Divisor Class Group Descent for Affine Krull Domains

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1. INTRODUCTION

Let $R$ be a Krull domain with the quotient field $K$, let $L$ be a finite algebraic extension of $K$, and let $S$ be the integral closure of $R$ in $L$. The inclusion of $R$ in $S$ induces a homomorphism of the divisor class groups $i: \text{Cl}(R) \to \text{Cl}(S)$. Descent theory for Krull rings investigates the kernel of this homomorphism. In most examples (see [1]) this kernel is a finitely generated group. In this paper we shall find some sufficient conditions for $\text{ker}(i)$ to be finitely generated and some examples where it is not so. We are mainly interested in the case when $R$ contains a fixed field $k$ and $S$ is affine (finitely generated $k$-algebra). If $k$ is algebraically closed then $\text{Cl}(S)$ corresponds to the divisor class group of a suitable algebraic variety and $\text{ker}(i)$ is finitely generated (Theorem 1) by some results of algebraic geometry. In general $\text{Cl}(S)$ may become much bigger (example 1), or much smaller (example 2) under an algebraic extension of the field of constants $k$. Therefore, if $k$ is not algebraically closed, we cannot use directly the methods of algebraic geometry. The result is still true if $\text{char}(k) = 0$ or if $k$ is algebraically closed in $L$. In the other cases $\text{ker}(i)$ need not be finitely generated.

2. GROUND FIELD ALGEBRAICALLY CLOSED

In Sections 2, 3, 4 we denote by $k$ a fixed field, $R$ is a Krull domain containing $k$, $K$ is the quotient field of $R$, $L$ is a finite algebraic extension of $K$, and $S$ is the integral closure of $R$ in $L$. We also assume that $S$ is finitely generated over $k$. We denote by $i$ the homomorphism $\text{Cl}(R) \to \text{Cl}(S)$ induced by the inclusion.

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**Lemma 1** [2, Chap. V, Theorem 2]. Let $A$ be an affine ring, let $F$ be a finite algebraic extension of the quotient field of $A$, and let $B$ be the integral closure of $A$ in $F$. Then $B$ is a finite $A$-module. In particular $B$ is affine.

**Lemma 2.** Let $R_0$ be a Krull subring of $R$ containing $k$ and assume that $R$ is algebraic over $R_0$. Then there exists an element $r$ of $R_0$, such that $R_0[1/r]$ is affine and $S[1/r]$ is integral over $R_0[1/r]$. Let

$$j_0: \text{Cl}(R_0) \to \text{Cl}(S) \quad \text{and} \quad j: \text{Cl}(R_0[1/r]) \to \text{Cl}(S[1/r])$$

be the homomorphisms induced by the inclusions. Then $\ker(j)$ is isomorphic to a factor group of $\ker(j_0)$ by a finitely generated group.

**Proof:** There exists an affine subring $R_1$ of $R_0$ with the same quotient field as $R_0$. By Lemma 1 we may assume that $R_1$ is integrally closed. Since $S$ is affine there exists an element $r$ of $R_1$, such that $S[1/r]$ is integral over $R_1[1/r]$. But $R_0[1/r]$ is contained in $S[1/r]$ and $R_1[1/r]$ is integrally closed, hence $R_0[1/r] = R_1[1/r]$ is affine. Let $H$ be the subgroup of $\text{Cl}(R_0)$ generated by the finitely many prime divisorial ideals containing $r$. Then $\ker(j)$ is isomorphic to $\ker(j_0)/(H \cap \ker(j_0))$.

It follows from Lemma 2 that, as far as the size of $\ker(i)$ is concerned, we may assume that $R$ is affine. Let us suppose that $k$ is algebraically closed. Then $R$ and $S$ are rings of regular functions on normal affine algebraic varieties $V'$ and $W'$, respectively. We can embed $V'$ and $W'$ in complete normal algebraic varieties $V$ and $W$. For an algebraic variety $U$ let $\text{Cl}(U)$ denote the group of the divisor classes of $U$. Then $\text{Cl}(R) = \text{Cl}(V')$ and $\text{Cl}(V')$ is the quotient of $\text{Cl}(V)$ by the subgroup generated by the finite number of the prime divisors in $V \setminus V'$. There are similar relations between $\text{Cl}(S)$, $\text{Cl}(W')$ and $\text{Cl}(W)$. Furthermore we have an exact sequence of groups

$$0 \to \text{Pic}(V) \to \text{Cl}(V) \to \text{NS}(V) \to 0$$

where $\text{NS}(V)$ is the finitely generated Neron–Severi group of $V$ and $\text{Pic}(V)$ is the group of points of the Picard variety of $V$ [3, Chap. III, Sect. 1]. We have a similar sequence for the variety $W$.

The inclusion of $R$ in $S$ induces a generically surjective rational map $\phi: W \to V$, which induces a homomorphism $j: \text{Cl}(V) \to \text{Cl}(W)$ and its restriction $j_0: \text{Pic}(V) \to \text{Pic}(W)$. The map $\phi$ also induces a surjective homomorphism of the Albanese varieties $\psi: A(W) \to A(V)$, which induces the same map $j_0$ of the Picard varieties

$$j_0: \text{Pic}(V) = \text{Pic}(A(V)) \to \text{Pic}(A(W)) = \text{Pic}(W).$$

By [3, Chap. V, Proposition 2], since $\psi$ is surjective, $\ker(j_0)$ is finite. It follows that $\ker(i)$ is finitely generated.
We have proven

**Theorem 1.** Let $k$ be an algebraically closed field. Let $R$ be a Krull domain containing $k$ and let $S$ be a Krull overring of $R$ integral over $R$ and finitely generated over $k$. Then the kernel of the homomorphism $i: \text{Cl}(R) \to \text{Cl}(S)$ is finitely generated.

### 3. Galois Descent

Let the notation and assumptions be as at the beginning of Section 2. We shall also assume $L$ separable over $K$. By [11, Theorem 16.1] we have:

**Lemma 3.** Suppose $L$ is Galois over $K$ with the Galois group $G$. Then $\ker(i)$ is naturally isomorphic to a subgroup of $H^1(G, S^*)$, where $S^*$ is the multiplicative group of the invertible elements of $S$.

We shall prove that for an affine ring $S$ the group $S^*$ is finitely generated modulo the maximal subfield of $S$. This will imply that $H^1(G, S^*)$ and $\ker(i)$ are finitely generated.

**Lemma 4.** Let $F$ be the maximal subfield of $S$. Then $S^*/F^*$ is a finitely generated free abelian group.

*Proof.* $F$ is the set of the elements of $S$ algebraic over $k$. By the normalization theorem there exists a ring of polynomials $R_0 = k[x_1, \ldots, x_m]$, such that $S$ contains $R_0$ and is integral over $R_0$. Let $v$ be the "infinite" valuation of $k(x_1, \ldots, x_m)$, i.e., $v(f/g) = \deg(g) - \deg(f)$. There are only a finite number of extensions of $v$ to valuations of $L$. Let $v_1, \ldots, v_k$ be the valuations of $L$ extending $v$. Then $S^*$ is generated by $S^* / F^*$. Consider a homomorphism $h: S^* \to Z^k$ defined by $h(s) = (v_1(s), \ldots, v_k(s))$. If $s$ belongs to $\ker(h)$, i.e., $v_i(s) = 0$ for every $i$, then $s$ belongs to $V'$ and it is integral over $V'$. Whenever an element $u$ is integral over a unique factorization domain $B$ the irreducible equation of $u$ over $B$ is monic. Therefore if $s$ belongs to $\ker(h)$, its irreducible monic equation over the field $k(x_1, \ldots, x_k)$ has coefficients in $k = V \cap k[x_1, \ldots, x_k]$ and $s$ is algebraic over $k$. It follows that $\ker(h) \subset F^*$ and $S^*/F^*$ is isomorphic to a factor group of a subgroup of $Z^k$. If $s$ has a finite order modulo $F^*$ then it is algebraic over $F^*$, hence belongs to $F^*$. Therefore $S^*/F^*$ is a free finitely generated abelian group.

**Lemma 5.** Suppose that $L$ is Galois over $K$ with the Galois group $G$. Then $H^1(G, S^*)$ is finitely generated.
Proof. Let $G = \{g_1, \ldots, g_m\}$ and let $s_1, \ldots, s_n$ represent free generators of $S^*/F^*$. Let $\sigma: G \rightarrow S^*$ be a cocycle, such that $\sigma(g_i) = a_i \Pi(s_j)^{k_{ij}}$, $a_i \in F^*$, for $i = 1, \ldots, m$. Let us define a homomorphism of the group of cocycles into a free abelian group $\mathbb{Z}^{m \times n}$ letting $\phi(\sigma) = \{(k_{ij})\}$. Suppose that $\sigma$ belongs to the kernel of $\phi$, i.e., $\sigma(g_i)$ belongs to $F^*$ for all $i$. Then $\sigma$ represents an element of the group $H^1(G, F^*)$, which is trivial by Hilbert Theorem 90. Therefore $\sigma$ is a coboundary in $H^1(G, S^*)$. It follows that $H^1(G, S^*)$ is isomorphic to a factor group of a subgroup of $\mathbb{Z}^{m \times n}$.

**Theorem 2.** Let $k$ be a field, $R$ a Krull ring containing $k$, and $K$ the quotient field of $R$. Let $L$ be a finite separable extension of $K$, and let $S$ be the integral closure of $R$ in $L$. Assume also that $S$ is finitely generated over $k$. Then the kernel of the homomorphism $i: \text{Cl}(R) \rightarrow \text{Cl}(S)$ is finitely generated.

Proof. Let $L_1$ be a Galois extension of $K$ containing $L$, and let $S_1$ be the integral closure of $R$ in $L_1$. Then, by Lemmas 3 and 5, the kernel of the homomorphism $j: \text{Cl}(R) \rightarrow \text{Cl}(S_1)$ is finitely generated and $\ker(i)$ is a subgroup of $\ker(j)$.

4. Radical Descent

Let the notation and assumptions be as at the beginning of Section 2. By [1, Proposition 17.5], we have:

**Lemma 6.** Let $\text{char}(k) = p \neq 0$, let $[L: K] = p$, and let $D$ be a derivation of $L$, such that $D(S) \subset S$, $D(K) = 0$, $1 \in D(S)$. Then $\ker(i)$ is isomorphic to the additive group $\{x \in S; x = U(s)/s, s \in S\}$ modulo the group $\{D(u)/u; u \in S^*\}$.

**Lemma 7.** Let $\text{char}(k) = p \neq 0$, let $[L: K] = p$, and let $L$ be a purely inseparable extension of $K$. Assume also that $k$ is algebraically closed in $L$. Then $\ker(i)$ is finitely generated.

Proof. Since $S$ is affine, we can find $r \in R$ and $z \in S$ such that $R[1/r]$ is affine and $S[1/r] = R[z, 1/r]$. Therefore, by Lemma 2, we may assume that $R$ contains $1/r$ and $S = R[z]$, and $z^p \in R$. There is a unique derivation $D$ of $L$, such that $D(K) = 0$, $D(z) = 1$, $D(S) \subset S$. Let $\overline{L}$ be the algebraic closure of $L$ and $\overline{R} = k \cdot R$, and $\overline{S} = k \cdot S$ in $\overline{L}$. Clearly $\overline{S} = \overline{R}[z]$. Suppose that $\overline{S}$ is integrally closed, and let $R_1$ be the integral closure of $\overline{R}$. There exists a unique derivation $\overline{D}$ of $\overline{L}$ such that $\overline{D}$ extends $D$, $\overline{D}(R_1) = 0$, and $\overline{D}(\overline{S}) \subset \overline{S}$. Suppose that $\ker(i)$ is not finitely generated. Then, by Lemma 6, there are infinitely many distinct elements of the form $D(s)/s = \overline{D}(s)/s \in \overline{S}$. By
Theorem 1. $\ker(Cl(R) \to Cl(S))$ is finitely generated. The maximal subfield of $\mathcal{S}$ equals $\mathcal{k}$, $D(\mathcal{k}) = 0$, $\mathcal{S}^{*}/\mathcal{k}^{*}$ is finitely generated, and therefore the group $\{D(u)/u; u \in \mathcal{S}^{*}\}$ is a finite group of exponent $p$. Therefore $\ker(i)$ cannot be infinite by Lemma 6. If $\mathcal{S}$ is not integrally closed then, by Lemma 1, its integral closure is finitely generated over $\mathcal{R}$ and there exists $r_{0} \in \mathcal{R}$, such that $\mathcal{S}[1/r_{0}]$ is integrally closed. $r_{0}$ belongs to a finite extension $L_{1}$ of $K$, $L_{1} \subset K$. Then there exists $r_{1} \in L_{1}$, $r_{1}$ integral over $\mathcal{R}$, and the norm $N_{L_{1}/K}(r_{0}) = r_{0} \cdot r_{1} = r \in \mathcal{R}$. By Lemma 2 we may assume that $\mathcal{R}$ contains $1/r$, and then $\mathcal{S}$ is integrally closed and is generated by $z$ over the integral closure $\mathcal{R}_{1}$ of $\mathcal{R}$. It follows as before that $\ker(i)$ has to be finitely generated.

**Theorem 3.** Let $k$ be a field, $\mathcal{R}$ a Krull domain containing $k$, and $K$ the quotient field of $\mathcal{R}$. Let $L$ be a finite extension of $K$ and let $S$ be the integral closure of $\mathcal{R}$ in $L$. Suppose that $k$ is algebraically closed in $L$, and that $S$ is finitely generated over $k$. Then the kernel of the homomorphism $i: Cl(\mathcal{R}) \to Cl(S)$ is finitely generated.

**Proof.** By Lemma 2 we may assume that $\mathcal{R}$ is affine. Let $K_{1}$ be the separable closure of $K$ in $L$. If $\text{char}(k) = p \neq 0$ let $K_{1} \subset K_{2} \subset \cdots \subset K_{n} = L$ be such that $[K_{i}: K_{i-1}] = p$ for $i = 2, 3, \ldots, n$. The theorem follows from Lemma 7 and Theorem 2.

**Theorem 4.** Let $S$ be a Krull ring finitely generated over a field $k$ and such that $Cl(S)$ is finitely generated. Let $\mathcal{R}$ be a Krull subring of $S$ containing $k$. If either $\text{char}(k) = 0$ or $k$ is algebraically closed in $S$ then $Cl(\mathcal{R})$ is finitely generated.

**Proof.** If we add variables to $\mathcal{R}$ its divisor class group does not change. Therefore we may assume $S$ algebraic over $\mathcal{R}$. By Lemma 2 we may assume $S$ integral over $\mathcal{R}$. Theorem 4 follows from Theorem 2 and Theorem 3.

As a corollary we get the following:

**Theorem 5.** A Krull subring of a ring of polynomials has a finitely generated divisor class group.

5. **Examples**

**Example 1.** $Cl(\mathcal{R})$ is trivial but becomes infinite after an extension of the field of constants.

Let $F$ be an algebraically closed field, let $x$ and $y$ be algebraically independent over $F$, and let $t = y^2 + y + x^3 + x$. Let $k = F(t)$. Then $k[x, y]$ is a quotient ring of $F[x, y]$, hence $k[x, y]$ is a UFD. On the other hand the
equation $y^2 + y + x^3 + x - t = 0$ defines a nonsingular elliptic curve over $k$, hence over the algebraic closure $\overline{k}$ of $k$ the ring $\overline{k}[x,y]$ has a very large divisor class group.

**Example 2.** $\text{Cl}(R)$ is not finitely generated but becomes trivial after an extension of the field of constants.

Let $F$ be an algebraically closed field of characteristic $p \neq 0$. Let $t$ be transcendental over $F$ and let $k$ be the separable closure of $F(t^p)$. Let $x, y$ be algebraically independent over $k$ and let $u = y + x^{p+1} + tx$. Let $K = k(u)$, $R = K[y, x^p, 1/x^p]$, $S = R[x]$. Then $S$ is the integral closure of $R[t]$, and $S = K(t)[x, 1/x]$, hence $\text{Cl}(S) = 0$. By Lemma 6 $\text{Cl}(R)$ is an abelian group of exponent $p$. We shall prove that it is infinite, hence not finitely generated.

Let $D$ be the unique derivation of $S$, such that $D(x) = 1$, $D(R) = 0$. By Lemma 6 $\text{Cl}(R)$ is isomorphic to the group $V = \{D(s)/s \in S; s \in S\}$ modulo the group $V_0 = \{D(u)/u; u \in S^*\}$. The group $S^*$ of the invertible elements of $S$ is generated by $x$ and by the elements of $K[t]$. We have $D(x)/x = 1/x$ and $D(t) = D((u - y - x^{p+1})/x) = (y - u)/x^2 = -(x^p + t)/x$. Therefore the elements of $V_0$ have the form $(ax^p + b)/x$, $a, b \in K(t)$. Let $c \neq 0$ be any element of $F$. There exists a solution $a \in k$ of the equation $\alpha^{p+1} + t\alpha + c^p = 0$. Then $a = -c^p/\alpha^p + t^p$. Let $\beta = -c/\alpha + t \in k(t)$. Then $\beta \in k$ and $\beta^{p+1} + t\beta + c = 0$. We have

$$D(\beta) = (c/(\alpha + t)^2) \cdot D(t) = (x^p + t)/x(\alpha + t)$$

$$D(x - \beta)/(x - \beta) = (x(\beta^p + t) - \beta(x^p + t))/x(x - \beta)(\alpha + t)$$

$$= (1 + x\beta^{p-1} + x^2\beta^{p-2} + \cdots + x^{p-1}\beta)/x(\beta^p + t).$$

Therefore every element $c \in F$ produces an element of $V$. Let $d \neq 0$ be another element of $F$, let $\gamma^{p+1} + t\gamma + d = 0$, and let $D(x - \gamma)/(x - \gamma) = (t + xy^{p-1} + x^2y^{p-2} + \cdots + x^{p-1}\gamma)/x(y^p + t)$ be the corresponding element of $V$. If $(D(x + \gamma)/(x + \gamma) = D(x - \beta)/(x - \beta)) \in V_0$ then all positive powers of $x$ in the numerator cancel, i.e., $(\gamma^p + t)\beta^{p-i} = (\beta^p + t)\gamma^{p-i}$ and $d\beta^{p+1-i} = c\gamma^{p+1-i}$ for $i = 1, 2, \ldots, p-1$. If $p > 2$ we get $\beta = \gamma$, which contradicts $d \neq c$. If $p = 2$ we get $(\beta - \gamma)(\beta\gamma - t) = 0$ and $d\beta^2 = c\gamma^2$, hence $\beta^2 = c\gamma^2/d$. But we also have $\beta^3 + t\beta + c = 0$ which contradicts the fact that $t$ is not algebraic over $F$. Therefore elements of $F$ induce distinct elements of $V$ modulo $V_0$ and $\text{Cl}(R)$ is not finitely generated.

**Example 3.** If $S$ is not affine $\text{ker}(i)$ may be not finitely generated also in characteristic 0.

We shall construct an example by the method of Leedham–Green [4]. Let $\overline{Q}$ be the field of the algebraic numbers and let $Q_0$ be the field of the real algebraic numbers. Let $A = Q^2$ be the plane over $Q$. For $P \in A$ let $\overline{P}$ denote
the point complex conjugate to $P$. Let $(C_n : f_n(x, y) = 0)$ be a sequence of all irreducible curves defined over $\overline{Q}$. For each $i = 1, 2, \ldots$ we shall choose a curve $C_n(i)$ and a point $P_i \in A$ such that

(i) the coordinates of $P_i$ do not both lie in $Q_0$;
(ii) for all $j < i$ $P_i$ is not equal to $P_j$ or $\overline{P}_j$;
(iii) neither $P_i$ nor $\overline{P}_i$ lie on any curve $C_n(j)$ for $j < i$;
(iv) if $i$ is odd we shall choose $C_n(i)$ to be the first curve neither equal nor conjugate to any $C_n(j)$ for $j < i$, and we shall choose on $C_n(i)$ a simple point $P_i$ satisfying (i), (ii), (iii);
(v) if $i$ is even we shall choose a point $P_i$ satisfying (i), (ii), (iii), and we shall choose a curve $C_n(i)$ not defined over $Q_0$, not containing any $P_j$ or $\overline{P}_j$ for $j < i$, and containing both $P_i$ and $\overline{P}_i$ as simple points.

Then every irreducible curve or its conjugate is equal to some $C_n(i)$. Let $S$ be the ring of rational functions $f/g \in \overline{Q}(x, y)$, such that for every point $P = P_i$ or $P = \overline{P}_i$ we have $\text{ord}_P(f) \geq \text{ord}_P(g)$. Let $R = S \cap Q_0(x, y)$. Then $R$ and $S$ are Dedekind domains and $S$ is integral over $R$. Prime ideals of $S$ correspond to points $P_i$ and $\overline{P}_i$ while prime ideals of $R$ correspond to pairs $(P_i, \overline{P}_i)$. Ideals $p_{2i}$ corresponding to distinct pairs $(P_{2i}, \overline{P}_{2i})$ represent distinct elements of $\text{Cl}(R)$. Indeed if a divisor of a rational function $f/g$ contains $P_{2i}$ and does not contain any $P_{2k+1}$ or $\overline{P}_{2k+1}$ then $f$ and $g$ have only prime factors of the form $f_{n(2i)}$ or $f_{n(2i)}$. If also $f/g \in Q_0(x, y)$ then these factors appear in pairs. The product $f_{n(2i)} \cdot f_{n(2i)}$ generates the square $(p_{2i})^2$ in $R$. It follows that the ideal $p_{2i}$ has order 2 in $\text{Cl}(R)$ and is different from any other $p_{2k}$ in $\text{Cl}(R)$. But the image of $p_{2i}$ in $S$ (i.e., $p_{2i} \cdot S$) is principal, generated by $f_{n(2i)}^{-1}$. Therefore $\ker(\text{Cl}(R) \to \text{Cl}(S))$ is not finitely generated.

References