( \tau' \) coincides with the translation of \( \Gamma' \), or \( \Gamma_t \), respectively. If \( t = 0 \), the new translation quiver \( \Gamma'' \) is obtained from \( \Gamma' \) by inserting only the sectional path consisting of the vertices \( x'_i, i \geq 0 \).

1. **Suppose that a vertex \( x \) in \( \Gamma' \) is injective and that \( \Gamma' \) admits two sectional paths starting at \( x \), one infinite and the other finite with at least one arrow

   \[ y_t \leftarrow \cdots \leftarrow y_2 \leftarrow y_1 \leftarrow x = x_0 \rightarrow x_1 \rightarrow x_2 \rightarrow \cdots \]

   such that any sectional path starting at \( x \) is a subpath of one of these paths. Then \( \Gamma'' \) is the translation quiver having as vertices those of \( \Gamma' \), additional vertices denoted by \( x'_0, z_j, x'_i \) (where \( i \geq 1, 1 \leq j \leq t \)), and having arrows as in the figure below:

The translation \( \tau' \) of \( \Gamma'' \) is defined as follows: \( x'_0 \) is projective–injective, \( \tau'z_{ij} = z_{i-1,j-1} \) if \( i \geq 2, j \geq 2 \), \( \tau'z_{1j} = x_{i-1} \) if \( i \geq 1 \), \( \tau'z_{0j} = y_{j-1} \) if \( j \geq 2 \), \( \tau'x'_i = z_{i-1,t} \) if \( i \geq 2 \), \( \tau'X_i = X'_i \) provided \( x_i \) is not injective in \( \Gamma' \); otherwise \( x'_i \) is injective in \( \Gamma'' \). For the remaining vertices of \( \Gamma'' \), \( \tau' \) coincides with the translation \( \tau \) of \( \Gamma' \).

We denote by \( \{ad 1^*\} \) and \( \{ad 2^*\} \) the admissible operations dual to the admissible operations \( \{ad 1\} \) and \( \{ad 2\} \), respectively.

A connected translation quiver \( \Gamma' \) is said to be a quasitube if \( \Gamma' \) can be obtained from a stable tube by an iterated application of admissible operations \( \{ad 1\}, \{ad 2\}, \{ad 1^*\}, \) or \( \{ad 2^*\} \). A tube (in the sense of [40]) is a quasitube having the property that each admissible operation in the sequence defining it is of the form \( \{ad 1\} \) or \( \{ad 1^*\} \). Finally, if we apply only operations of type \( \{ad 1\} \) (respectively, of type \( \{ad 1^*\} \)), then such a quasitube \( \Gamma' \) is called a ray tube (respectively, a coray tube). Observe that a quasitube without injective (respectively, projective) vertices is a ray tube (respectively, a coray tube). A quasitube \( \Gamma' \) whose all nonstable vertices are projective–injective is said to be smooth.
The following proposition provides a characterization of quasitubes in the Auslander–Reiten quivers of selfinjective algebras ([34, Theorem A],[33,61]).

**Proposition 2.1.** Let $A$ be a selfinjective algebra and $\Gamma$ a connected component of $\Gamma_A$. The following statements are equivalent.

(i) $\Gamma$ is a quasitube.
(ii) $\Gamma^s$ is a stable tube.
(iii) $\Gamma$ contains an oriented cycle.

Here, $\Gamma^s$ denotes the stable part of $\Gamma$, obtained from $\Gamma$ by removing the projective–injective modules and the arrows attached to them.

The following characterization of generalized standard stable tubes of an Auslander–Reiten quiver has been established in [45, Corollary 5.3] (see also [47, Lemma 3.1]).

**Proposition 2.2.** Let $A$ be an algebra and $\Gamma$ a stable tube of $\Gamma_A$. The following statements are equivalent.

(i) $\Gamma$ is generalized standard.
(ii) The mouth of $\Gamma$ consists of pairwise orthogonal bricks.
(iii) $\text{rad}^\infty(X, X) = 0$ for any module $X$ in $\Gamma$.

Recall that an indecomposable $A$-module $X$ is called a **brick** if its endomorphism algebra $\text{End}_A(X)$ is a division algebra. We note that the division algebras of all modules lying on the mouth of a generalized standard stable tube of $\Gamma$ are isomorphic.

Let $A$ be an algebra, and let $\Gamma$ be a stable tube of $\Gamma_A$. Then $\Gamma$ has two types of arrow: arrows pointing to infinity and arrows pointing to the mouth. Hence, for any module $Z$ lying in $\Gamma$, there is a unique sectional path $X_1 \rightarrow X_2 \rightarrow \cdots \rightarrow X_m = Z$ in $\Gamma$ with $X_1$ lying on the mouth of $\Gamma$ (consisting of arrows pointing to infinity) and there is a unique sectional path $Z = Y_1 \rightarrow Y_2 \rightarrow \cdots \rightarrow Y_m$ with $Y_0$ lying on the mouth of $\Gamma$ (consisting of arrows pointing to the mouth), and $m$ is called the **quasi-length** of $Z$ in $\Gamma$, denoted by $\text{ql}(Z)$. Observe that, if $\Gamma$ is of rank 1 and $X$ its unique module lying on the mouth, then for any module $Z$ in $\Gamma$ we have $\{Z\} = \text{ql}(Z)[X]$, and hence $\Gamma$ consists of modules with pairwise different classes in the Grothendieck group $K_0(A)$.

For stable tubes of ranks bigger that one, we have following theorem (see [47, Theorem 4.3]).

**Theorem 2.3.** Let $A$ be an algebra, $\Gamma$ a generalized standard stable tube of $\Gamma_A$ of rank $r > 1$, and $M,N$ nonisomorphic modules from $\Gamma$. Then $[M] = [N]$ if and only if $\text{ql}(M) = \text{ql}(N) = cr$ for some $c \geq 1$.

For stable tubes consisting of modules which do not lie on infinite short cycles, we have the following results, established in [47, Corollaries 4.4 and 4.6].

**Theorem 2.4.** Let $A$ be an algebra, $\Gamma$ a stable tube of rank $r > 1$ in $\Gamma_A$ consisting of modules which do not lie on infinite short cycles in mod $A$, and $M$ a module in $\Gamma$. Then $M$ is uniquely determined (up to isomorphism) by $[M]$ if and only if $r$ does not divide $\text{ql}(M)$.

**Theorem 2.5.** Let $A$ be an algebra, and let $\Gamma$ and $\Gamma'$ be different stable tubes in $\Gamma_A$ consisting of modules which do not lie on infinite short cycles in mod $A$. Let $r$ be the rank of $\Gamma$ and $r'$ be the rank of $\Gamma'$. Assume that $[M] = [N]$ for some modules $M$ in $\Gamma$ and $M'$ in $\Gamma'$. Then $r$ divides $\text{ql}(M)$, $r'$ divides $\text{ql}(M')$, and the tubes $\Gamma$ and $\Gamma'$ are orthogonal.

Observe that, by Proposition 2.2, every stable tube $\Gamma$ of an Auslander–Reiten quiver $\Gamma_A$ consisting of modules which do not lie on infinite short cycles is generalized standard [47, Corollary 3.2]. We shall show that it is also the case for the smooth quasitubes. We need some results on the degrees of irreducible homomorphisms proved by Liu in [33].

By a result of Bautista [8], a homomorphism $f : M \rightarrow N$ between indecomposable modules in mod $A$ is irreducible if and only if $f \in \text{rad}(M, N) \setminus \text{rad}^2(M, N)$. The following more general result has been established by Igusa and Todorov in [21].

**Proposition 2.6.** Let $A$ be an algebra, and let

\[
X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} X_{n-1} \xrightarrow{f_n} X_n
\]

be a path of irreducible homomorphisms between indecomposable modules in mod $A$ corresponding to a sectional path of $\Gamma_A$. Then we have $f_n \ldots f_2 f_1 \in \text{rad}^n(X_0, X_n) \setminus \text{rad}^{n+1}(X_0, X_n)$.

Let $A$ be an algebra, and let $f : X \rightarrow Y$ be an irreducible homomorphism in mod $A$, with $X$ and $Y$ indecomposable modules. Following Liu [33], $f$ is said to be of infinite left degree if, for any integer $n \geq 1$ and a homomorphism $g : M \rightarrow X$ in $\text{rad}^n(M, X) \setminus \text{rad}^{n+1}(M, X)$, we have $fg \in \text{rad}^{n+1}(M, Y) \setminus \text{rad}^{n+2}(M, Y)$. Dually, $f$ is said to be of infinite right degree if, for any integer $n \geq 1$ and a homomorphism $h : Y \rightarrow N$ in $\text{rad}^n(Y, N) \setminus \text{rad}^{n+1}(Y, N)$, we have $hf \in \text{rad}^{n+1}(X, N) \setminus \text{rad}^{n+2}(X, N)$.

The following facts are consequences of [33, Corollary 1.6 and its dual].
Proposition 2.7. Let $A$ be an algebra. The following statements hold.

(i) Assume that $\Gamma_A$ admits a full translation subquiver

\[
\cdots \rightarrow X_{i+1} \overset{f_{i+1}}{\rightarrow} X_{i} \rightarrow \cdots \rightarrow X_1 \overset{f_1}{\rightarrow} X_0 = X
\]

\[
\cdots \rightarrow Y_{i+1} \overset{g_{i+1}}{\rightarrow} Y_{i} \rightarrow \cdots \rightarrow Y_1 \overset{g_1}{\rightarrow} Y_0 = Y
\]

where the upper and lower infinite paths are sectional. Then every irreducible homomorphism $f: X \rightarrow Y$ in $\mod A$ is of infinite left degree.

(ii) Assume that $\Gamma_A$ admits a full translation subquiver

\[
M = M_0 \rightarrow M_1 \rightarrow \cdots \rightarrow M_j \rightarrow M_{j+1} \rightarrow \cdots
\]

\[
N = N_0 \rightarrow N_1 \rightarrow \cdots \rightarrow N_j \rightarrow N_{j+1} \rightarrow \cdots
\]

where the upper and lower infinite paths are sectional. Then every irreducible homomorphism $g: M \rightarrow N$ in $\mod A$ is of infinite right degree.

Let $A$ be an algebra, and let $C$ be a smooth quasitube in $\Gamma_A$. Then the stable part $C^s$ of $C$ is a stable tube, and we may define the stable quasi-length $\text{sql}(X)$ of a stable module $X$ in $C$ as the quasi-length $\text{ql}(X)$ of $X$ in $C^s$. Moreover, the stable quasi-length of a projective–injective module in $C$ is defined to be 0.

Lemma 2.8. Let $A$ be an algebra, and let $C$ be a smooth quasitube in $\Gamma_A$. Moreover, let $r$ be the rank of $C^s$ and $m$ be the maximum of stable quasi-length of the radicals of projective–injective modules in $C$. Then, for all modules $X$ and $Y$ in $C$ of stable quasi-length bigger than $m + r$, we have $\text{rad}(X, Y) \neq 0$.

Proof. Let $X$ and $Y$ be modules in $C$ with $\text{sql}(Y)$ and $\text{sql}(X)$ bigger than $m + r$. Then there are in $C$ sectional paths

\[
X = U_0 \rightarrow U_1 \rightarrow \cdots \rightarrow U_{p-1} \rightarrow U_p = Z,
\]

consisting of arrows of $C^s$ pointing to the mouth, and

\[
Z = V_0 \rightarrow V_1 \rightarrow \cdots \rightarrow V_{q-1} \rightarrow V_q = Y,
\]

consisting of arrows of $C^s$ pointing to infinity. Moreover, $C$ admits full translation subquivers

\[
\cdots \rightarrow W_{s+1}^{(j-1)} \rightarrow W_s^{(j-1)} \rightarrow \cdots \rightarrow W_1^{(j-1)} \rightarrow W_0^{(j-1)} = V_{j-1}
\]

\[
\cdots \rightarrow W_{s+1}^{(j)} \rightarrow W_s^{(j)} \rightarrow \cdots \rightarrow W_1^{(j)} \rightarrow W_0^{(j)} = V_j
\]

for $j \in \{1, \ldots, q\}$, formed by parallel infinite sectional paths, consisting of indecomposable modules of stable quasi-length bigger than $m$. Take irreducible homomorphisms in $\mod A$

\[
\varphi_i: U_{i-1} \rightarrow U_i \text{ and } \psi_j: V_{j-1} \rightarrow V_j,
\]

for $i \in \{1, \ldots, p\}$ and $j \in \{1, \ldots, q\}$. Then, it follows from Proposition 2.6 that $\varphi = \varphi_p \cdots \varphi_1 \in \text{rad}^p(X, Z) \setminus \text{rad}^{p+1}(X, Z)$. On the other hand, by Proposition 2.7, the irreducible homomorphisms $\psi_1, \ldots, \psi_q$ are of infinite left degree. Then, for $\psi = \psi_q \cdots \psi_1 \in \text{Hom}_A(Z, Y)$, we obtain that $\psi \varphi \in \text{rad}^{p+q}(X, Y) \setminus \text{rad}^{p+q+1}(X, Y)$. Therefore, we conclude that $\text{rad}(X, Y) \neq 0$. \[\square\]

We need also the following lemma (see [46, Lemma 2.1]).

Lemma 2.9. Let $A$ be an algebra, and let $X, Y$ be indecomposable modules in $\mod A$ with $\text{rad}^\infty(X, Y) \neq 0$. Then the following statements hold.

(i) There exist an infinite path

\[
X = X_0 \overset{f_1}{\rightarrow} X_1 \overset{f_2}{\rightarrow} X_2 \rightarrow \cdots \rightarrow X_{i-1} \overset{f_i}{\rightarrow} X_i \rightarrow \cdots
\]

of irreducible homomorphisms between indecomposable modules in $\mod A$ and homomorphisms $g_i \in \text{rad}^\infty(X_i, Y)$, $i \geq 1$, such that $g_if_{i+1}f_i \neq 0$ for all $i \geq 1$. 

(ii) There exist an infinite path

$$\cdots \longrightarrow Y_j \xrightarrow{h_j} Y_{j-1} \longrightarrow \cdots \longrightarrow Y_2 \xrightarrow{h_2} Y_1 \xrightarrow{h_1} Y_0 = Y$$

of irreducible homomorphisms between indecomposable modules in $\text{mod } A$ and homomorphisms $u_j \in \text{rad}^\infty(X, Y_j)$, $j \geq 1$, such that $h_1, \ldots, h_n u_j \neq 0$ for all $j \geq 1$.

**Proposition 2.10.** Let $A$ be an algebra, and let $C$ be a smooth quasitube in $\Gamma_A$ consisting of modules which do not lie on infinite short cycles in $\text{mod } A$. Then $C$ is a generalized standard component of $\Gamma_A$.

**Proof.** Since $C$ is a smooth quasitube of $\Gamma_A$, the stable part $C^0$ of $C$ is a stable tube, say of rank $r$. Denote by $m$ the maximum of stable quasi-length $s$ of the radicals $\text{rad } P$ of the radicals $P$ of projective–injective modules $P$ in $C$. Consider the positive integer $n = m + 2r$, and denote by $\Gamma$ the full translation subquiver of $C$ consisting of all modules of stable quasi-length $\geq n$. Moreover, let $M$ be the direct sum of all indecomposable modules in $C \setminus \Gamma$. Clearly, $M$ is a module in $\text{mod } A$, and hence $\text{End}_A(M)$ is an artin algebra over $k$.

Assume that there are modules $X$ and $Y$ in $C$ such that $\text{rad}^\infty(X, Y) \neq 0$. Then, it follows from **Lemma 2.9(i)** that there exist an infinite path

$$X = X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \longrightarrow \cdots \longrightarrow X_{s-1} \xrightarrow{f_{s-1}} X_s \longrightarrow \cdots$$

of irreducible homomorphisms between indecomposable modules in $\text{mod } A$ and homomorphisms $g_i \in \text{rad}^\infty(X_i, Y_i)$, $i \geq 1$, such that $g_i f_i \neq 0$ for any $i \geq 1$. Since $\text{rad}^\infty(M)$ is nilpotent and $f_i \in \text{rad}(X_i, X_i)$ for all $i \geq 1$, we conclude that there is an integer $s_0 \geq 1$ such that all modules $X_i$, $s \geq s_0$, belong to $\Gamma$. Since $\text{rad}^\infty(X_{s_0}, Y) \neq 0$, applying **Lemma 2.9(ii)**, we conclude that there exist an infinite path

$$\cdots \longrightarrow Y_1 \xrightarrow{h_1} Y_{t-1} \longrightarrow \cdots \longrightarrow Y_2 \xrightarrow{h_2} Y_1 \xrightarrow{h_1} Y_0 = Y$$

of irreducible homomorphisms between indecomposable modules in $\text{mod } A$ and homomorphisms $u_i \in \text{rad}^\infty(Y_i, Y_i)$, $t \geq 1$, such that $h_1, \ldots, h_n u_t \neq 0$ for all $t \geq 1$. Moreover, we conclude as above that, for some integer $t_0 \geq 1$, all modules $Y_i$, $t \geq t_0$, belong to $\Gamma$. Clearly, by our choice of $\Gamma$, the modules $X_{t_0}$ and $Y_{t_0}$ have stable quasi-length bigger than $m + r$. Then, it follows from **Lemma 2.8** that there is a nonzero homomorphism $v \in \text{rad}(Y_{t_0}, X_{t_0})$. Summing up, there is an infinite short cycle in $\text{mod } A$ of the form

$$X_{t_0} \xrightarrow{u} Y_{t_0} \xrightarrow{v} X_{t_0},$$

where $u = u_{t_0}$, with $X_{t_0}$ and $Y_{t_0}$ in $C$, a contradiction. Therefore, $C$ is a generalized standard component of $\Gamma_A$. \qed

**Lemma 2.11.** Let $A$ be an algebra, and let $C$ be a quasitube in $\Gamma_A$. Assume that there exist indecomposable modules $X, Y, M$ in $\text{mod } A$ such that $\text{rad}^\infty(X, M) \neq 0$, $\text{rad}^\infty(M, Y) \neq 0$, and $X$ and $Y$ lie in $C$. Then there is an infinite short cycle $N \to M \to N$ in $\text{mod } A$ with $N$ in $C$.

**Proof.** Since $\text{rad}^\infty(X, M) \neq 0$, it follows from **Lemma 2.9(i)** that there exist an infinite path

$$\Theta: \quad X = X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \longrightarrow \cdots \longrightarrow X_{s-1} \xrightarrow{f_{s-1}} X_s \longrightarrow \cdots$$

of irreducible homomorphisms between indecomposable modules in $\text{mod } A$ and homomorphisms $g_i \in \text{rad}^\infty(X_i, M_i)$, $s \geq 1$, such that $g_i f_i \neq 0$ for any $s \geq 1$. Now, suppose that there is a finite family $\{Z_t\}_{t \geq 0}$ of indecomposable modules in $C$ which are isomorphic with infinitely many modules from the family $\{X_t\}_{t \geq 0}$. Let $Z$ be the direct sum of all modules from the family $\{Z_t\}_{t \geq 0}$. Clearly, $Z$ is a module in $\text{mod } A$, and hence $\text{End}_A(Z)$ is an artin algebra over $k$. Since $f_i \in \text{rad}(X_{i-1}, X_i)$ for all $i \geq 1$, we get then arbitrary large nonzero compositions of homomorphisms from $\text{rad } \text{End}_A(Z)$, and hence, because $\text{rad}_Z(Z)$ is nilpotent, a contradiction. Moreover, since $\text{rad}^\infty(M, Y) \neq 0$, applying **Lemma 2.9(ii)**, we conclude that there exist an infinite path

$$\Sigma: \quad \cdots \longrightarrow Y_1 \xrightarrow{h_1} Y_{t-1} \longrightarrow \cdots \longrightarrow Y_2 \xrightarrow{h_2} Y_1 \xrightarrow{h_1} Y_0 = Y$$

of irreducible homomorphisms between indecomposable modules in $\text{mod } A$ and homomorphisms $u_i \in \text{rad}^\infty(M, Y_i)$, $t \geq 1$, such that $h_1, \ldots, h_n u_t \neq 0$ for all $t \geq 1$. Similarly as above, we conclude that there is no finite family $\{Z_t\}_{t \geq 0}$ of indecomposable modules from $C$ which are isomorphic with infinitely many modules from the family $\{Y_t\}_{t \geq 0}$. Therefore, we conclude that the path $\Theta$ intersects the path $\Sigma$.

Let $N$ be a module in $\Theta \cap \Sigma$. Then there are $s \geq 0$ and $t \geq 0$ such that $X_s = N = Y_t$, and hence we obtain an infinite short cycle $N \xrightarrow{g_t} M \xrightarrow{u_t} N$. \qed
Let $A$ be an algebra, and let $C$ be a family of components of $\Gamma_A$. Then $C$ is said to be sincere if any simple $A$-module occurs as a composition factor of a module in $C$, and faithful if its annihilator $\text{ann}_A(C)$ in $A$ (the intersection of the annihilators of all modules in $C$) is zero. Observe that if $C$ is faithful then $C$ is sincere. Moreover, in general, $\text{ann}_A(C)$ is an ideal of $A$ and $C$ is a faithful family of components in the Auslander–Reiten quiver $I_A/\text{ann}_A(C)$ of the quotient algebra $A/\text{ann}_A(C)$ of $A$. Further, by an external short path in $\text{mod } A$, with respect to a family $C$ of components in $\Gamma_A$, we mean a sequence $X \to Y \to Z$ of nonzero nonisomorphisms between indecomposable modules with $X$ and $Z$ from $C$ but $Y$ not in $C$ [37].

**Lemma 2.12.** Let $A$ be an algebra, and let $C$ and $C'$ be two different ray tubes in $\Gamma_A$ having infinitely many modules with common composition factors and consisting of modules which do not lie on infinite short cycles. Then there are no external short paths in $\text{mod } A$ with respect to the components $C$ and $C'$.

**Proof.** Assume that there is an external short path $M \to L \to M'$, where $M$ is in $C$, $M'$ is in $C'$, and $L$ is neither in $C$ nor in $C'$. First, we will show that then there is an external short path $M \to L \to N$ with $N$ in $C$. It follows from Lemma 2.9(i) that there exist an infinite path

$$\Theta: \cdots \to X_0 \xrightarrow{h_0} X_{-1} \to \cdots \xrightarrow{h_2} X_1 \xrightarrow{h_1} X_0 = M'$$

of irreducible homomorphism between indecomposable modules from $C'$ and homomorphisms $u_s \in \text{rad}^\infty(l, X_s)$, $s \geq 1$, such that $h_1 \ldots h_su_s = 0$ for all $s \geq 1$. Now, assume that there is a finite family $(Z_s)_{s \geq 1}$ of indecomposable modules in $C'$ which are isomorphic to infinitely many modules from the family $X = (X_s)_{s \geq 1}$. Let $Z$ be the direct sum of all modules from the family $(Z_s)_{s \geq 1}$. Clearly, $Z$ is a module in $\text{mod } A$, and hence $\text{End}_A(Z)$ is an artin algebra over $k$. Since $h_1 \in \text{rad}(X_s, X_{s-1})$ for all $s \geq 1$, we get then arbitrary large nonzero compositions of homomorphisms from $\text{rad}_A(Z)$, and hence, because $\text{rad}_A(Z)$ is nilpotent, a contradiction. Therefore, we conclude that the path $\Theta$ intersects each ray in $C'$ at least once. Moreover, it follows from our assumption that there is a ray in $C'$ with infinitely many modules $N'$ such that $[N'] = [N]$ for a module $N$ in $C$. Using the fact that the irreducible homomorphisms lying on rays of the ray tube $C'$ are monomorphisms, we conclude that there is an external short path $M \to L \to M'$, with $M$ in $C$, $M'$ in $C'$, $L$ neither in $C$ nor in $C'$, and there is a module $N$ in $C$ such that $[M'] = [N]$.

Because $[M'] = [N]$, applying [47, Proposition 4.1], we obtain the equality

$$|\text{Hom}_A(L, N)| - |\text{Hom}_A(N, \tau L)| = |\text{Hom}_A(L, M')| - |\text{Hom}_A(M', \tau L)|,$$

where $|V|$ denotes the length of a $k$-module $V$. If $\text{Hom}_A(M', \tau L) \neq 0$, then, by [38, Theorem 1.6], $M'$ is the middle of a short chain, and thus is on a short cycle $M' \to E \to M''$, with $E$ an indecomposable module such that the composition of irreducible monomorphism corresponding to arrowsof the subpath of $E$ from $M'$ to $M''$ does not belong to $C'$. Hence this cycle is infinite, which contradicts our assumption. Thus $\text{Hom}_A(M', \tau L) = 0$, and $\text{Hom}_A(L, N) = 0$. Therefore, we get an external short path $M \to L \to N$, with $M$ and $N$ in $C$. Obviously, then we have $\text{rad}^\infty(M, L) \neq 0$ and $\text{rad}^\infty(L, N) \neq 0$, and hence, applying Lemma 2.11, we conclude that there exists an infinite short cycle $X \to L \to X$ in $\text{mod } A$ with $X$ in $C$. □

**Lemma 2.13.** Let $A$ be an algebra, $B = A/I$ a quotient algebra of $A$, and $T$ a stable tube of $\Gamma_B$. Assume that the modules of $T$ belong to a stable tube $C$ of $\Gamma_A$. Then $C = T$.

**Proof.** In order to prove that $C = T$, it suffices to show that every module $M$ in $C$ is a $B$-module. Because $T \subseteq C$ and the stable tube $T$ consists of infinitely many $B$-modules, then, for every $A$-module $M$ in $C$, there is an $A$-module monomorphism $f : M \to N$, where $N$ is a module lying on a ray in $C$ containing $M$, and there is an $A$-module epimorphism $g : Z \to N$, where $Z$ is a $B$-module from $T$ lying on a ray in $C$ containing $N$. Therefore, $Ml = g(Z)l = g(0) = 0$. Hence, $f(Ml) = f(M)l = 0$, and so $Ml = 0$, because $f$ is a monomorphism. Therefore, $M$ is a $B$-module. □

**Lemma 2.14.** Let $A$ be an algebra, $\Lambda$ a quotient algebra of $A$, and $T$ a stable tube of $\Gamma_\Lambda$. Assume that the modules of $T$ belong to a family $C$ of smooth quasitubes of $\Gamma_\Lambda$ consisting of modules which do not lie on infinite short cycles. Then the modules of $T$ belong to one quasitube of $C$.

**Proof.** Assume that there are two different quasitubes $C_x$ and $C_y$ in $C$ and modules $M, N \in T$ such that $M \in C_x$ and $N \in C_y$. Let $\Theta$ be the infinite sectional path in $T$ starting at $M$ and pointing to infinity, and let $\Sigma$ be the infinite sectional path in $T$ from infinity to $N$. Let $Z$ be a module in $\Theta \cap \Sigma$ and $f : M \to Z$ the composition of irreducible nonisomorphism corresponding to arrows of the subpath of $\Theta$ from $M$ to $Z$ and the composition of irreducible nonisomorphisms corresponding to arrows of the subpath of $\Sigma$ from $Z$ to $N$. Then $f \in \text{rad}_A^\infty(M, Z)$ or $g \in \text{rad}_A^\infty(Z, N)$, because $Z \in C_x$ or $Z \in C_y$ or $Z \in C$, where $Z \neq x, y$. Assume, without lost of generality, that $Z$ is not in $C_x$, and hence $f \in \text{rad}_A^\infty(M, Z)$. Let $L$ be a module in $T$, lying on $\Theta$, with $q(Z)l = q(L)l$, and such that there is a sectional path in $T$ from $L$ to $M$. Then the composed monomorphism $h : M \to L$ belongs to $\text{rad}_A^\infty(M, L)$. Hence, we have the infinite short cycle $M \xrightarrow{h} L \xrightarrow{\nu} M$ in $\text{mod } A$, where $\nu$ is the composition of irreducible nonisomorphism corresponding to the arrows of the sectional path from $L$ to $M$, which contradicts our assumption on $C$. □

**Lemma 2.15.** Let $A$ be an algebra, $\Lambda$ be a quotient algebra of $A$, and let $T$ and $T'$ be orthogonal stable tubes of $\Gamma_\Lambda$. Assume that there exist smooth quasitubes $C$ and $C'$ of $\Gamma_\Lambda$ such that $C$ contains all modules of $T$ and $C'$ contains all modules of $T'$. Then $C$ is different from $C'$. □
Proof. Suppose that \( C = C' \). Let \( r \) be the rank of \( C' \) and \( m \) the maximum of stable quasi-lengths of the radicals of projective-injective modules in \( C \). Since \( T \) and \( T' \) have infinitely many modules, there exist \( X \in T \) and \( X' \in T' \) of stable quasi-lengths \( > m + r \) in \( C' \). Then, by Lemma 2.8, we have \( \text{Hom}_A(X, X') \neq 0 \), which contradicts the orthogonality of \( T \) and \( T' \) in mod \( A \). □

Lemma 2.16. Let \( A \) be an algebra, let \( A \) be a quotient algebra of \( A \), and let \( T = (T_x)_{x \in X} \) be a generalized standard family of stable tubes of \( \Gamma_x \) with common composition factors and consisting of modules which do not lie on infinite short cycles. Assume that there exists a stable tube \( T_x \) in \( T \) such that the modules of \( T_x \) belong to a family \( C \) of smooth quasitubes of \( \Gamma_x \), closed under composition factors, and consisting of modules which do not lie on infinite short cycles. Then all modules of \( T_x \) belong to the family \( C \).

Proof. For each \( x \in X \), we denote by \( r_x \) the rank of \( T_x \). It follows from Lemma 2.14 that the modules of \( T_x \) belong to one quasitube \( C_x \) of \( C \). Because \( T \) is a family of stable tubes with common composition factors, then, for any \( y \in X \), there are modules \( M_y \) in \( T_y \) and \( N_y \) in \( T_y \) such that \( [M_y] = [N_y] \). Now, using the fact that the modules from \( T \) do not lie on infinite short cycles in mod \( A \), we conclude, by Theorem 2.5, that \( r_x \) divides \( \text{ql}(N_y) \) and \( r_x \) divides \( \text{ql}(M_y) \). It follows from Theorem 2.3 that, for any two modules \( N, N' \) in a stable tube \( T_y \) with \( \text{ql}(N) = \text{ql}(N') = c \), for some \( c \geq 1 \), we have \( [N] = [N'] \). Moreover, for any \( N \in T_y \) and any module \( M \) in \( T_y \) with \( \text{ql}(N) = c \text{ql}(N) \) and \( \text{ql}(M) = c \text{ql}(M) \), we get \([N] = [M]\). Now, because the family \( C \) is closed on composition factors, we conclude that, for all \( y \in X \) and for any \( N \in T_y \) with \( \text{ql}(N) = c \text{ql}(N) \), for some \( c \geq 1 \), \( N \) is in the family \( C \).

Let \( N \) be any module in \( T_y \) with \( y \in X \). We will show that \( N \) belongs to the family \( C \). Suppose that \( N \) is not in \( C \). We may choose a module \( M_y \) in \( T_y \) such that \([N] = [M_y]\) for a module \( M_y \in T_y \) and with \( \text{ql}(N) < n = \text{ql}(M_y) \). Then there are sectional paths in \( T_y \)

\[
N_y'' = M_0 \rightarrow M_1 \rightarrow \cdots \rightarrow M_m \rightarrow N
\]

and

\[
N \rightarrow N_m \rightarrow \cdots \rightarrow N_1 \rightarrow N_0 = N_y',
\]

where \( \text{ql}(N_y') = n = \text{ql}(N_y'') \). Since \( r_y \) divides \( n \), we conclude that the modules \( N_y' \) and \( N_y'' \) belong to the family \( C \). Hence, we obtain a nonzero homomorphism from \( \text{rad}^\infty N_y', N_y'' \), which contradicts Proposition 2.10. □

For the convenience of the reader, we give a proof of the following well-known fact, which we need in further considerations.

Lemma 2.17. Let \( A \) be an algebra and \( T \) a faithful stable tube of \( \Gamma_A \). Then all but finitely many indecomposable modules in \( T \) are faithful \( A \)-modules.

Proof. First, notice that, for any ray \( \Sigma \) in \( T \) and a module \( M \) lying on \( \Sigma \), we have \( \text{ann}(M) \subseteq \text{ann}(M') \) for every module \( M' \) lying on \( \Sigma \) with \( \text{ql}(M') > \text{ql}(M) \). Now, because \( T \) is a faithful stable tube of \( \Gamma_A \), there are indecomposable modules \( M_1, \ldots, M_r \) in \( T \) such that \( \text{ann}(\bigoplus_{i=1}^r M_i) = 0 \). Let \( n \) be the maximum of quasi-lengths of modules \( M_1, \ldots, M_r \). Then, for every module \( Z \) with quasi-length bigger than \( n + r \), where \( r \) is the rank of \( T \), the unique sectional path in \( T \), starting at \( Z \) and pointing to the mouth, intersects every ray containing modules from \( \{M_1, \ldots, M_r\} \) and consists of epimorphisms. Note that, for an epimorphism \( f : X \rightarrow Y \), we have \( \text{ann}(X) \subseteq \text{ann}(Y) \), because \( Y \text{ann}(X) = f(X) \text{ann}(X) = f(\text{ann}(X)) = \text{f}(0) = 0 \). Therefore, \( \text{ann}(Z) \subseteq \text{ann}(\bigoplus_{i=1}^r M_i) = 0 \), and hence \( Z \) is a faithful module. □

3. Quasitilted algebras of canonical type

The purpose of this section is to present characterizations of quasitilted algebras of canonical type.

Let \( A \) be an algebra. Then a family \( C \) of components of \( \Gamma_A \) is said to be separating in mod \( A \) if the indecomposable modules in mod \( A \) split into three disjoint classes \( \mathcal{P}^A, \mathcal{C}^A, \mathcal{Q}^A \) such that

(S1) \( \mathcal{C}^A \) is a sincere generalized standard family of components; (S2) \( \text{Hom}_A(\mathcal{Q}^A, \mathcal{P}^A) = 0, \text{Hom}_A(\mathcal{Q}^A, \mathcal{C}^A) = 0 \), and \( \text{Hom}_A(\mathcal{C}^A, \mathcal{P}^A) = 0 \); and

(S3) any homomorphism from \( \mathcal{P}^A \) to \( \mathcal{Q}^A \) factors through the additive category add \( \mathcal{C}^A \) of \( \mathcal{C}^A \).

Algebras with a separating family of stable tubes have attracted much attention. A prominent class of algebras with this property is formed by the canonical algebras, introduced by Ringel (see [40, 41]). Hence, for a canonical algebra \( \Lambda, \Gamma_A \) admits a decomposition \( \Gamma_A = \mathcal{P}^A \vee \mathcal{T}^A \vee \mathcal{Q}^A \), where \( \mathcal{T}^A \) is a (canonical) family of stable tubes separating \( \mathcal{P}^A \) from \( \mathcal{Q}^A \). Following [28], an algebra \( C \) is called concealed canonical of type \( \Lambda \) if \( C \) is the endomorphism algebra \( \text{End}_A(T) \) for a tilting module from add \( \mathcal{P}^A \). Then the images of all modules from \( T \) via the functor \( \text{Hom}_A(T, -) \) form a separating family \( \mathcal{C}^A \) of stable tubes of \( \Gamma_C \). In particular, we have a decomposition \( \Gamma_C = \mathcal{P}^C \vee \mathcal{T}^C \vee \mathcal{Q}^C \). We note that \( \mathcal{T}^C \) is a family of stable tubes \( \mathcal{T}^C_x, x \in \mathcal{X} \), where the index set \( \mathcal{X} \) is in a natural bijection with the set of stable tubes of a tame hereditary algebra \[
\begin{bmatrix}
F & M \\
0 & G
\end{bmatrix}
\]
where \( F \) and \( G \) are finite central skew fields, \( F \)-modules \( M \) satisfies \( \dim_F M = 4 \) (see [10, 39, 41]). Moreover, if \( k \) is an algebraically closed field, then \( \mathcal{X} \) is in a natural bijection with the projective line \( \mathbb{P}_1(k) \) [40], and is equipped with the structure of a weighted projective line [14]. It has been proved in [30, Theorem 1.1] that an algebra \( C \) is a concealed canonical algebra if and only if \( \Gamma_C \) admits a separating family of stable tubes.
An algebra $A$ is said to be a quasitilted algebra of canonical type if $A = \text{End}_\mathcal{H}(T)$, where $T$ is a tilting object in an abelian hereditary category $\mathcal{H}$ whose derived category $D^b(\mathcal{H})$ is equivalent (as a triangulated category) to the derived category $D^b(\text{mod } A)$ of the module category mod $A$ of a canonical algebra $A$.

An algebra $A$ is said to be an almost concealed canonical algebra if $A$ is the endomorphism algebra $\text{End}_\mathcal{H}(T)$ of a tilting module $T$ from the additive category $\text{add}(P^A \vee T^A)$, for the canonical decomposition $\Gamma'_A = P^A \vee T^A \vee Q^A$, with $\mathcal{T}^A$ the canonical family of stable tubes separating $P^A$ from $Q^A$ over a canonical algebra $A$. It has been proved in [31, Theorem 3.4] that $A$ is quasitilted if and only if $\Gamma'_A$ admits a separating family of semiregular (ray or coray) tubes. Moreover, the class of almost concealed canonical algebras coincides with the class of tubular extensions of concealed canonical algebras (using modules from the canonical family of stable tubes), and with the class of algebras having a separating family of ray tubes (see [28, Theorem 3.1] and [31, Theorem 3.4]).

We will need the following deeper characterization of almost concealed canonical algebras (see [47, Proposition 3.5] and [49, Theorem 1.6]).

**Theorem 3.1.** An algebra $A$ is an almost concealed canonical algebra if and only if $\Gamma'_A$ has a sincere generalized standard family of ray tubes without external short paths.

We have also the following characterization of concealed canonical algebras (see [37, Theorem 3.1], [48, Theorem C], and [49, Theorem 1.6]).

**Theorem 3.2.** An algebra $A$ is a concealed canonical algebra if and only if $\Gamma'_A$ has a sincere family of pairwise orthogonal stable tubes without external short paths.

We will need the following consequence of the above theorem (see [49, Corollary 1.7]).

**Corollary 3.3.** Let $A$ be an algebra and $\mathcal{T}$ a sincere stable tube in $\Gamma'_A$ without external short cycles. Then $\mathcal{T}$ is a faithful generalized standard stable tube and $A$ is a concealed canonical algebra.

### 4. Quasitube enlargements of algebras

In this section, we introduce quasitube enlargements of algebras, essential for our further considerations.

Let $A$ be an algebra, $F$ a division algebra, and $fM_A$ an $F$-$A$-bimodule such that $M_A$ is in mod $A$ and $k$ acts centrally on $fM_A$. Then the one-point extension of $A$ by $M$ is the matrix algebra of the form

$$A[M] = \begin{bmatrix} F & fM_A \\ 0 & A \end{bmatrix} \quad \text{with } \begin{cases} f \in F, \ a \in A, \ m \in M \end{cases}$$

with the usual addition and multiplication. Dually, one also defines the one-point coextension of $A$ by $fM_A$ as the matrix algebra

$$[M]A = \begin{bmatrix} A & D(fM_A) \\ 0 & F \end{bmatrix} \quad ,$$

Let $A$ be an algebra and $\Gamma'$ a generalized standard component of $\Gamma'_A$. For each indecomposable module $X$ in $\Gamma'$ which is a pivot of an admissible operation of type (ad 1), (ad 2), (ad 1'), or (ad 2'), we shall define the corresponding admissible operation on $A$ in such a way that the modified translation quiver $\Gamma''$ is a component of the Auslander–Reiten quiver $\Gamma'_A$ of the modified algebra $A'$ (see [45]). Since $\Gamma'$ is generalized standard, such a pivot $X$ is necessarily a brick, and we denote by $F$ the division algebra $\text{End}_A(X)$.

Clearly, $X$ is an $F$-$A$-bimodule. Suppose that $X$ is the pivot of an admissible operation of type (ad 1) and that $t \geq 1$. Denote $D = D_t$ the full $t \times t$ upper triangular matrix algebra over the division algebra $F$ and by $Y$ the unique indecomposable projective-injective $D$-module, which we consider as an $F$-$D$-bimodule. Then $A' = (A \times D)[X \oplus Y]$ is the required modified algebra. If $X$ is the pivot of an admissible operation of type (ad 2), then the modified algebra $A'$ is defined to be $A' = A[X]$. Dually, invoking the one-point coextensions, one defines the modified algebra $A'$, if $X$ is a pivot of an admissible operation of type (ad 1') or (ad 2'). Then the following fact mentioned above holds (see [4, Section 2]).

**Lemma 4.1.** The modified translation quiver $\Gamma''$ of $\Gamma'$ is a component of $\Gamma'_A$.

Let $C$ be an algebra and $\mathcal{T}$ a generalized standard family of stable tubes in $\Gamma'_C$. Following [5], an algebra $B$ is said to be a quasitube enlargement of $C$ using modules from $\mathcal{T}$ if there is a finite sequence of algebras $A_0 = C, A_1, \ldots, A_m = B$ such that, for each $0 \leq j < m, A_{j+1}$ is obtained from $A_j$ by an admissible operation of type (ad 1), (ad 2), (ad 1'), or (ad 2'), with pivot either in a stable tube of $\mathcal{T}$ or in a quasitube of $\Gamma'_B$ obtained from a stable tube of $\mathcal{T}$ by means of the sequence of admissible operations of types (ad 1), (ad 2), (ad 1'), (ad 2')) done so far. We note that a tubular extension (respectively, tubular coextension) of $C$ (in the sense of [40]), using modules from $\mathcal{T}$, is just an enlargement of $C$ invoking only admissible operations of type (ad 1) (respectively, type (ad 1')).

We have the following proposition (see [4, Lemmas 2.2 and 2.3] and [35, Theorem C]).

**Proposition 4.2.** Let $B$ be a quasitube enlargement of an algebra $C$ using modules from a generalized standard family $\mathcal{T}$ of stable tubes of $\Gamma'_C$, and $\mathcal{C}$ the family of components of $\Gamma'_B$ obtained from $\mathcal{T}$ by means of admissible operations leading from $C$ to $B$. Then $\mathcal{C}$ is a generalized standard family of quasitubes of $\Gamma'_B$. 

Moreover, we have the following theorem (see [5, Theorem 3.5] and [35, Theorem C]).

**Theorem 4.3.** Let $C$ be a concealed canonical algebra and $T$ a separating family of stable tubes of $\Gamma_C$. Let $B$ be a quasitube enlargement of $C$, using modules from $T$, and $E$ the associated generalized standard family of quasitubes of $\Gamma_B$. Then the following statements hold.

(i) There is a unique maximal tubular coextension $B_1$ of $C$ inside $B$ and a generalized standard family $E^1$ of coray tubes of $\Gamma_B$ such that $B$ is obtained from $B_1$ (respectively, $C$ is obtained from $E^1$) by a sequence of admissible operations of types (ad 1) and (ad 2), using modules from $E^1$.

(ii) There is a unique maximal tubular extension $B_2$ of $C$ inside $B$ and a generalized standard family $E^2$ of ray tubes of $\Gamma_B$ such that $B$ is obtained from $B_2$ (respectively, $C$ is obtained from $E^2$) by a sequence of admissible operations of types (ad 1') and (ad 2'), using modules from $E^2$.

For a quasitube enlargement $B$ of a concealed canonical algebra $C$, the maximal tubular extension $B_2$ of $C$ inside $B$ is an almost concealed algebra, called the right quasitilted part of $B$. Similarly, the maximal tubular coextension $B_1$ of $C$ inside $B$ is the opposite algebra of an almost concealed algebra, called the left quasitilted part of $B$.

We note that a quasitube of an Auslander–Reiten quiver is an almost cyclic coherent component in the sense of [34]. The following theorem is then a special case of a characterization of algebras with separating families of almost cyclic coherent Auslander–Reiten components established in [35, Theorem A].

**Theorem 4.4.** Let $A$ be a basic, connected, artin algebra. The following statements are equivalent.

(i) $I_A$ admits a separating family of quasitubes.

(ii) $I_A$ admits a sincere generalized standard family of quasitubes without external short paths.

(iii) $A$ is a quasitube enlargement of a concealed canonical algebra $C$.

5. Selfinjective orbit algebras

For an algebra $A$, we denote by $D$ the standard duality $\text{Hom}_k(-, E)$ on mod $A$, where $E$ is a minimal injective cogenerator in mod $k$. Then an algebra $A$ is selfinjective if and only if $A \cong \text{D}(A)$ in mod $A$. If $A$ is selfinjective, then the left socle and the right socle of $A$ coincide, and we denote them by soc $A$. Two selfinjective algebras $A$ and $B$ are said to be socle equivalent if the factor algebras $A$/soc $A$ and $B$/soc $A$ are isomorphic.

Let $A$ be a selfinjective algebra and $\{e_i \mid 1 \leq i \leq s\}$ a complete set of orthogonal primitive idempotents of $A$. We denote by $v = v_{A,e}$ the Nakayama automorphism of $A$ inducing an $A$-$A$-bimodule isomorphism $A \cong \text{D}(A)_v$, where $A_v$ denotes the right $A$-module obtained from $A$ by changing the right operation of $A$ as follows: $f \cdot a = f(v(a))$ for each $a \in A$ and $f \in D(A)$. Hence we have $\text{soc}(v(e_i)A) \cong \text{top}(e_iA) = e_iA/\text{rad}(e_iA))$ as right $A$-modules for all $i \in \{1, \ldots , s\}$. Since $\{v(e_i)A \mid 1 \leq i \leq s\}$ is a complete set of representatives of indecomposable projective right $A$-modules, there is a (Nakayama) permutation of $\{1, \ldots , s\}$, denoted again by $v$, such that $v(e_i)A \cong e_{v(i)}A$ for all $i \in \{1, \ldots , s\}$. Invoking the Krull–Schmidt theorem, we may assume that $v(e_i)A = v(e_j)A = e_{v(i)}A$ for all $i \in \{1, \ldots , s\}$.

Let $B$ be an algebra. The *repetitive algebra* $B$ of $B$ [20] is an algebra (without identity) whose $k$-module structure is that of

$$\bigoplus_{m \in \mathbb{Z}}(B_m \oplus D(B)_m)$$

where $B_m = B$ and $D(B) = D(B)$ for all $m \in \mathbb{Z}$, and the multiplication is defined by

$$(a_m, f_m)(b_m, g_m) = (a_mb_m, a_mb_m + f_m b_{m-1})$$

for $a_m, b_m \in B_m, f_m, g_m \in D(B)_m$. For a fixed set $\mathcal{E} = \{e_i \mid 1 \leq i \leq n\}$ of orthogonal primitive idempotents of $B$ with $1_B = e_1 + \cdots + e_n$, consider the canonical set $\hat{\mathcal{E}} = \{e_{m,i} \mid m \in \mathbb{Z}, 1 \leq i \leq n\}$ of orthogonal primitive idempotents of $B$ such that $e_{m,i}B = (e_mB_m + e_iD(B)_m)$ for $m \in \mathbb{Z}$ and $1 \leq i \leq n$. By an automorphism of $\hat{B}$ we mean a $k$-algebra automorphism of $B$ which fixes the chosen set $\hat{\mathcal{E}}$ of orthogonal primitive idempotents of $B$. A group $G$ of automorphisms of $B$ is said to be admissible if the induced action of $G$ on $\hat{\mathcal{E}}$ is free and has finitely many orbits. Then the *orbit algebra* $\hat{B}/G$ is a finite-dimensional selfinjective algebra and the $G$-orbits in $\hat{\mathcal{E}}$ form a canonical set of orthogonal primitive idempotents of $B/G$ whose sum is the identity of $B/G$. We denote by $\hat{v}_G$ the Nakayama automorphism of $B$ such that $\hat{v}_G(e_{m,i}) = e_{m+1,i}$ for all $m \in \mathbb{Z}, 1 \leq i \leq n$. Then the infinite cyclic group $(\mathbb{Z}/\mathbb{Z})$ generated by $\hat{v}_G$ is admissible and $\hat{B}/(\hat{v}_G)$ is the trivial extension $B \times D(B)$ of $B$ by $D(B)$. An automorphism $\varphi$ of $\hat{B}$ is said to be positive (respectively, rigid) if $\varphi(B_m) \subseteq \sum_{m \in \mathbb{Z}}B_j$ (respectively, $\varphi(B_m) = B_m$) for any $m \in \mathbb{Z}$. Finally, $\varphi$ is said to be strictly positive if $\varphi$ is positive but not rigid.

Let $A$ be a selfinjective algebra, an ideal of $A$, $B = A/I$, and $e$ an idempotent of $A$ such that $e + I$ is the identity of $B$. We may assume that $e = e_1 + \cdots + e_n$, where $\{e_i \mid 1 \leq i \leq n\}$ is a complete set of orthogonal primitive idempotents of $A$ which are not in $I$. Then such an idempotent $e$ is uniquely determined by $I$, up to an inner automorphism of $A$, and we call it a *residual identity of $B$* [52]. Note that $B \cong eAe/ee$ and $1 - e \in I$. We denote by $l_{A}(I)$ and $r_{A}(I)$ the left and right annihilators of $I$ in $A$, respectively. Following [52, (2.1)] the ideal $I$ is said to be deforming if $\text{id}_{e} = l_{A}(I) = r_{A}(I)$ and $A/I$ is triangular (the ordinary quiver of $A/I$ has no oriented cycles). The following lemma has been proved in [57, Lemma 4.1].
Lemma 5.1. Let $A$ be a selfinjective algebra, $e$ an idempotent of $A$, and assume that $I_A(I) = eI$ or $r_A(I) = eI$. Then $e$ is a residual identity of the factor algebra $A/I$.

Moreover, the following proposition has been proved in [52, Proposition 2.3].

Proposition 5.2. Let $A$ be a selfinjective algebra, $I$ an ideal of $A$, $B = A/I$, $e$ a residual identity of $B$, and assume that $Ie = 0$. Then the following conditions are equivalent.

(i) $le$ is an injective cogenerator in mod $B$.
(ii) $el$ is an injective cogenerator in mod $B^{op}$.
(iii) $I_A(I) = le$.
(iv) $r_A(I) = el$.

Moreover, under these equivalent conditions, we have $ele = I_{eAe}(I) = r_{eAe}(I)$.

Let $A$ be a selfinjective algebra, $I$ a deforming ideal of $A$, and $e$ a residual identity of $A/I$. Then $I$ can be considered as a (not necessarily unitary) $(eAe/eAe)$-(eAe/eAe)-bimodule. Denote by $A[I]$ the direct sum of $k$-modules $(eAe/eAe) \oplus I$ with the multiplication

$$(b, x) \cdot (b', x') = (bb', bx' + xb' + xx')$$

for $b, b' \in eAe/eAe$ and $x, x' \in I$. Then $A[I]$ is an algebra with the identity $(e, \ 1 - e)$ and, by identifying $x \in I$ with $(0, x) \in A[I]$, we may consider $I$ as an ideal of $A[I]$.

The following combination of results proved in [52, Theorem 4.1], [53, Theorem 3], and [56, Proposition 3.2] establishes the relationship between $A$ and $A[I]$.

Theorem 5.3. Let $A$ be a selfinjective algebra, $I$ a deforming ideal of $A$, and $e$ a residual identity of $A/I$. Then the following statements hold.

(i) $A[I]$ is a selfinjective algebra, $I$ is a deforming ideal of $A[I]$, and the Nakayama permutations of $A$ and $A[I]$ are the same.
(ii) $A$ and $A[I]$ are socle equivalent.
(iii) Assume that $IeI = 0$ and $e_i \neq e_{r(i)}$ for any primitive summand $e_i$ of $e$. Then $A$ and $A[I]$ are isomorphic.

The following criterion is a direct consequence of [54, Theorems 3.8 and 4.1] and Proposition 5.2.

Theorem 5.4. Let $A$ be a selfinjective algebra, $I$ an ideal of $A$, $B = A/I$, and $e$ a residual identity of $B$. Assume that $B$ is triangular and that $I_A(I) = le$. Then $A[I]$ is isomorphic to an algebra $B/(\psi \psi B)$, for some positive automorphism $\psi$ of $B$.

6. Selfinjective algebras of canonical type

A selfinjective algebra $A$ is said to be a selfinjective algebra of canonical type if $A$ is isomorphic to an orbit algebra $\widehat{B}/G$, where $B$ is a quasitilted algebra of canonical type and $G$ is an admissible torsion-free automorphism group of $B$.

The following general result is a consequence of results proved in [1,12,32,36,44].

Theorem 6.1. Let $B$ be a quasitilted algebra of canonical type, $G$ an admissible torsion-free group of automorphisms of $\widehat{B}$, and $A = \widehat{B}/G$ the associated orbit algebra. Then the following statements hold.

(i) $G$ is an infinite cyclic group generated by a strictly positive automorphism $\psi$ of $\widehat{B}$.
(ii) The push-down functor $\Gamma_\psi : \text{mod} \ B \to \text{mod} \ A$ associated to the Galois covering $F : \widehat{B} \to \widehat{B}/G = A$ with Galois group $G$ is dense.
(iii) The Auslander–Reiten quiver $\Gamma_\psi$ of $A$ is isomorphic to the orbit quiver $\Gamma_{\widehat{B}}/G$ of the Auslander–Reiten quiver $\Gamma_{\widehat{B}}$ of $\widehat{B}$ with respect to the induced action of $G$ on $\Gamma_{\widehat{B}}$.

The following proposition (see [1,32,36,44]) relates the selfinjective algebras of canonical type with almost concealed canonical algebras.

Proposition 6.2. Let $B$ be a quasitilted algebra of canonical type. Then there exists an almost concealed canonical algebra $B^*$ such that $B = \widehat{B}^*$.

We note that in general we may have several almost concealed canonical algebras whose repetitive algebras are isomorphic.

The class of selfinjective algebras of canonical type may be divided into three disjoint classes, according to the natural division of almost concealed canonical algebras into three disjoint classes.

Let $B$ be an almost concealed canonical algebra, $G$ an admissible infinite cyclic automorphism group of $\widehat{B}$, and $A = \widehat{B}/G$. Then $A$ is said to be

- a selfinjective algebra of Euclidean type, if $B$ is a tilted algebra of Euclidean type;
- a selfinjective algebra of tubular type, if $B$ is a tubular algebra;
- a selfinjective algebra of wild canonical type, if $B$ is of wild canonical type

(see [58, Section 7]).
Let $B$ be a tubular algebra, $G$ an infinite cyclic admissible group of automorphisms of $B$, and $A = \widehat{B}/G$. Then the following hold.

Theorem 6.3. Let $A$ be a selfinjective algebra of canonical type. Then the Auslander–Reiten quiver $\Gamma_A$ of $A$ has the form

\[
\begin{array}{c}
\cdots \\
\chi^{(r-2)} \\
\chi^{(r-1)} \\
\chi^{(0)} \\
\chi^{(1)} \\
\chi^{(2)} \\
\chi^{(3)} \\
\chi^{(4)} \\
\cdots
\end{array}
\]

for some integer $r \geq 1$, where each $\chi^{(i)}$, $i \in \{0, \ldots, r - 1\}$, is an infinite family of quasitubes, and

(i) if $A$ is of Euclidean type, then every $\chi^{(i)}$, $i \in \{0, \ldots, r - 1\}$, is an acyclic component of Euclidean type (the stable part is of the form $\mathbb{Z}\Delta$ for an Euclidean quiver $\Delta$);

(ii) if $A$ is of tubular type, then every $\chi^{(i)}$, $i \in \{0, \ldots, r - 1\}$, is a disjoint union $\bigvee_{q \in \hat{Q}^{i+1}} \chi^{(i)}_q$, where $\chi^{(i)}_q$ is an infinite family of stable tubes for each $q \in \hat{Q}^{i+1} = \hat{Q} \cap (i, i + 1)$; and

(iii) if $A$ is of wild canonical type, then every $\chi^{(i)}$, $i \in \{0, \ldots, r - 1\}$, is an infinite family of components whose stable parts are of the form $\mathbb{Z}\Delta_{\infty}$.

We call the above decomposition of $\Gamma_A$ a canonical decomposition of $\Gamma_A$.

The main aim of the remaining part of this section is to prove two propositions which show the implication (ii)$\Rightarrow$(i) of Theorem 1.1.

Proposition 6.4. Let $B$ be a tubular algebra, $G$ an infinite cyclic admissible group of automorphisms of $\widehat{B}$, and $A = \widehat{B}/G$. Then the following statements are equivalent.

(i) $\Gamma_A$ admits a family of quasitubes with common composition factors, closed under composition factors, and consisting of modules which do not lie on infinite short cycles.

(ii) $\Gamma_A$ admits a family of stable tubes with common composition factors, closed under composition factors, and consisting of modules which do not lie on infinite short cycles.

(iii) $G = (\varphi_\gamma^B)$ for a positive automorphism $\varphi$ of $B$.

Proof. It follows from the results established in [14, 19, 36, 44] (see also [9, 24]) that the Auslander–Reiten quiver $\Gamma_B$ of $\widehat{B}$ has a decomposition

\[\Gamma_B = \bigvee_{q \in \hat{Q}} \chi^{(i)}_q = \bigvee_{q \in \hat{Q}, x \in \hat{X}_q} \chi^{(i)}_{q,x}\]

such that the following hold.

(1) For each $q \in \mathbb{Z}$, $\chi^{(i)}_q$ is an infinite family $\chi^{(i)}_{q,x}$, $x \in \hat{X}_q$, of quasitubes containing at least one projective module.

(2) For each $q \in \mathbb{Q} \setminus \mathbb{Z}$, $\chi^{(i)}_q$ is an infinite family $\chi^{(i)}_{q,x}$, $x \in \hat{X}_q$, of stable tubes.

(3) For each $q \in \mathbb{Q}$, $\chi^{(i)}_q$ is a family of pairwise orthogonal generalized standard quasitubes with common composition factors, closed under composition factors, and consisting of modules which do not lie on infinite short cycles in mod $\widehat{B}$.

(4) There is a positive integer $m$ such that $3 \leq m \leq \rk K_0(B)$ and $v_B(\chi^{(i)}_q) = \chi^{(i+1)}_q$ for any $q \in \mathbb{Q}$.

(5) $\Hom_B(\chi^{(i)}_p, \chi^{(i)}_r) = 0$ for all $q > r$ in $\mathbb{Q}$.

(6) $\Hom_B(\chi^{(i)}_{q+p}, \chi^{(i)}_{q+m}) = 0$ if and only if $q \in \mathbb{Z}$.

(7) For $q \in \mathbb{Q}$, we have $\Hom_B(\chi^{(i)}_{q+p}, \chi^{(i)}_{q+m}) \neq 0$ for any $r \in \mathbb{Q}$ with $p \leq r \leq q$.

(8) For $p < q$ in $\mathbb{Q}$, we have $\Hom_B(\chi^{(i)}_{q+p}, \chi^{(i)}_q) \neq 0$ and $\Hom_B(\chi^{(i)}_r, \chi^{(i)}_q) \neq 0$ for any $r \in \mathbb{Q}$.

(9) For all $p \in \mathbb{Q} \setminus \mathbb{Z}$ and all $q \in \mathbb{Q}$, we have $\Hom_B(\chi^{(i)}_{q+p}, \chi^{(i)}_q) \neq 0$ and $\Hom_B(\chi^{(i)}_{q+p}, \chi^{(i)}_q) \neq 0$ for all $x \in \hat{X}_p$ and $y \in \hat{X}_q$.

(10) For all $p \in \mathbb{Q}$ and all $q \in \mathbb{Q} \setminus \mathbb{Z}$, we have $\Hom_B(\chi^{(i)}_{q+p}, \chi^{(i)}_q) \neq 0$ and $\Hom_B(\chi^{(i)}_{q+p}, \chi^{(i)}_q) \neq 0$ for all $x \in \hat{X}_p$ and $y \in \hat{X}_q$. 

\[\]
We know also from [44] (see Theorem 6.1(i)) that $G$ is generated by a strictly positive automorphism $g$ of $\hat{B}$. Consider the canonical Galois covering $F: \hat{B} \to \hat{B}/G = A$ and the associated push-down functor $F_\chi: \text{mod } \hat{B} \to \text{mod } A$. Since $F_i$ is dense, we obtain natural isomorphisms of $k$-modules

$$\bigoplus_{i \in \mathbb{Z}} \text{Hom}_B(X, s^i Y) \cong \text{Hom}_A(F_i(X), F_i(Y)),$$

$$\bigoplus_{i \in \mathbb{Z}} \text{Hom}_B(s^i X, Y) \cong \text{Hom}_A(F_i(X), F_i(Y)),$$

for all indecomposable modules $X$ and $Y$ in $\text{mod } \hat{B}$.

We first show that (iii) implies (ii). Assume that $g = \varphi v^2$ for some positive automorphism $\varphi$ of $\hat{B}$. Then it follows from (4) that there is a positive integer $l \geq 2m$ such that $g(C^x_q) = C^x_{q+l}$ for any $q \in \mathbb{Q}$. Since $g = \varphi v^2 = (\varphi v^2)v^2$ with $\varphi v^2$ a strictly positive automorphism of $\hat{B}$, invoking the knowledge of the supports of indecomposable modules in $\text{mod } \hat{B}$ (see [36, Section 3]), we conclude that the images $F_i(S)$ and $F_i(T)$ of any nonisomorphic simple $\hat{B}$-modules $S$ and $T$ which occur as composition factors of modules in a fixed family $C^x_q$, are nonisomorphic simple $A$-modules. Therefore, it follows from Theorem 6.1 and properties (1)--(4) that, for each $q \in \mathbb{Q}$, $C^x_q = F_i(C^x_q)$ is an infinite family $C^x_{q+r, x} = F_i(C^x_{q+r, x})$, $x \in X_q$, of quasitubes of $\Gamma^x_{\chi}$, with common composition factors, and closed under composition factors. Take now $p \in \mathbb{Q} \setminus \mathbb{Z}$, from property (2), $C^x_p = (C^x_{p, x})_x$ is a family of stable tubes of $\Gamma^x_{\chi}$. We claim that $C^x_p$ consists of indecomposable $A$-modules which do not lie on infinite short cycles in mod $A$. Observe first that, for two indecomposable modules $M$ and $N$ in $C^x_p$, we have $M = F_i(X)$ and $N = F_i(Y)$, for some indecomposable modules $X$ and $Y$ in $C^x_p$, and $F_i$ induces an isomorphism of $k$-modules $\text{Hom}_A(M, N) \cong \bigoplus_{i \in \mathbb{Z}} \text{Hom}_B(X, s^i Y)$.

Since $\text{Hom}_A(M, L) \neq 0$, we may choose, invoking property (5), a minimal $r > p$ and $Z \in C^x_r$ such that $L = F_i(Z)$ and $\text{Hom}_B(X, Z) \neq 0$. Since $p \in \mathbb{Q} \setminus \mathbb{Z}$ and $X$ lies in $C^x_p$, applying properties (6) and (7), we infer that $p < r < p + m$. Further, we have also an isomorphism of $k$-modules, induced by $F_i$,

$$\bigoplus_{i \in \mathbb{Z}} \text{Hom}_A(L, M) \cong \bigoplus_{i \in \mathbb{Z}} \text{Hom}_B(Z, s^i Y).$$

Observe that, for each $i \in \mathbb{Z}$, $s^i X$ is an indecomposable module from $C^x_{p+i}$, and clearly with $F_i(s^i X) = F_i(X) = M$. Since $\text{Hom}_A(L, M) \neq 0$, $L = F_i(Z)$ for $Z \in C^x_r$ with $r > p$ and $X \in C^x_p$, applying property (5), we conclude that $\text{Hom}_B(Z, s^i X) \neq 0$, for some $i \geq 1$. But then $p + li > p + l > p + m > r + m$, because $r < p + m$, and we obtain a contradiction with property (6).

Summing up, we have proved that $C^x_p = F_i(C^x_p)$ is a family of stable tubes of $\Gamma^x_{\chi}$ with common composition factors, closed under composition factors, and consisting of modules which do not lie on infinite short cycles in mod $A$. Therefore, (iii) implies (ii).

Since clearly (ii) implies (i), it remains to show that (i) implies (iii). Assume that $\Gamma^x_{\chi}$ admits a family $C = (C^x_{p, x})_x$ of quasitubes with common composition factors, closed under composition factors, and consisting of modules which do not lie on infinite short cycles in mod $A$. We know from property (3) that, for each $q \in \mathbb{Q}$, $C^x_q = F_i(C^x_q)$ is a family $C^x_{q+r, x} = F_i(C^x_{q+r, x})$, $x \in X_q$, of quasitubes with common composition factors. Moreover, the push-down functor $F_i$ induces an isomorphism of translation quivers $\Gamma^x_{\hat{B}}/G \cong \Gamma^x_{\chi}$ (see Theorem 6.1), and hence every component of $\Gamma^x_{\chi}$ is a quasitube of the form $C^x_{q, x} = F_i(C^x_{q, x})$ for some $q \in \mathbb{Z}$ and $x \in X_q$. Then, since the family $C$ is closed under composition factors, we conclude that there is $r \in \mathbb{Q}$ such that $C$ contains all quasitubes $C^x_{i, x} \in X_r$ of $C^x_i$. In particular, we conclude that the family $C^x_r = (C^x_{r, x})_{x \in X_r}$ consists of modules which do not lie on infinite short cycles in mod $A$. We claim that this forces $g$ to be of the form $g = \varphi v^2$ for some positive automorphism $\varphi$ of $\hat{B}$. Suppose that this is not the case. Since $g$ is a strictly positive automorphism of $\hat{B}$ and all projective $\hat{B}$-modules lie in $\bigcup_{p \in \mathbb{Z}} C^x_p$, invoking property (4), we conclude that there exists a positive integer $s < 2m$ such that $g(C^x_q) = C^x_{q+s}$ for any $q \in \mathbb{Q}$. Let $p$ be the natural number such that $r \in [p, p + 1) \cap \mathbb{Q}$. We have two cases to consider.
Assume that $s < m$. Take $q \in (Q \setminus Z) \cap (p + 1, p + s)$. Since $m \geq 3$, we have the inequalities

$$p \leq r \leq p + 1 < q \leq p + s \leq r + s < p + m$$

which, together with properties (7) and (8), imply that $\text{Hom}_B(e_{r,q}^B, e_{r+1,q}^B) \neq 0$ and $\text{Hom}_B(e_{r,s}^B, e_{r+1,s}^B) \neq 0$. Moreover, from properties (9) and (10), we obtain that

$$\text{Hom}_B(e_{r,s}^B, e_{r+1,s}^B) \neq 0 \quad \text{for any } x \in X_r \text{ and } y \in X_q,$$

$$\text{Hom}_B(e_{r,s}^B, e_{r+1,s}^B) \neq 0 \quad \text{for any } x' \in X_{r+s} \text{ and } y' \in X_q,$$

because $q \in (Q \setminus Z)$. Since $\text{Hom}_B(e_{s+1,q}^B, e_{s+2,q}^B) = 0$, there exist $x \in X_r, y \in X_q$ and modules $X \in e_{r,s}^B, Y \in e_{r,y}^B$ and $X' \in e_{s+1,q}^B$ such that $\text{Hom}_B(X, Y) \neq 0, \text{Hom}_B(Y, X') \neq 0$ and $F_s(X) = F_s(X')$. Hence we have an infinite short cycle $F_s(X) \rightarrow F_s(Y) \rightarrow F_s(X') = F_s(X)$ in mod $A$ with $F_s(X)$ in $e_{r,s}^B$, which contradicts our assumption.

Finally, assume that $m \leq s < 2m$. Take $q = p + m$. We have the inequalities

$$p \leq r \leq p + 1 < q \leq p + s \leq r + s < p + 2m$$

Because $p + m - 1 \in Z$, $\text{Hom}_B(e_{p+m-1, q}^B, e_{p+2m-1, q}^B)$ is non-empty, and hence from property (8) we get $\text{Hom}_B(e_{q, p+m-1}^B, e_{p+2m-1, q}^B) \neq 0$, and so $\text{Hom}_B(e_{q, p+m-1}^B, e_{p+2m-1, q}^B) \neq 0$. Using properties (9) and (10), we obtain

$$\text{Hom}_B(e_{r,s}^B, e_{r+1,s}^B) \neq 0 \quad \text{for any } x \in X_r \text{ and } y \in X_q,$$

$$\text{Hom}_B(e_{r,s}^B, e_{r+1,s}^B) \neq 0 \quad \text{for any } x' \in X_{r+s} \text{ and } y' \in X_q,$$

because $q \in (Q \setminus Z)$. Similarly as above, we conclude that there is an infinite short cycle $F_s(X) \rightarrow F_s(Y) \rightarrow F_s(X') = F_s(X)$ in mod $A$ with $F_s(X)$ in $e_{r,s}^B$, a contradiction with our assumption. \(\square\)

**Proposition 6.5.** Let $B$ be an almost concealed canonical algebra of Euclidean or wild type, $G$ an infinite cyclic admissible group of automorphisms of $B$, and $A = B/G$. Then the following statements are equivalent.

(i) $\Gamma_A$ admits a family of quasitubes with common composition factors, closed under composition factors, and consisting of modules which do not lie on infinite short cycles.

(ii) $G$ is one of the forms

(a) $G = (\varphi^2)_B$, for a strictly positive automorphism $\varphi$ of $\widehat{B}$,

(b) $G = (\varphi^2)_B$, for a rigid automorphism $\varphi$ of $\widehat{B}$ whose restriction to $B$ does not fix any nonstable ray tube of the unique separating family $T$ of ray tubes of $\Gamma_B$.

**Proof.** It follows from [1,3,32] that the Auslander–Reiten quiver $\Gamma_B$ of $\widehat{B}$ has a decomposition

$$\Gamma_B = \bigvee_{q \in Z} (e_{q, X_q} \lor X_{q}^B)$$

such that

(1) For each $q \in Z$, $e_{q, X_q}^B$ is an infinite family $e_{q, X_q}^B, x \in X_q$, of quasitubes.

(2) For each $q \in Z$, $X_{q}^B$ is either an acyclic component of Euclidean type, if $B$ is of Euclidean type, or an infinite family of components whose stable parts are of the form $\mathbb{Z} B_{\infty}$, if $B$ is of wild type.

(3) For each $q \in Z$, $e_{q, X_q}^B$ is a family $e_{q, X_q}^B, x \in X_q$, of pairwise orthogonal generalized standard quasitubes with common composition factors, closed under composition factors, and consisting of modules which do not lie on infinite short cycles in mod $B$.

(4) For each $q \in Z$, we have $v_B(e_{q, X_q}^B) = e_{q+2, X_{q+2}}^B$ and $v_B(X_{q}^B) = X_{q+2}^B$.

(5) For each $q \in Z$, we have $\text{Hom}_B(X_{q+2}^B, e_{q, X_{q+2}}^B \lor \bigvee_{r < q} (e_{r, X_{r+2}}^B \lor X_{r+2}^B)) = 0$, and $\text{Hom}_B(e_{q, X_q}^B, \lor \bigvee_{r < q} (e_{r, X_{r+2}}^B \lor X_{r+2}^B)) = 0$.

(6) For each $q \in Z$, we have $\text{Hom}_B(e_{q, X_q}^B, X_{q+2}^B \lor \bigvee_{r > q+2} (e_{r, X_{r+2}}^B \lor X_{r+2}^B)) = 0$ and $\text{Hom}_B(X_{q}^B, \lor \bigvee_{r > q+2} (e_{r, X_{r+2}}^B \lor X_{r+2}^B)) = 0$.

(7) For $q \in Z, x \in X_q$ and $y \in X_{q+2}$, we have $\text{Hom}_B(e_{q, X_q}^B, e_{q+2, y}^B) \neq 0$ if and only if the quasitube $e_{q, X_q}^B$ is nonstable and $v_B(e_{q, X_q}^B) = e_{q+2, y}^B$.

(8) For all $q \in Z, x \in X_q$ and $y \in X_{q+1}$, we have $\text{Hom}_B(e_{q, X_q}^B, e_{q+1, y}^B) \neq 0$.

(9) For each $q \in Z$ and any stable tubes $e_{q, X_q}^B, e_{q, X_q+3, y}^B$ and $e_{q, X_q+3, y}^B$ in $e_{q, X_q}^B$ and $e_{q, X_q+3, y}^B$ in $e_{q, X_q+3, y}^B$, there is an indecomposable projective $\widehat{B}$-module $P$ in $X_{q+1}^B$ such that $\text{Hom}_B(e_{q, X_q}^B, P) \neq 0$ and $\text{Hom}_B(P, e_{q, X_q+3, y}^B) \neq 0$. 

We know also from [1,3,32,44] that $G$ is generated by a strictly positive automorphism $g$ of $\widehat{B}$. Hence there exists a positive integer $l$ such that $g(\mathcal{C}_q^B) = \mathcal{C}_{q+l}^B$ and $g(\mathcal{X}_q^B) = \mathcal{X}_{q+l}^B$ for any $q \in \mathbb{Z}$. Consider the canonical Galois covering $F : \widehat{B} \rightarrow \widehat{B}/G = A$ and the associated push-down functor $F_* : \text{mod} \widehat{B} \rightarrow \text{mod} A$. Since $F_*$ is dense, we obtain natural isomorphisms of $k$-modules

$$\bigoplus_{i \in \mathbb{Z}} \text{Hom}_A(X_i, Y_i) \sim \text{Hom}_A(F_*(X_i), F_*(Y_i)),$$

$$\bigoplus_{i \in \mathbb{Z}} \text{Hom}_A(\mathcal{C}_i, \mathcal{D}_i) \sim \text{Hom}_A(F_*(\mathcal{C}_i), F_*(\mathcal{D}_i)),$$

for all indecomposable modules $X$ and $Y$ in $\text{mod} \widehat{B}$.

We show first that (i)$\Rightarrow$(ii). Assume that $\Gamma$ admits a family $C = (C_\lambda)_{\lambda \in \Gamma}$ of quasitubes with common composition factors, closed under composition factors, and consisting of modules which do not lie on infinite short cycles in $\text{mod} A$. It follows from Proposition 2.10 that the quasitubes $C_x, x \in \Gamma$, are generalized standard. In fact, they are also pairwise orthogonal. Indeed, because $A = \widehat{B}/G$, where $B$ is an almost concealed canonical algebra with a separating family of ray tubes $\mathcal{T} = (T_i)_{i \in \mathbb{N}}$, we infer that the quasitubes $C_x, x \in \Gamma$, are obtained from the ray tubes $T_i$ by admissible operations of types $(ad^1)$ and $(ad^2)$. Therefore, using Lemma 2.9 and arguments as in the proof of Lemma 2.11, we obtain that the quasitubes $C_x, x \in \Gamma$, are pairwise orthogonal, because the ray tubes $T_i$ are pairwise orthogonal.

We know from property (3) that, for each $q \in \mathbb{Z}$, $C_{q,0} = F_q(\mathcal{C}_{q,0})$ is an infinite family $C_{q,0}^A = F_q(\mathcal{C}_{q,0})$, $x \in \mathcal{X}_q$, of quasitubes with common composition factors. Moreover,

$$\Gamma_x = C_{q,0}^A \cap \Gamma_x \cap C_{q,1}^A \cap \cdots \cap C_{q,-1}^A \cap \mathcal{X}_q,$$

with $\mathcal{X}_q = F_q(\mathcal{X}_{q,0})$ for $q \in \{0, 1, \ldots, l - 1\}$, since $F_q$ induces an isomorphism of translation quivers $\Gamma_q/G \sim \Gamma_x$. Therefore, using Lemma 2.10, we conclude that $\mathcal{C}_x$ contains all quasitubes $\mathcal{C}_x \cap C_{q,0}^A$ in $\mathcal{C}_x$, of $C_{q,0}^A$. In particular, we conclude that $\mathcal{C}_x^A = (C_x \cap C_{q,0})_{x \in \Gamma}$ is a family of pairwise orthogonal generalized standard quasitubes consisting of modules which do not lie on infinite short cycles in $\text{mod} A$. We may assume, without loss of generality, that $r = 0$.

We claim that this forces $G$ to be one of the two forms (a) and (b) required in (ii). We show first that $g = \psi_\beta \psi^2$ for a positive automorphism $\varphi$. Suppose that this is not the case. Then, by property (4), we conclude that $I \in \{1, 2, 3\}$. We have three cases to consider.

Assume that $I = 1$. Then we have $F_q(\mathcal{C}_{q,0}^A) = C_{q,0}^A = F_q(\mathcal{C}_{q,0}^A)$. Applying then property (8), we conclude that, for any $x \in \mathcal{X}_0$, the quasitube $C_{q,0}^A$ is not generalized standard, a contradiction.

Assume that $I = 2$. Then we have $F_q(\mathcal{C}_{q,0}^A) = C_{q,0}^A = F_q(\mathcal{C}_{q,2}^A)$. We know from property (1) that $C_{q,0}^A = (C_{q,0})_{q \in \mathcal{X}_0}$ and $C_{q,2}^A = (C_{q,2})_{q \in \mathcal{X}_0}$ are infinite families of quasitubes. Since $F_q$ has only finitely many projective modules, we may choose $x_0 \in \mathcal{X}_0$ and $x_1 \in \mathcal{X}_1$ such that $C_{x_0,0}^A$ and $C_{x_1,1}^A$ are stable tubes. Observe that $C_{x_0,0}^A = F_q(\mathcal{C}_{x_0,0}^A)$, $C_{x_1,1}^A = F_q(\mathcal{C}_{x_1,1}^A)$, and $C_{x_0,0}^A = F_q(\mathcal{C}_{x_2,0}^A) = \mathcal{C}_{x_2,0}^A$ such that $\varphi_2(\mathcal{C}_{x_0}^A) = \mathcal{C}_{x_2}^A$, by property (4). Applying property (8), we conclude that $\text{Hom}_x(\mathcal{C}_{x_0}^A, \mathcal{C}_{x_1}^A) \neq 0$ and $\text{Hom}_x(\mathcal{C}_{x_1}^A, \mathcal{C}_{x_2}^A) \neq 0$, and hence $\text{Hom}_x(\mathcal{C}_{x_0}^A, \mathcal{C}_{x_1}^A) \neq 0$ and $\text{Hom}_x(\mathcal{C}_{x_1}^A, \mathcal{C}_{x_0}^A) \neq 0$. Then it follows from Lemma 2.11 that there is in $\text{mod} A$ an infinite short cycle $M \rightarrow N \rightarrow M$ with $M$ in $C_{x_0}^A$, and $N$ in $C_{x_1}^A$, a contradiction, because $C_{x_0}^A$ is a quasitube of the family $C = (C_x)_{x \in \Gamma}$.

Assume that $I = 3$. Then we have $F_q(\mathcal{C}_{q,0}^A) = C_{q,0}^A = F_q(\mathcal{C}_{q,2}^A)$. Since $C_{q,0}^A = (C_{q,0})_{q \in \mathcal{X}_0}$ is an infinite family of quasitubes and the number of projective modules in $\Gamma_q$ is finite, we may choose $x_0 \in \mathcal{X}_0$ such that $C_{x_0,0}^A$ is a stable tube of $\Gamma_q$. Observe that then $C_{x_0,0}^A = F_q(\mathcal{C}_{x_0,0}^A)$, $C_{x_0,0}^A$ is a stable tube of $\Gamma_q$, and hence $g(\mathcal{C}_{x_0}^A)$ is a stable tube $\mathcal{C}_{x_0}^A$, for some $x_0 \in \mathcal{X}_3$, of $\Gamma_q$. Applying now property (9), we conclude that there is an indecomposable projective module $P$ in $\mathcal{C}_{x_0}^A$ such that $\text{Hom}_P(\mathcal{C}_{x_0}^A, P) \neq 0$ and $\text{Hom}_P(\mathcal{C}_{x_0}^A, \mathcal{C}_{x_0}^A) \neq 0$. Then we have $F_q(\mathcal{C}_{x_0}^A) = F_q(\mathcal{C}_{x_0}^A)$, and $(P, \mathcal{C}_{x_0}^A)$ is an indecomposable projective $A$-module in $F_q(\mathcal{C}_{x_0}^A)$ such that $\text{Hom}_P(F_q(\mathcal{C}_{x_0}^A), \mathcal{C}_{x_0}^A) \neq 0$ and $\text{Hom}_P(F_q(\mathcal{C}_{x_0}^A), \mathcal{C}_{x_0}^A) \neq 0$. Then it follows from Lemma 2.11 that there is in $\text{mod} A$ an infinite short cycle $M \rightarrow F_q(P) \rightarrow M$ with $M$ in $C_{x_0}^A$, again a contradiction, since $C_{x_0}^A$ is a quasitube of the family $C = (C_x)_{x \in \Gamma}$.

Summing up, we have proved that indeed $g = \psi_\beta \psi^2$ for a positive automorphism $\varphi$ of $\widehat{B}$.

Assume now that $\psi$ is a rigid automorphism of $\widehat{B}$ and that $B$ is an almost concealed canonical algebra (of Euclidean or wild type) whose unique separating family $\Gamma^B$ of ray tubes contains at least one projective module, or, equivalently (see [30,32]), $B$ is not a concealed canonical algebra. Then the family $\mathcal{C}_x^B$ of quasitubes of $\Gamma^B_x$, and hence the family $\mathcal{C}^B_x = F_x(\mathcal{C}_x^B)$ of quasitubes in $\Gamma^B_x$, contains at least one projective module. We also note that, since $\psi$ is a rigid automorphism of $\widehat{B}$, its restriction $\varphi_\psi$ to $B = B_\psi$ is a $k$-algebra automorphism of $B$ and $\varphi_\psi$ acts on the unique separating family $\Gamma^B$ of ray tubes of $\Gamma^B_x$. Suppose that $\varphi_\psi$ fixes a nonstable tube (a ray tube containing projective module) of $\Gamma^B$. Then there is $x_0 \in X_0$ such that $\mathcal{C}_{x_0}^B$ is a quasitube.
containing at least one projective module such that \( \varphi(C_{0,x}) = C_{0,x}^\delta \). Since \( g = \varphi v_2^\delta \) applying property (4), we then obtain that \( g(C_{0,x}^\delta) = C_{0,x}^\delta \). Take now an indecomposable projective \( \hat{B} \)-module \( P \) in \( C_{0,x}^\delta \). Then, by property (4), we conclude that \( v_2^\delta(P) \in E_{2,x}^\delta \) and \( v_2^\delta(P) \in E_{4,x}^\delta \). Clearly, we have Hom\(_{B}(P, v_\delta^\delta(P)) \neq 0 \) and Hom\(_{B}(v_\delta^\delta(P), v_2^\delta(P)) \neq 0 \). Moreover, \( g(P) \) and \( v_2^\delta(P) \) belong to the same quasitube \( C_{4,x}^\delta \). Therefore, we conclude that there are indecomposable projective \( A \)-modules \( F_3(P) \) and \( F_1(v_2^\delta(P)) \) in \( E_{4,x}^\delta \) and an indecomposable projective \( A \)-module \( F_2(v_\delta^\delta(P)) \) in \( E_{2,x}^\delta \) such that Hom\(_{X}(F_3(P), F_1(v_\delta^\delta(P))) \neq 0 \) and Hom\(_{X}(F_2(v_\delta^\delta(P)), F_1(v_\delta^\delta(P))) \neq 0 \). Applying now Lemma 2.11, we conclude that there is in mod A an infinite short cycle \( M \to F_2(v_\delta^\delta(P)) \to M \) with \( M \in C_{0,x}^\delta \), a contradiction, since \( C_{0,x}^\delta \) is in \( \mathcal{C} = (C_{x})_{x \in \mathbb{X}} \). This finishes the proof that (i) implies (ii).

Assume now that (ii) holds. In particular, we have \( g = \varphi v_2^\delta \) for a positive automorphism of \( \hat{B} \). Then it follows from property (4) that there is a positive integer \( I \geq 4 \) such that \( g(C_q^\delta) = C_{q+1}^\delta \) for any \( q \in \mathbb{Z} \). Let \( C = (C_x)_{x \in \mathbb{X}} \) with \( X = X_0 \) and \( C_x = C_{0,x} = \mathcal{F}_x(C_{0,x}^\delta) \) for any \( x \in \mathbb{X} \). Since \( g = \varphi v_2^\delta = (\varphi v_\delta^\delta)v_\delta^\delta \) with \( v_\delta^\delta \) a strictly positive automorphism of \( \hat{B} \), invoking the knowledge of the supports of indecomposable modules in mod \( \hat{B} \) (see [1,32]), we conclude that the images \( F_i(S) \) and \( F_i(T) \) of any nonisomorphic simple \( \hat{B} \)-modules \( S \) and \( T \) which occur as composition factors of modules in a fixed family \( C_q^\delta \) are nonisomorphic simple \( A \)-modules. Therefore, it follows from Theorem 6.1 and properties (1)--(4) that \( C \) is an infinite family of quasitubes with common composition factors and closed under composition factors. We show now that \( C \) consists of indecomposable \( A \)-modules which do not lie on infinite short cycles in mod \( A \). Observe that, for two indecomposable modules \( M \) and \( N \) in \( C \), we have \( M = F_i(X) \) and \( N = F_j(Y) \), for some indecomposable \( B \)-modules \( X \) and \( Y \) in \( C_q^\delta \), and \( F_i \) induces an isomorphism of \( k \)-modules Hom\(_{A}(M, N) \to \) Hom\(_{A}(X, Y) \), by properties (5) and (6), and since \( I \geq 4 > 2 \). In particular, by properties (2) and (3), \( C = (C_{x})_{x \in \mathbb{X}} \) is a family of pairwise orthogonal generalized standard quasitubes of \( I_{\mathcal{A}} \). Suppose that there is an infinite short cycle \( M \to L \to M \) in mod \( A \) with \( M \in C_{0,x}^\delta \) for some \( x \in \mathbb{X} = X_0 \).

Clearly, then \( L \) does not belong to \( C_{x} \). Then \( M = F_i(X) \) for some \( X \in C_{0,x}^\delta \) and \( L = F_i(Z) \) for some indecomposable module \( Z \) in mod \( \hat{B} \) such that Hom\(_{A}(X, Z) \neq 0 \). Applying properties (5) and (6), we conclude that \( Z \in C_{0,x}^\delta \vee C_{0,x}^\delta \vee C_{0,x}^\delta \). Since Hom\(_{A}(M, l) \neq 0 \), applying properties (5) and (6) again, we infer that Hom\(_{A}(Z, l) \neq 0 \). Observe that \( l \neq \mathbb{X} \in g(C_0^\delta) = C_\delta^\delta \) with \( l \geq 4 \). Hence, invoking properties (5) and (6), we obtain that \( Z \) belongs to \( C_\delta^\delta \) and \( l = 4 \). But then property (7) forces \( C_{0,x}^\delta \) to be nonstable, \( Z \in C_{0,x}^\delta \vee C_{0,x}^\delta \vee C_{0,x}^\delta \). In particular, we obtain that

\[
(v_2^\delta \varphi)(C_{0,x}^\delta) = (\varphi v_2^\delta)(C_{0,x}^\delta) = g(C_{0,x}^\delta) = v_2^\delta (C_{0,x}^\delta),
\]

and hence \( \varphi(C_{0,x}^\delta) = C_{0,x}^\delta \). Therefore, \( \varphi \) is a rigid automorphism of \( \hat{B} \) which fixes the nonstable quasitube \( C_{0,x}^\delta \) of \( I_{\hat{B}} \). Then the restriction \( \varphi_{B} \) of \( \varphi \) to \( B \) is a \( k \)-algebra automorphism of \( B \) which fixes the nonstable tube \( T_{\hat{B}}^\delta \) of the unique separating family \( T_{\hat{B}}^\delta \) of ray tubes of \( T_{\hat{B}} \) all of whose modules belong to the quasitube \( C_{0,x}^\delta \) of \( I_{\hat{B}}^\delta \). This contradicts assumption (ii).

Therefore, the family \( C = C_{0} \) of quasitubes \( C_{x} = C_{0,x} \), \( x \in \mathbb{X} = X_0 \), consists of the indecomposable \( A \)-modules which do not lie on infinite short cycles in mod \( A \). This completes the proof that (ii) implies (i).

7. Proof of Theorem 1.1

The aim of this section is to complete the proof of Theorem 1.1, by showing the implication (i)⇒(ii).

Assume that \( A \) is a basic, connected, selfinjective algebra and \( C = (C_{x})_{x \in \mathbb{X}} \) a family of quasitubes in \( I_{\mathcal{A}} \) with common composition factors, closed under composition factors, and consisting of modules which do not lie on infinite short cycles. Then it follows from Proposition 2.10 that all quasitubes \( C_{x} \) in \( C \) are generalized standard components of \( I_{\mathcal{A}} \).

We will show first that \( C \) is a family of quasitubes of a quasitube enlargement \( \Delta \) of a concealed canonical algebra \( \mathcal{C} \).

Fix \( i \in I \), and consider the quotient algebra \( A_i = A / \text{ann}_A(C_{i}) \). Then the quasitube \( C_{i} \) is a generalized standard faithful, hence sincere, component of \( I_{\mathcal{A}} \). Moreover, it follows from the proof of Lemma 2.12 that \( C_{x} \) is a quasitube without external short paths. Applying Theorem 4.4, we conclude that \( A_i \) is a quasitube enlargement of a concealed canonical algebra \( C_i \), there is a separating family \( T_i^\mathcal{C} = (T_x^i)_{x \in \mathbb{X}_{i}} \) of stable tubes of \( I_{C_i} \) and a stable tube \( T_{\mathcal{C}_{i}} \), for some \( x_i \in \mathbb{X}_{i} \), such that \( C_i \) is obtained from \( T_{\mathcal{C}_{i}} \) by a sequence of admissible operations of types (ad 1), (ad 2), (ad 1*), and (ad 2*), corresponding to those admissible operations leading from \( C_i \) to \( A_i \). We recall that the index set \( \mathbb{X}_{i} \) is infinite. Hence \( T^\mathcal{C} \) is an infinite family of pairwise orthogonal stable tubes consisting of modules which do not lie on infinite short cycles in mod \( C_{i} \), because \( T^\mathcal{C} \) is a separating family of stable tubes of \( I_{\mathcal{C}_{i}} \). Observe also that \( C_{i} \) is a quotient algebra of \( A \), say \( C_{i} = A_j / \text{ideal} \), for an ideal \( j \) of \( A \), since \( C_{i} \) is a quotient algebra of \( A_i \). We note that \( T^\mathcal{C} = (T_x^i)_{x \in \mathbb{X}_{i}} \) is a family of stable tubes of \( I_{\mathcal{C}_{i}} \) with common composition factors (see [30,48]). Since the quasitube \( C_{i} \), containing all modules of \( T_{\mathcal{C}_{i}} \), belongs to \( C \) and \( C \) is closed under composition factors, we conclude that all modules of the family \( T^\mathcal{C} \) belong to \( C \). Applying Lemma 2.14, we conclude that, for each \( x \in \mathbb{X}_{i} \), there exists a quasitube \( C_x^{(i)} \) in \( C \) containing all modules of the stable tube \( T_x^{(i)} \) of \( I_{\mathcal{C}_{i}} \). Moreover, by Lemma 2.12, we have
Lemma 7.1. Observe first that the annihilator for a finite family $M$ is
additive. Moreover, we have $\eta_j = \operatorname{ann}(M_i)$, because $T_x$ is a faithful component of $\Gamma_C$.

We claim now that all concealed canonical algebras $C_i$, $i \in I$, coincide. Take $i \neq j$ in $I$. Since the sets $\mathcal{X}_i$ and $\mathcal{X}_j$ are infinite, we may take $x \in \mathcal{X}_i$ and $y \in \mathcal{X}_j$ such that $T_x^C = C_x^i$ and $T_y^C = C_y^j$. In particular, we have $\eta_x = \operatorname{ann}(C_x)$ and $\eta_y = \operatorname{ann}(C_y)$. We may assume that $T_x$ and $T_y$ are different, because $T_x$ forces $\eta_x = \eta_y$ and then $C_i = A/\eta_x = A/\eta_y = C_i$. Observe that $T_x^C$ and $T_y^C$ are stable tubes of $\Gamma_A$ with common composition factors and consist of modules which do not lie on infinite short cycles in $\operatorname{mod} A$, because $T_x^C$ and $T_y^C$ belong to the family $C = (C_i)_{i \in I}$. Applying Theorem 2.5 and Lemma 2.17, we conclude that there exist indecomposable $A$-modules $M_i \in T_x^C$ and $M_j \in T_y^C$ such that $[M_i] = [M_j]$ in $\mathcal{K}_0(A)$ and $\eta_j = \operatorname{ann}(M_i, M_j)$. Since $[M_i] = [M_j]$, there is a quotient algebra $D = A/L$, for an ideal $L = \operatorname{Ag}A$ of $A$ given by an idempotent $g$ of $A$, such that $M_i$ and $M_j$ are sincere indecomposable $D$-modules. Clearly, then $M_i \subseteq L$ and $M_j \subseteq L$, so $T_x^C = C_x^i$ and $T_y^C = C_y^j$ are sincere stable tubes of $\Gamma_B$ consisting of indecomposable $D$-modules which do not lie on infinite short cycles in $\operatorname{mod} D$. Moreover, we have $\operatorname{ann}(T_x^C) = L/\eta_x$ and $\operatorname{ann}(T_y^C) = L/\eta_y$. Applying Corollary 3.3, we conclude that $D$ is a concealed canonical algebra, $T_x^C$ and $T_y^C$ are faithful stable tubes of $\Gamma_D$, and consequently $J_i = L_j$. Therefore, indeed we have $C_i = A/\eta_x = A/\eta_y = C_i$ for all $i, j \in I$.

Summing up, we have proved that there exists a concealed canonical algebra $C$ such that $C$ is a quotient algebra of $A$ and, for each $i \in I$, $A_i = A/\operatorname{ann}_A(C_i)$ is a quasitube enlargement of $C$, and $C_i$ is obtained from a stable tube $T_x^C, x \in \mathcal{X}_i$, by the corresponding iterated application of admissible operations of types $(ad 1)$, $(ad 2)$, $(ad 1^*)$, and $(ad 2^*)$, where $\mathcal{X}_i$ is the index set of a separating family of stable tubes of $\Gamma_C$. Since the family $C = (C_i)_{i \in I}$ consists of quasitubes with common composition factors and is closed under composition factors, we conclude that $C$ has a canonical decomposition

$$
\Gamma_C = \Gamma^C \vee T \vee T
$$

where $T = (T_x^C)_{x \in \mathcal{X}_i}$ is a separating family of stable tubes such that, for any $i \in I$, $T_x^C$ is a stable tube of $T$. In particular, we have $\mathcal{X}_i = \emptyset$ for any $i \in I$. Moreover, we proved that, for a fixed $x \in \mathcal{X}_i$, all modules of the stable tube $T_x^C$ are contained in a quasitube $C_x$, from the family $C$. Therefore, we conclude that $\Gamma = A/\operatorname{ann}_A(C)$ is a quasitube enlargement of the concealed canonical algebra $C$, using modules from the separating family $T = (T_x^C)_{x \in \mathcal{X}_i}$ of stable tubes of $\Gamma$, and $C$ is the separating family of quasitubes of $\Gamma_A$, obtained from the family $T$ by the corresponding iterated application of admissible operations of types $(ad 1)$, $(ad 2)$, $(ad 1^*)$, and $(ad 2^*)$. Applying Theorem 4.3, we conclude also that there is a unique almost concealed canonical quotient algebra $B = A_1$ of $A$ (the right quasitubed part of $A$), which is a tubular extension of $C$ and that $\Gamma_B$ admits the separating family $C' = (C_i)_{i \in \mathcal{X}_i}$ of quasitubes of $\Gamma_B$ by the corresponding iterated application of admissible operations of type $(ad 1)$. Moreover, the family $C = (C_i)_{i \in \mathcal{X}_i}$ of quasitubes of $\Gamma_A$ (and $\Gamma_B$) is obtained from the family $C'$ by a sequence of admissible operations of types $(ad 1^*)$ and $(ad 2^*)$.

Let $I$ be the annihilator $\operatorname{ann}_A(C')$ of the family $C'$ (of modules) in $A$. Since $C'$ is a faithful family of ray tubes of $\Gamma_B$, we conclude that $B = A/I$. We may assume that there exists a complete set of pairwise orthogonal idempotents $e_1, \ldots, e_n$ of $A$ such that $1_A = e_1 + \cdots + e_n$ and $e = e_1 + \cdots + e_m$, for some $m \leq n$, is a residual identity of $B = A/I$. We will show that $I$ is a deforming ideal of $A$ with $I = le$ and $r_j(I) = el$.

In order to prove that $I$ is a deforming ideal of $A$ we need several technical results.

Denote by $\Gamma$ the trace ideal of the family $C'$ in $A$, that is, the sum of the images of all homomorphisms from modules in $C'$ to the right $A$-module $A$. Similarly, by $\Gamma'$ we denote the trace ideal of the dual family $D(C')$ of left $A$-modules in $A$.

**Proposition 7.1.** $I \cup \Gamma' \subseteq \Gamma$.

**Proof.** Observe first that the annihilator $I = \operatorname{ann}_A(C')$ of $C'$ is the annihilator $\operatorname{ann}_A(M)$ of a module from the additive closure $\operatorname{add}(C')$ of $C'$. Indeed, since $A$ is of finite length over $k$, we have

$$
I = \operatorname{ann}_A(C') = \bigcap_{X \in C'} \operatorname{ann}_A(X) = \bigcap_{i=1}^{r} \operatorname{ann}_A(M_i) = \operatorname{ann}_A \left( \bigoplus_{i=1}^{r} M_i \right)
$$

for a finite family $M_1, \ldots, M_r$ of indecomposable modules from $C'$, so we may take $M = M_1 \oplus \cdots \oplus M_r$. We also note that $I$ is the annihilator of the left $A$-module $D(M)$, from add $D(C')$. In particular, we obtain that $M$ is a faithful right $B$-module and $D(M)$ is a faithful left $B$-module.

Invoking again the fact that $A_k$ is of finite length over $k$, we obtain also that

$$
J = \sum_{h \in \operatorname{Hom}(Y, A_k)} Y_i \in C' \bigcap \sum_{i=1}^{k} \operatorname{im} h_i
$$
for some homomorphisms $h_i \in \text{Hom}_A(Y_i, A)$ with $Y_i \in \mathcal{C}^r$, for $i \in \{1, \ldots, s\}$, and hence an epimorphism of right $A$-modules

$$[h_1 \ldots h_s] : Y \rightarrow \bigoplus_{i=1}^s Y_i \rightarrow J.$$ 

Then $N = M \oplus Y$ is a module from add $\mathcal{C}^r$ with $\text{ann}_A(N) = \text{ann}_A(M) = I$: hence $N$ is a faithful right $B$-module, and there exists an epimorphism of right $A$-modules $g : N \rightarrow J$. Clearly, then $J$ is a right $B$-module, because $J = \text{Im}(N) = \text{Im}(N) = g(0) = 0$.

We will show now the inclusion $J \subseteq I$. Suppose that we have $J \nsubseteq I$. Since $I = \text{ann}_A(N)$ is the intersection of the kernels of all homomorphisms from $\text{Hom}_A(A, N)$, we conclude that there is a homomorphism $f : A \rightarrow N$ in mod $A$ such that $f(J) \neq 0$. Then there are indecomposable direct summands $U$ and $V$ of $N$ and $P$ of $A_A$ such that $f((U \cap P) \cap V) \neq 0$, and consequently we obtain a short path in mod $A$

$$U \xrightarrow{u} P \xrightarrow{v} V,$$

with $U$ and $V$ in $\mathcal{C}^r$, $P$ an indecomposable projective right $A$-module, and $vu \neq 0$. Moreover, $\text{Im} u$ contains $\text{soc} P$, and so $P$ is a simple right $B$-module, because $\text{Im} u$ is a right $B$-module. On the other hand, the family of quasitubes $\mathcal{C}$ is obtained from the family of ray tubes $\mathcal{C}^r$ by a sequence of admissible operations of types $(ad\,1^*)$ and $(ad\,2^*)$, so we then infer that $P \notin \mathcal{C}^r$. Hence $u$ and $v$ belong to $\text{rad}^\infty(mod\,A)$, and so $0 \neq vu \in \text{rad}^\infty(U, V)$, a contradiction, since $\mathcal{C}^r$ is a generalized standard family of modules in mod $B$, and hence in mod $A$. Therefore, we have indeed $J \subseteq I$.

Further, since $A_A$ is of finite length over $k$, we obtain that

$$J' = \sum_{h' \in \text{Hom}_{A}(D(Y'), A), Y' \in \mathcal{C}^r} \text{Im} h'$$

for some homomorphisms $h'_j \in \text{Hom}_{A}(D(Y'_j), A)$ with $Y'_j \in \mathcal{C}^r$, for $j \in \{1, \ldots, t\}$, and hence an epimorphism of left $A$-modules

$$[h'_1 \ldots h'_t] : D(Y') = \bigoplus_{j=1}^t D(Y'_j) \rightarrow A_A.$$

Then $N' = M \oplus Y'$ is a module from $\mathcal{C}^r$, $D(N')$ is a module in add $D(\mathcal{C}^r)$, and $\text{ann}_{A_D}(D(N')) = \text{ann}_{A_D}(D(M)) = I$. Hence $D(N')$ is a faithful left $B$-module, and there exists an epimorphism $g' : D(N') \rightarrow J'$ of left $A$-modules. Obviously, then $J'$ is a left $B$-module, because $J' = \text{Im}(D(N')) = g'(D(N')) = g'(0) = 0$.

We claim now that $J' \subseteq I$. Suppose that $J' \nsubseteq I$. Since $I = \text{ann}_A(D(N'))$ is the intersection of the kernels of all homomorphisms from $\text{Hom}_{A}(A_A, D(N'))$, there exists a homomorphism $f' : A_A \rightarrow D(N')$ of left $A$-modules such that $f'(J') \neq 0$. Then we have the sequence of homomorphisms of left $A$-modules

$$D(N') \xrightarrow{g'} J' \xrightarrow{w'} A_A \xrightarrow{f'} D(N'),$$

where $w'$ is the canonical embedding, with $f'w'g' \neq 0$. Applying the duality, we obtain homomorphisms in mod $A$

$$N' \xrightarrow{D(f')} D(A_A) \xrightarrow{D(w'g')} N'$$

with $D(w'g')(D(f')(U') \cap P') \cap V' \neq 0$, and consequently a short path in mod $A$

$$U' \xrightarrow{u'} P' \xrightarrow{v'} V',$$

with $U'$ and $V'$ in $\mathcal{C}^r$, $P'$ an indecomposable projective right $A$-module, and $vu' \neq 0$. Since $\text{Im} u'$ is a nonzero right $B$-module, and $P'$ is a simple right $B$-module, and so we infer as above that $P' \notin \mathcal{C}^r$. Hence $u'$ and $v'$ belong to $\text{rad}^\infty(mod\,A)$, and then $0 \neq v'u' \in \text{rad}^\infty(U', V')$, a contradiction, since $\mathcal{C}^r$ is a generalized standard family of modules in mod $A$. $\square$

**Lemma 7.2.** We have $I_A(I) = J$, $r_A(I) = J'$ and $I = I_A(J) = I_A(J')$.

**Proof.** Because $J$ is a right $B$-module, $I \subseteq r_A(J)$. Let $N$ be a module from add $\mathcal{C}^r$ such that $I = r_A(N)$. Let $\rho : N \rightarrow A^t$ be an embedding of $N$ into a finite-dimensional free right $A$-module. Denote by $\rho_i : N \rightarrow A$, for $i \in \{1, \ldots, t\}$, the composite of $\rho$ with the projection on the $i$-th component of $A^t$. Then there is an embedding of $N$ into the direct sum $\bigoplus_{i=1}^t \rho_i(N)$, which is contained in $\bigoplus_{i=1}^t J$. Hence we have

$$I = r_A(N) \supseteq r_A \left( \bigoplus_{i=1}^t \rho_i(N) \right) \supseteq r_A \left( \bigoplus_{i=1}^t J \right) = r_A(J).$$

Consequently, we obtain $I = r_A(J)$. Applying now a theorem by Nakayama [60, Theorem 2.3.3], we get $J = I_A(J) = I_A(I)$.

We will show now that $J' = r_A(I)$. First, notice that, because $J'$ is a left $B$-module, $I \subseteq I_A(J')$. Let $N'$ be a module from add $\mathcal{C}^r$ such that $I = I_A(D(N'))$. Let $\rho' : D(N') \rightarrow A^t$ be an embedding of $D(N')$ into a finite-dimensional free left $A$-module.
Denote by $\rho'_i : \mathbb{D}(N') \to A'$, for $i \in \{1, \ldots, s\}$, the composite of $\rho'^i$ with the projection on the $i$-th component of $A'$. Then there is an embedding of $\mathbb{D}(N')$ into the direct sum $\bigoplus_{i=1}^{s} \rho'_i(\mathbb{D}(N'))$, which is contained in $\bigoplus_{i=1}^{s} J'$. Hence we have

$$I = l_{A}(\mathbb{D}(N')) \supseteq l_{A} \left( \bigoplus_{i=1}^{s} \rho'_i(\mathbb{D}(N')) \right) \supseteq l_{A} \left( \bigoplus_{i=1}^{s} J' \right) = l_{A}(J').$$

Thus we obtain $I = l_{A}(J')$. Applying now the theorem by Nakayama mentioned above, we get $J' = r_{A} l_{A}(J') = r_{A}(I)$. □

**Lemma 7.3.** We have $e e = e e e = e f' e$. In particular, $(e e)^2 = 0$.

**Proof.** Since $e$ is a residual identity of $B = B/l$, we have $B \cong e A / e e$. Thus $C' = e$ is a faithful generalized standard family of ray tubes in $I'_{c e} / e e$. Further, $J$ is a right $B$-module, $1 - e \in I$, and so $J = e J + (1 - e) = e J$, because $J(1 - e) \subseteq J(1 - e) = 0$. Then $e f'$ is an ideal of $e A / e e$ with $e f' \subseteq e e$, by Proposition 7.1.

Consider the algebra $B' = e A / e e / e f'$. Then $C' = e$ is a sincere generalized standard family of ray tubes in $I'_{c e}$. Because the family $C' = e$ in $I'_{c e}$ consists of modules which do not lie on infinite short cycles in $mod A$, the modules from the family $C'$ in $I'_{c e}$ do not lie on infinite short cycles in $mod B'$. Moreover, for any $x \neq y$ in $X$, the ray tubes $C'_{x}$ and $C'_{y}$ have infinitely many modules with common composition factors, since $C'_{x}$ contains all modules of $T'_{x}$, and $C'_{y}$ contains all modules of $T'_{y}$. Therefore, by Lemma 2.12, the family $C'$ consists of modules which do not lie on external short paths in $mod B'$. Hence, applying Theorem 3.1, we conclude that $B'$ is an almost concealed canonical algebra and that $C'$ is a separating family of ray tubes of $I'_{c e}$. But then the sincere generalized standard family $C'$ of ray tubes of $I'_{c e}$ is faithful in $mod B'$. This implies that $e e / e f' = \mathrm{ann}_{B'}(C') = 0$, and hence $e e = e f'$. In a similar way we show that $e e = e f'$. Applying Lemma 7.2, we obtain the equalities $(e e)^2 = e J e e = e e e = (e e) e = e e = 0$. □

We shall use also the following general lemma on almost split sequences over triangular algebras (see [52, Lemma 5.6]).

**Lemma 7.4.** Let $R$ and $S$ be algebras and $N$ an $S$-$R$-bimodule. Let $\Gamma' = \begin{pmatrix} S & N \\ 0 & R \end{pmatrix}$ be the triangular matrix algebra defined by the bimodule $S_{N} R$. Then an almost split sequence $0 \to X \to Y \to Z \to 0$ in $mod R$ is an almost split sequence in $mod \Gamma'$ if and only if $\Gamma'_{0}(R, X) = 0$.

**Lemma 7.5.** Let $f$ be a primitive idempotent in $I$ such that $f f \neq f A$. Then $K = f A f + f A f f A + e A f + e A$ is an ideal of the algebra $F = (e + f) A (e + f)$, and $N = f A f / f K$ is a right $B$-module such that $\mathrm{Hom}_{B}(C', N) \neq 0$ and $\mathrm{Hom}_{B}(N, C') = 0$.

**Proof.** It follows from Lemma 7.3 that $e e$. Since $e e \subseteq J$, we obtain the inclusions $f A e \subseteq f (e e) \subseteq J$. Therefore $K$ is an ideal of $F = (e + f) A (e + f)$. Observe also that $f K e = f f f A f f A e + f K f e$, because $(f K f) = \begin{pmatrix} f A f & f f A f \\ f f A f & f K f \end{pmatrix} \leq (e e) e = e e = e e = e e$, and $e f K = e A f$. Moreover, $N \neq 0$. Indeed, if $f A e = f K e$, then, since $f A e \subseteq f A$, we have from Lemma 7.3 that $f A e = f f f A f f A e$, and so $f A e = f A$, a contradiction with our assumption. Further, $B = e A / e e$ and $(f A e)(e e) = f A e \subseteq f A e$, and hence $N$ is a right $B$-module. Finally, $N$ is a left module over $S = f A f / f K f$ and $\Gamma' = F / K$ is isomorphic to the triangular matrix algebra $\Gamma' = \begin{pmatrix} S & N \\ 0 & B \end{pmatrix}$. Invoking now the structure of the family $C' = (C'_{x})_{x \in X}$ of quasitubes of $I'_{A}$, we conclude that the family $C' = (C'_{x})_{x \in X}$ of ray tubes of $I'_{B}$ is the image of the family $C'$ via the restriction functor $(e + f) : \mathrm{mod} A \to \mathrm{mod} F$, and consequently $C'$ is a family of ray tubes of $I'_{B}$. We note also that the ray tubes $C'_{x}$, $x \in I$, do not contain injective modules, and hence for any module $X$ in $C'$ there exists an almost split sequence $0 \to X \to Y \to Z \to 0$ in $\mathrm{mod} F$ consisting entirely of $B$-modules. Therefore, applying Lemma 7.4, we obtain $\mathrm{Hom}_{B}(N, X) = 0$ for any module $X$ in $C'$, and so $\mathrm{Hom}_{B}(N, C') = 0$. Further, $C'$ is a separating family of ray tubes of $I'_{B}$, and hence every indecomposable module in $\mathrm{mod} B$ is either generated or cogenerated by $C'$. This implies that $\mathrm{Hom}_{B}(C', N) \neq 0$. □

Denote by $\nu$ the Nakayama automorphism of $A$ and by $\nu^{-1}$ its inverse. Then for any primitive idempotent $f$ of $A$ we have $\text{soc}(\nu(f) A) \cong \text{top}(fA) / \text{rad}(fA)$. We have then the following two lemmas, proved in [52, Lemmas 1.1 and 5.11].

**Lemma 7.6.** The right ideal $\nu(e) l_{A}(I)$ is a minimal injective cogenerator in $\text{mod} B$, and the left ideal $r_{A}(I) \nu^{-}(e)$ is a minimal injective cogenerator in $\text{mod} B^{op}$.

**Lemma 7.7.** We have $\nu(e) = l_{A} l_{e} e(e e)$ and $f \nu^{-}(e) = r_{A} \nu^{-}(e) e(e e)$.

**Lemma 7.8.** We have $\nu(e) e = \nu(e) e$ and $e \nu^{-}(e) = f \nu^{-}(e)$.

**Proof.** Let $e$ be a primitive direct summand of $e$, and put $f = \nu(e)$. We shall show that $f e = f f$. It is enough to prove that $f e = f f$. Let $f e = f f$. Then we have $f e = f f$. It is enough to prove that $f e = f$. Then $f e = f f$. Further, we have $f e = f$. Then $f e = f$. Therefore, we get $f e = f e$. □

Now consider $K$ and $N$ as in Lemma 7.5. Then we have $\mathrm{Hom}_{B}(C', N) \neq 0$ and $\mathrm{Hom}_{B}(N, C') = 0$. Take a module $M$ from $C'$ such that $\mathrm{Hom}_{B}(M, N) \neq 0$. □
Let $f = fAe / J$. Observe that $f$ is a right $B$-module, because $B \cong eAe / eI$ and $eI = eI$ from Lemma 7.3. We claim that $\text{Hom}_{B}(f, A) = 0$. It is enough to show that $f$ is generated by $N$, because $\text{Hom}_{B}(N, M) = 0$. In fact,

$$L \cong (fAe)/fAe / (J \cap fAeAe)$$

as $B$-modules, and the module on the right-hand side is generated by $N = fAe / (J + fAeAe)$, where we note that

$$(fAe)J \subseteq fAe \cap A$$

and $(eI)^2 = 0$ by Lemma 7.3. Since $\tau_{B}M = 0$ or $\tau_{B}M$ belongs to $C'$, we have also $\text{Hom}_{B}(N, \tau_{B}M) = 0$, and so $\text{Hom}_{B}(f, \tau_{B}M) = 0$.

(2) We show that $\text{Hom}_{e}(f, \tau_{e}(M)) = 0$. Applying now the functor $\text{Hom}_{e}(M, -)$ to the exact sequence $0 \rightarrow J \rightarrow fAe \rightarrow N \rightarrow 0$, we obtain the exact sequence

$$\text{Hom}_{e}(f, \tau_{e}(M)) = 0.$$}

We have $\text{le} = 0$. Then $\nu(eI)le = 0$, because $\text{soc}(\nu(eI)) \cong \text{dim}(\text{Av}(eI))$. But, by Lemma 7.8, we have $\nu(eI)le = 0$, a contradiction. Hence $\le = 0$. □

**Lemma 7.10.** Let $f$ be a primitive idempotent in $I$ with $fAe \neq fI$. Then $\text{Hom}_{B}(C', fAe / fJ) = 0$ and $\text{Hom}_{B}(eAe / eJ', D(C')) = 0$.

**Proof.** Consider $K$ and $N$ as in Lemma 7.5. Observe that $fAeAe = (fAe)AeAe \subseteq eI$. Since $\le = 0$, by Lemma 7.9, we then have $N = fAe / fJ = fAe / fJ$. The claim follows from Lemma 7.5, and from the left–right dual argument. □

**Lemma 7.11.** Let $f$ be a primitive idempotent in $I$ such that $\nu^{-1}(f) \in I$. We have $\text{Hom}_{B}(C', fAe) = 0$.

**Proof.** We note that $fAe$ is a right $B$-module, because $B \cong eAe / eI$ and $(fAe)(eI) \subseteq eI$ and $eI = 0$, by Lemma 7.9. As a restriction of the isomorphism $D(A) \cong A_{-} \times A_{-}$-bimodules, we obtain the isomorphism $D(fAe) \cong eAe / eI$-modules. Further, since $\text{dim}(\nu^{-1}(f)) \cong \text{dim}(A)$, we obtain $fAe / eI$-modules, and $\nu(f) \in I$, we obtain $fAe / eI$-modules, and $\nu^{-1}(f) = 0$. Thus we have the isomorphism of left $(eAe/eI)$-modules $eAe / eI \cong D(fAe)$, where we note that $\nu^{-1}(f) = 0$, and $eI = eI$, by Lemma 7.3. Consequently, it follows from Lemma 7.10 that $\text{Hom}_{B}(D(fAe), D(\nu^{-1}(f))) = 0$, which implies that $\text{Hom}_{B}(C', fAe) = 0$. □

**Lemma 7.12.** Let $f$ be a primitive idempotent in $I$. Then we have $eI = eJ / f$, and $eAe / eJ' = fAe / fJ$. □

**Proof.** It is enough to show the first equality. We assume that $f \neq 0$, since the assertion is obvious in the case $f = 0$. Suppose that $f \neq 0$. Take $K$ and $N$ as in Lemma 7.5. Observe that, as in the proof of Lemma 7.10, we have $N = fAe / fJ = fAe / fJ$. Applying Lemma 7.5 we obtain $\text{Hom}_{B}(C', fAe / fJ) = 0$. Note that $\nu^{-1}(f) \in I$. Indeed, if $\nu^{-1}(f) \notin I$, then $fAe / fJ$ by Lemma 7.8, and hence $fAe = fAe$, a contradiction. But $\nu^{-1}(f) \in I$ implies that $\nu^{-1}(f) = 0$, because $\nu^{-1}(f)$ is a right ideal of $A$, $\nu^{-1}(f) \subseteq fAe / fJ$, and $\nu^{-1}(f) \notin I$. Therefore, $\nu^{-1}(f) / fJ = fAe$ and, applying Lemma 7.11, we get $\text{Hom}_{B}(C', fAe) = 0$, a contradiction to the fact established above. □

Now we are in position to prove the following crucial result.

**Proposition 7.13.** We have $le = J$, $eI = eJ'$, and $eI = eI' \cap eJ'$. □

**Proof.** Observe that $le = eI + (1 - e)I$. From Lemma 7.3 we have $eI = eI$, and $eI = eI$, further, by Lemma 7.12, we obtain that $(1 - e)I = (1 - e)le = (1 - e)J = (1 - e)J$, because $1 - e \in I$. Hence $le = 0$. Invoking Lemma 7.2, we then get $le \subseteq I_{e}(I) = J$, and so $eI = J$. The equality $eI = eI$ follows in a similar way. Finally, observe that $J \cap J' = eI / eJ$, and $eJ = eI$. □

**Theorem 7.14.** $I$ is a deforming ideal of $A$ with $l_{A}(I) = le$ and $r_{A}(J) = le$. □
Proof. From Lemma 7.2 and Proposition 7.13 we know that $l_A(I) = J = le$ and $r_A(I) = J' = el$. In particular, we have $lel = 0$. Therefore, from Proposition 5.12, we get $ele = l_{ele}(I) = r_{ele}(I)$. Finally, $B = A/I$ is an almost concealed canonical algebra, and hence a quasitilted algebra. Then the ordinary quiver $Q_B$ of $B$ is acyclic, by [17, Proposition III.1.1]. This shows that $I$ is a deforming ideal of $A$.

We complete now the proof of the implication (i)$\Rightarrow$(ii) of Theorem 1.1. We know that $I = \ann_{A}(E')$ is a deforming ideal of $A$, with $l_A(I) = le$, and that $B = A/I$ is an almost concealed canonical algebra. Then it follows from Theorem 5.4 that the deformed selfinjective algebra $A[I]$ is isomorphic to the orbit algebra $B/(\psi_{V_3})$ for some positive automorphism $\psi$ of $B$. Moreover, by Theorem 5.3(iii), the algebras $A$ and $A[I]$ are socle equivalent, and consequently the module categories $\mod(A/\soc A)$ and $\mod(A[I]/\soc A[I])$ coincide. We note also that the Auslander–Reiten quivers $\Gamma_{A}$ and $\Gamma_{A[I]}$ are isomorphic. Then our assumption (i) on $A$ forces that $\Gamma_{A[I]}$ admits a family $C_{e}' = (C_{i})_{i\in I}$ of quasitubes with common composition factors, closed under composition factors, and consisting of indecomposable $A[I]$-modules which do not lie on infinite short cycles in $\mod A[I]$. Namely, for each $i \in I$, the quasitube $C_{i}'$ is obtained from the quasitube $C_{i}$ by replacing any indecomposable projective $A$-module $P$ by the corresponding indecomposable projective $A[I]$-module $P'$, and keeping the remaining indecomposable $A$-modules in $C_{i}$. Then it follows from Propositions 6.4 and 6.5 that $G = (\psi_{V_3})$ satisfies conditions (ii) of Theorem 1.1. In particular, we conclude that $e_{i} \neq e_{e(i)}$ for any primitive summand $e_{i}$ of the residual identity $e$. Applying Theorem 5.3(iii), we conclude that $A$ and $A[I]$ are isomorphic $k$-algebras. Therefore, $A$ is isomorphic to the orbit algebra $B/G$ with $G$ satisfying conditions (ii) of Theorem 1.1. This finishes the proof of the implication (i)$\Rightarrow$(ii) of Theorem 1.1.

Acknowledgements

The authors gratefully acknowledge support from the Research Grant No. N N201 263 135 of Polish Ministry of Science and Higher Education and the Grant-in-Aid for Scientific Research (B) 21340003 of Japan. The research of the second and third named authors was supported through the programme “Research in Pairs” by the Mathematisches Forschungsinstitut Oberwolfach in 2009. Moreover, the first named author gratefully acknowledges support from the Warmińsko-Mazurskie Przedsiebiorstwo Drogowe Sp. z o.o.

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