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On Finding and Cancelling Variables in $k[X, Y, Z]$

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INTRODUCTION

Let k be a ring (*commutative and with unity as all other rings in this paper*) *and* $C = k^{[3]}$. (*For any ring* R *by* $R^{[n]}$ *we denote a polynomial ring in* n *variables over* R.)

Let A be a subring of C such that $C = A^{[1]}$. There then arise the questions:

(1) *The cancellation problem*: Is $A \approx k^{[2]}$?

(2) *The embedded plane problem*: Suppose that $A = k^{[2]}$ and $F \in C \setminus A$ is a *plane* over k, i.e., $C/(F) \approx k^{[2]}$. Then is F an *embedded plane over* k, i.e., is $C = k[F]^{[2]}$?

In case k is a field of characteristic zero, it is a conjecture that the answer to (2) is "yes" and the answer to (2) is known to be "no" if k has positive characteristic.

No counterexamples to (1) are known when k is a field.

If k is a field, special cases of (1) and (2) have been recently studied by the following method.

Let $A^{[1]} = A[T]$. For the cancellation problem take F to be a suitable variable in C so that $C = k[F]^{[2]}$ and $F \notin A$. Now identify A as a subring of $A[T]/(F) = B$. Then one explicitly constructs variables for A in terms of judiciously chosen variables for $B \approx k^{[2]}$, exploiting the fact that B is a simple ring extension of A. In [9] and [7] this was done for $F = bT + a$ with $a, b \in A$ and in [10] the case $F = bT^n + a$ with $n > 1$ and coprime with the characteristic of k was treated.

In [8], the ideas of [10] have been extended to equations F such that B is *Galois* over A, i.e., A and B^G have the same quotient field, where $G = \mathrm{Aut}_A B$.

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In this paper, we consider the condition that there exists a variable $x \in B$ such that

$$k[x] \cap A \not\subset k[x^p], \qquad p = \operatorname{char} k.$$

This gives further partial results to (1) and (2) as follows. (For full details see Section 3.)

(I) Let $A^{[1]} = k^{[3]} = k[X, Y, Z]$ such that for some $a \in A$ we have $a = f(X) + Zh(X, Y, Z)$, where $f(X) \in k[X]$ and $h(X, Y, Z) \in k[X, Y, Z]$. Moreover assume that either

(i) k is a field and $f(X)$ is separable or

(ii) $k[f(X)] = k[X]$ and k is factorial.

Then $A \approx k^{[2]}$. (See Corollary 3.7.) (This generalizes [4, Theorem 4.1]).

(II) Let A be an affine domain over a field of characteristic zero. Let $F \in A[T] \approx A^{[1]}$ be written as $F = \sum_0^n a_i T^i$ with $a_i \in A$ and suppose that $a_1, \ldots, a_n$ have a nonunit common factor $a \in A$. Then $A[T] = L[F]^{[2]} \Leftrightarrow A[T]/(F) \approx L^{[2]}$ and $A \approx L^{[2]}$. (See Section 3.8.2).

(This generalizes most of the results of [7, 9, 10]).

An important step in establishing the above results is to find criteria in terms of fibers over prime ideals of k which ensure that an affine over-ring A of k is a polynomial ring in one variable over k.

This we do in some detail in Section 2 by exploiting the notion of S-inertness, where S is a multiplicative subset of k. (See Definition 2.1.2.)

The results, which are related to and generalize theorems from [4, 5] are of interest in their own right. Questions about when $F \in k^{[2]}$ is a variable in $k^{[2]}$ can be handled by the same technique. In this direction we obtain the following analog of the epimorphism theorem of Abhyankar and Moh. (See 1.1.)

(III) If k is a locally factorial Krull domain of characteristic zero and $F \in k^{[2]}$ such that $k^{[2]}/(F) \approx k^{[1]}$, then $k^{[2]} = (k[F])^{[1]}$. (See Theorem 2.6.2.)

Let us note that the main ingredient of the proof is the theorem of Abhyankar and Moh which ensures that $L^{[2]} = L[F]^{[1]}$, where L is the quotient field of k. If $\operatorname{char} k \neq 0$, we can prove the same theorem by adding the hypothesis that

$$L(F) \bigotimes_{k[F]} k^{[2]} = L(F)^{[1]}.$$

1. Notations and Preliminary Results

We will use the following notation.

(1) If K is a ring, K^* is the group of units of K.

(2) If K is a domain, qt K is the field of quotients of K.

(3) UFD and PID stand for *unique factorization domain* and *principal ideal domain*, respectively. Also *factorial* has the same meaning as UFD.

(4) $\mathrm{Sym}_K(Q)$ is the symmetric algebra of a K-module Q.

(5) A statement "$A = K^{[n]}$" means that K is in an obvious way a subring of A and A is K-isomorphic to a polynomial ring in n-variables over K (denoted by $K^{[n]}$).

The following results are mostly well known. We collect them here for easy reference.

1.1. Epimorphism Theorem [2, Theorem 1.2]. *Let L be a field of characteristic zero. If $F \in L^{[2]}$ is such that $L^{[2]}/(F) \approx L^{[1]}$, then $L^{[2]} = L[F]^{[1]}$.*

1.2.1. Parallel Lines Lemma [7, 1.6; 9, Corollary 1]. *Let L be a field and $0 \neq F \in A = L^{[2]}$ such that $A/(F) = B^{[1]}$, where B is an L-algebra.*
Suppose that either one of the following holds:

(i) *There is a separable algebraic extension E of L and a factor F_1 of F in $E \otimes_L A$ such that $E \otimes_L A = E[F_1]^{[1]}$.*

(ii) *L has characteristic zero.*

Then $A = L[X, Y]$ with $F \in L[X]$ for some $X, Y \in A$.

Proof. B has Krull dimension zero and hence $B = \prod B_i$, where each B_i is a local Artinian L-algebra. We see that:

(1) There is a one–one correspondence between prime ideals of B and distinct irreducible factors of F.

(2) Any two irreducible factors of F are comaximal.

Analogous statements hold if L is replaced by any extension field E of L.
If $\mathrm{char}\, L = 0$, then there is a separable algebraic extension $E \supset L$ such that one of the residue fields of $E \otimes_L B$ is E. If P_1 is the corresponding prime ideal of $E \otimes_L B$ and F_1 the corresponding irreducible factor of F, then

$$E \underset{L}{\otimes} A/(F_1) \approx \left(E \underset{L}{\otimes} B/P_1\right)^{[1]} = E^{[1]}$$

and

$$E \underset{L}{\otimes} A = E[F_1]^{[1]} \qquad \text{by Theorem 1.1.}$$

Thus in either case we can assume that (i) holds. By [7, Lemma (1.6)], it is enough to prove the lemma in case $E = L$.
Now put $X = F_1$, $A = L[X, Y]$. Let F' be any other irreducible factor of F. Then $A/(F') = L'^{[1]}$, where L' is a finite field extension of L. Since $(X, F')A = A$,

the image $\bar{X}$ of X module (F') is a unit. Hence $\bar{X} \in L'$ and there exists $0 \neq f(X) \in L[X]$ such that $f(\bar{X}) = 0$. Then F' divides $f(X)$ and hence $F' \in L[X]$. If follows that $F \in L[X]$.

1.2.2. COROLLARY. (i) $\deg_X F = \text{length}_L B$.

(ii) *If $Z \in A$ such that $F \in L[Z]$, then $Z \in L[X]$ and*

$$L[X] = L[Z] \Leftrightarrow \deg_Z(F) = \text{length}_L B$$
$$\Leftrightarrow L[\bar{Z}] = B,$$

where $\bar{Z}$ is the image of Z modulo (F).

Proof. Clear.

1.3. LÜROTH LEMMA [1, 2.10]. *Let L be a field and $E \supset L$ a separable algebraic field extension. If A is a normal domain such that*

$$L \subsetneq A \subset E^{[1]}$$

and if L is algebraically closed in A (or equivalently if $A^* = L^*$), then

$$A = L^{[1]}.$$

Remark. The case when $L = E$ can be generalized. See Section 3.2.

2. CRITERIA FOR ONE-VARIABME POLYNOMIAL RINGS OVER A SUBRING

2.1. *Notation and definitions*

2.1.1. We will consider the following situation throughout this section:

$$S \subset k \subset K \subset A.$$

Here A is a domain, k, K are subrings, and S is a multiplicative set not containing 0.

2.1.2. DEFINITION. K is said to be S-inert in A relative to k if the following conditions hold:

(i) $A \cap S^{-1}K = K$.

(ii) For every height one prime ideal P of k containing some element of S we have

(a) PA is prime,

(b) $\text{qt}(\bar{K})$ is algebraically closed in $\text{qt}(A/PA)$, where $\bar{K}$ is the image of K in A/PA,

(c) $A/PA \cap \text{qt}(\bar{K}) = \bar{K}$.

Remark. We will drop the reference to k whenever it is clear from the context.

2.2. MODIFICATION LEMMA. *Assume the setup in* 2.1.1 *and assume that K is S-inert in A. Let $x_1, \ldots, x_r$, $t \in A$ such that*

$$p_i x_i \in K[t], \qquad i = 1, \ldots, r,$$

where each p_i is a product of elements in S which are prime elements in k. Then there exists $a \in K$ and $b \in S$ such that

$$t' = (t - a)/b \in A,$$

and

$$x_i \in K[t'].$$

Proof. We call $t' \in A$ a *modification* of t if $t' = (t - a)/b$ with $a \in K$ and $b \in S$. Note that then $K[t] \subset K[t']$. Also a modification of t' is clearly a modification of t and hence, by induction on r, we only need to prove the case $r = 1$.

Write

$$p_1 x_1 = f(t) = \sum a_j t^j, \qquad a_j \in K. \tag{1}$$

Since p_1 is a product of prime elements of k it suffices to show that by replacing t by some modification we can get a relation similar to (1) with a smaller number of prime factors for p_1.

Let p be a prime factor of p_1. Since K is S-inert in A, the conditions of Definition 2.1.2 hold. By Definition 2.1.2(i), if $a_j \in pA$ for all j, then $a_j \in pK$ and hence p can be cancelled from both sides of (1). Otherwise, if "$\overline{}$" denotes images in A/pA, we get that

$$0 = \sum \bar{a}_j \bar{t}^j$$

is a nontrivial relation for $\bar{t}$ over $\bar{K}$. By Definition 2.1.2(ii)(a), A/pA is a domain and from Definition 2.1.2(ii)(b) and (c) we get that $\bar{t} \in \mathrm{qt}(\bar{K}) \cap A/pA = \bar{K}$. Hence for some $a \in K$ we get that $t - a = pt'$ for some $t' \in A$.

Now

$$p_1 x_1 = f(pt' + a) = \sum a'_j t'^j \qquad \text{with} \quad a'_j \in K. \tag{2}$$

Clearly $a'_j \in pA$ for $j > 0$ and since $p_1 x_1 \in pA$, we have $a'_0 \in pA$ as well.

As we have seen above, this means that p can be cancelled from both sides of (2) and hence the result.

2.3.1. THEOREM. *Assume the setup in Section* 2.1.1. *Assume that A is finitely generated over K. Moreover, assume that*

(i) $S^{-1}A = (S^{-1}K)^{[1]}$,

(ii) *S is generated by prime elements of k, and*

(iii) *K is S-inert in A.*

Then $A = K^{[1]}$.

Proof. Find $t \in A$ such that $S^{-1}A = (S^{-1}K)[t]$. If $x_1, \ldots, x_r$ generate A over K, then by (2.2) we get a modification t' of t such that $t' \in A$ and $x_i \in K[t']$. Then $A = K[t']$.

2.3.2. *Remark.* From the proofs of Lemma 2.2 and Theorem 2.3.1 it is clear that $A = K[t']$ precisely when $(S^{-1}K)[t'] = S^{-1}A$ and t' is residually transcendental modulo each prime $p \in S$. Also, if $S^{-1}A = (S^{-1}K)[t]$ with $t \in A$, then t' is necessarily a modification of t, i.e., $t' = (t - a)/b$ with $a \in K$ and $b \in S$.

2.4. Let L be a field. As is well known, if $F \in L[X, Y] \approx L^{[2]}$ is "generically a line," i.e., if $L(F)[X, Y] = L(F)^{[1]}$, then F is an "embedded line" or a variable, i.e., $L[X, Y] = L[F]^{[1]}$.

The following result (Theorem 2.4.2) gives a proof of this as well as a slight generalization of [4, Theorem 4.4.].

2.4.1. Definition. Let L be a field and A an L-algebra. A is said to be "*geometrically factorial*" over L if $E \otimes_L A$ is a UFD for any algebraic extension field $E \supset L$.

2.4.2. Theorem. *Let L be a field, A a finitely generated L-algebra, and $F \in A$. Assume that*

(i) $S^{-1}A = L(F)^{[1]}$, *where* $S = L[F] \backslash \{0\}$,

(ii) $L(F) \cap A = L[F]$, *and*

(iii) *A is geometrically factorial over L.*

Then $A = L[F]^{[1]}$.

Proof. We check the conditions of Theorem 2.3.1 for $S \subset k = K = L[F] \subset A$. Of these conditions (i) and (ii) are obvious and we only need to show that K is S-inert in A. Thus we check the conditions (i) and (ii) of Definition 2.1.2. Condition (i) of Definition 2.1.2 is just the hypothesis (ii) above.

Now let $p \in L[F]$ be irreducible. If $p = ab$ with $a, b \in A$, then a, b are units in $S^{-1}A = L(F)^{[1]}$ and hence $a, b \in L(F) \cap A = L[F]$. Thus p is irreducible in A and since A is a UFD by hypothesis (iii), pA is prime. Thus Definition 2.1.2(ii)(a) holds.

It is easy to see that the assumptions (i) and (ii) above hold for $E \subset E \otimes_L A$ for any algebraic extension E of L. By the argument given above, if $p_1 \in E[F]$ is irreducible, then $p_1 E \otimes_L A$ is prime.

Now let $p \in L[F]$ be irreducible. Put $E = L[F]/(p)$. Then $p = p_1{}^q p'$ with $p_1, p' \in E[F]$, p_1 linear in F, q a power of the characteristic exponent of L and p_1, p' relatively prime. Let $\bar{E}$ be an algebraic closure of E. Then

$$\bar{E} \underset{E}{\otimes} \left(E \underset{L}{\otimes} A/(p_1)\right) \approx \bar{E} \underset{L}{\otimes} A/(p_1)$$

is a domain (p_1 is linear and hence irreducible in $\bar{E}[F]$). Hence E is algebraically closed in $E \otimes_L A/(p_1)$. Now suppose that $E \subset E' \subset A/pA$, where E' is a field. Then

$$\left(E \underset{L}{\otimes} A/(p_1{}^q)\right) \times \left(E \underset{L}{\otimes} A/(p')\right) \approx \left(E \underset{L}{\otimes} A\right)\Big/(p) \supset E \underset{L}{\otimes} E' = \prod E_i ,$$

where each E_i is a local Artinian E'-algebra. Hence $E \otimes_L A/(p_1) \supset E' \supset E$ and hence $E' = E$. This proves condition (ii)(b) of Definition 2.1.2. Definition 2.1.2(ii)(c) is obvious since $K = L[F]$ is a PID. Thus all conditions of Definition 2.1.2 are checked, and the theorem is proved.

2.4.3. *Remark.* Condition (ii) of Theorem 2.4.2 above is satisfied if $A^* = L^*$.

2.4.4. *Remark.* A result closely related to Theorem 2.4.2 is proved in Section 3.2.

2.5. The following result is most likely well known. We include a proof for the convenience of the reader.

2.5.1. LEMMA. *Let k be a locally factorial Krull domain. Let $p_1 ,..., p_r \in k$. Then there exist $a_1 ,..., a_s \in k$ with $(a_1 ,..., a_s)k = k$ such that each p_i is a product of prime elements in each k_{a_j} = (k localized at the multiplicative system generated by a_j).*

Proof. Since k is a Krull domain, $p_1 p_2 \cdots p_r$ is contained in only finitely many height one primes of k. It suffices to show that given any height one prime ideal P of k and given any maximal ideal M of k, there exists $a \in k \backslash M$ such that Pk_a is principal.

Now Pk_M is principal since k_M is factorial by hypothesis and there exists $b \in k$ such that $Pk_M = bk_M$. The divisor of b is of the form $P + \sum Q_i$, where each Q_i is a prime divisor (height one prime) such that $Q_i \not\subset M$. Hence there exists $a \in \prod Q_i \backslash M$. Then the divisor of b on k_a is Pk_a so that Pk_a is principal (generated by b).

2.5.2. THEOREM. *Let the setup be as in* 2.1.1 *and let A be finitely generated over K. Suppose that*

(i) *k is a locally factorial Krull domain,*

(ii) $S^{-1}A = (S^{-1}K)^{[1]}$, *and*

(iii) *K is S-inert in A.*

Then A is K-isomorphic to $\mathrm{Sym}_K(Q)$ *where Q is a finitely generated rank* 1 *projective K-module such that Q is locally free on k.*

Proof. Write $S^{-1}A = (S^{-1}K)[t]$ with $t \in A$. Let $x_1, ..., x_r$ be generators for A over k. Then there exist $p_i \in S$ such that $p_i x_i \in K[t]$ for $1 \leqslant i \leqslant r$. Let $a_1, ..., a_s$ be as in Lemma 2.5.1. Clearly K_{a_j} is S-inert in A_{a_j}, $j = 1, ..., s$. By Theorem 2.3.1, there exist $t_j \in A_{a_j}$ such that $A_{a_j} = K_{a_j}[t_j]$, $j = 1, ..., s$. Now $(a_1, ..., a_s)K = K$ and $A = \mathrm{Sym}_K(Q)$ as stated. This is well known—see [4, Lemma 3.1] or [3, footnote]. Since $A_{a_j} = \mathrm{Sym}_{K_{a_j}}(Q \otimes K_{a_j}) \approx K_{a_j}$, we get that $Q \otimes_K K_{a_j}$ is free. Since $a_1, ..., a_s \in k$, Q is locally free on k.

2.5.3. COROLLARY. *With the assumptions as in Theorem* 2.5.2, *if we further assume that $K = k^{[m]}$ for some m, then A is K-isomorphic to* $\mathrm{Sym}_k(Q') \otimes_k K$, *where Q' is a rank one finitely generated projective module over k.*

Proof. The projective module Q in Theorem 2.5.2 is locally free on k, hence locally extended from k and hence is extended from k by [6, Theorem 1], i.e., $Q = Q' \otimes_k K$ for some finitely generated projective k-module Q' of rank one. Hence $A \approx \mathrm{Sym}_K(Q) \approx (\mathrm{Sym}_k Q') \otimes_k K$.

2.5.4. *Remark.* Theorem 2.5.2 is a strengthened version of [5, Theorem 1]. Obviously our conditions are tailor-made to make the rather direct method of proof go through. This method of *modification* of a generic choice of generator as in Lemma 2.2 is from [4, Theorem 4.4; 5, Lemma 1.3]. Conditions (ii)(a) and (b) of Definition 2.1.2 roughly correspond to the requirement of geometric integrality of fibers in [5, Theorem 1] (however we only need conditions for fibers over height one primes), whereas conditions 2.1.2(i) and (ii)(c) in essence replace the assumption of faithful flatness for A over k made in [5]. Certainly Theorem 2.1.3(i) and (ii)(c) are easy consequences of faithful flatness and are more convenient to check in our applications. Keeping track of the two subrings $k \subset K$ and inverting only elements of k has technical advantages in treating "embedded plane" type problems ((Theorem 2.6.2, for instance), and the corresponding stronger result does not need any more complicated proof.

2.6.1. LEMMA. *Let D be a domain and*

$$F \in D[X_1, ..., X_n] \approx D^{[n]}$$

such that $F(0, ..., 0) = 0$ *and* $c(F) =$ (*the ideal generated by the coefficients of* F) $= D$.
Let $E = \mathrm{qt}\, D$.
Then $E(F) \cap D[X_1, ..., X_n] = D[F]$.

Proof. Put $R = E(F) \cap D[X_1, ..., X_n]$. Then $R \subset E(F) \cap E[X_1, ..., X_n] = E[F]$. Suppose $G = a_0 + a_1 F + \cdots + a_s F^s \in R$ with $a_i \in E$. Since $F(0, ..., 0) = 0$, $a_0 \in D$ and

$$(G - a_0) = F(a_1 + \cdots + a_s F^{s-1}) \in R.$$

Choose $b \in D$ such that $ba_i = a_i' \in D$ for $i = 1, ..., s$. Then

$$F(a_1' + \cdots + a_s' F^{s-1}) \equiv 0 \bmod (b).$$

Since $c(F) = D$, F is not a zero divisor modulo b and hence

$$a_1' + a_2'F + \cdots + a_s'F^{s-1} \equiv 0 \pmod{(b)},$$

so $a_1' = ba_1 \in bD$ and hence $a_1 \in D$. It is now clear how to proceed by induction to show that $a_i \in D$ for all i and hence $G \in D[F]$. Thus $R \subset D[F]$ and $D[F] \subset R$ is obvious. Hence the result.

Remark. $c(F - F(0,..., 0)) = (1)$ if and only if F is transcendental modulo $MD[X_1, ..., X_n]$ for each maximal ideal M of D.

2.6.2. THEOREM. *Let k be a locally factorial Krull domain with quotient field L. Let $F \in A \approx k^{[2]}$.*
Assume that

(i) $A/FA \approx k^{[1]}$,

(ii) char $k = 0$ *or* $L(F) \otimes_k A = L(F)^{[1]}$.

Then $A = k[F]^{[1]}$.

Proof. Write $A = k[X, Y]$. We claim that

$$L \bigotimes_k A = L[X, Y] = L[F]^{[1]}. \tag{1}$$

If char $k = 0$, this follows by Theorem 1.1. If $L(F) \otimes_k A \approx L(F)^{[1]}$, then (1) follows by applying Theorem 2.4.2 to $L \subset L \otimes_k A$.
From (1) we get that condition (ii) of Theorem 2.5.2 holds for

$$S = k - \{0\} \subset k \subset K = k[F] \subset A.$$

We proceed to check the remaining conditions of Theorem 2.5.2.
Let P be a prime ideal of k. Let "$\overline{}$" denote images modulo PA and put $l = \mathrm{qt}(k/P)$.
Then

$$\bar{A} = (k/P)^{[2]}, \tag{2}$$

$$\bar{A}/\bar{F}\bar{A} = (k/P)^{[1]}, \tag{3}$$

$$\left(l \bigotimes_k \bar{A}\right)\Big/\bar{F}\left(l \bigotimes_k \bar{A}\right) = l^{[1]}. \tag{4}$$

Since $A/FA = k^{[1]}$, there exist $a, b \in k$ such that $F(a, b) = 0$, and hence replacing X, Y by $X + a$, $Y + b$ we may assume that $F(0, 0) = 0$.
By (2), PA is prime. By (3), $\bar{F} \neq 0$ and $c(F) \not\subset P$. Since this holds for all

prime ideals P we get $c(F) = (1)$ and by (2) and Lemma 2.6.1 we have $l(\bar{F}) \cap \bar{A} = (k/P)[\bar{F}]$. Let $\bar{l}$ be an algebraic closure of l. Then

$$\bar{l} \underset{l}{\otimes} \left(\left(l \underset{k}{\otimes} \bar{A}\right)\Big/\bar{F}\left(l \underset{k}{\otimes} \bar{A}\right)\right) \approx \bar{l}^{[1]}$$

is a domain (because of (4)) and hence $\operatorname{qt} \bar{K} = l(\bar{F})$ is algebraically closed in $\operatorname{qt} \bar{A}$.

We have now established that K is S-inert in A. By Theorem 2.5.2 and Corollary 2.5.3 we get

$$A \approx \operatorname{Sym}_k(Q) \underset{k}{\otimes} k[F]$$

(by a $k[F]$-isomorphism) where Q is a finitely generated projective k-module. However $\operatorname{Sym}_k(Q) \approx A/FA \approx k^{[1]}$ is free and hence $A = k[F, G]$ for some $G \in A$.

3. Applications to the Cancellation and Embedded Plane Problem

3.1. Theorem. *Let A be an affine domain over a field of characteristic p. Suppose $A \subset L[x, y] \approx L^{[2]}$ such that $L[x] \cap A \not\subset L[x^p]$.*

If A is geometrically factorial, then

$$L[x] \cap A = L[u] \quad \textit{for some} \quad u \in A, \tag{3.1.1}$$

$$A \approx L[u]^{[1]} \quad \textit{or} \quad A = L[u]. \tag{3.1.2}$$

Proof. We have $L \subset A \cap L[x] \subset L[x]$, $A \cap L[x]$ is clearly normal and L is relatively algebraically closed in $L[x]$ and hence in $A \cap L[x]$.

Therefore Lemma 1.3 gives that $A \cap L[x] \approx L^{[1]}$. This proves (3.1.1).

Now we have

$$L(u) \subset A \underset{L[u]}{\otimes} L(u) \subset L[x, y] \underset{L[u]}{\otimes} L(u) = L(x)[y].$$

We apply Lemma 1.3 again by checking conditions as follows:

Any element of $A \otimes_{L[u]} L(u)$ which is algebraic over $L(u)$ belongs to $L(x)$ and hence to

$$(A \cap L[x]) \underset{L[u]}{\otimes} L(u) = L(u).$$

$A \otimes_{L[u]} L(u)$ is normal, being a localization of A. Thus Lemma 1.3 gives $A \otimes_{L[u]} L(u) = L(u)$ and clearly $A = L[u]$ or $A \otimes_{L[u]} L(u) = L(u)^{[1]}$.

Then $L[u] \subset A \cap L(u) \subset A \cap L(x) = A \cap L[x] = L[u]$. Hence Theorem 2.4.2 is applicable with u replacing F. Thus $A = L[u]^{[1]}$.

Remark. Note that geometric factoriality is used only in the last step of applying Theorem 2.4.2.

3.2. The following is a generalization of the special case $u = x$ of Theorem 3.1.

THEOREM. *Let $k \subset A \subset k[y] \approx k^{[1]}$ be factorial domains. Then $k = A$ or $A = k^{[1]}$.*

Proof. Assume that $A \neq k$. Then there exists $h(y) \in A$ such that

(i) $h(y)$ has the least positive degree among the elements of A,

(ii) $h(0) = 0$, and

(iii) $h(y)$ is irreducible in A.

Let $E = \mathrm{qt}\, k$. Then $E \subset A \otimes_k E \subset E[y]$ and $A \otimes_k E$ is factorial. Hence by the Lüroth lemma (Lemma 1.3) we get that $A \otimes_k E \approx E^{[1]}$ and from (i) it follows that

$$A \bigotimes_k E = E[h]. \tag{1}$$

Then for every $a \in A$ we can find $b, b_0, \ldots, b_n \in k$ for some n such that

$$ab = b_0 + b_1 h + \cdots + b_n h^n. \tag{2}$$

By induction on n we will prove that $b_i \in bk$ for all i and hence b can be cancelled from both sides, which shows $a \in k[h]$. Then $A = k[h]$ is obvious.

Now assume the result for all values of $n < m$ and put $n = m$.

Comparing the constant terms with respect to y on both sides of (2) and using that $h(0) = 0$, we get that

$$b_0 = ba_0 \qquad \text{for some} \quad a_0 \in k. \tag{3}$$

Thus

$$(a - a_0)b = h[b_1 + \cdots + b_m h^{m-1}].$$

Since $b \in k$ and $h \notin k$ is irreducible in A we get that

$$a - a_0 = ha_1, \qquad a_1 \in A.$$

Moreover,

$$a_1 b = b_1 + \cdots + b_m h^{m-1}$$

and hence by the induction hypothesis $b_i \in bk$ for $i > 1$. From (3) $b_0 \in bk$ and hence the result.

3.3. *Remark.* In the above theorem, we do not resort to using the old theorem (Theorem 2.3) since it becomes difficult to check the inertness conditions, in particular Definition 2.1.2(ii)(b). However, just the fact that A is contained in a polynomial ring $k[y]$ over k helps out in producing, in fact, a simpler proof.

Unfortunately we do not know how to generalize the above situation to $k \subset A \subset R[y] \approx R^{[1]}$, where R is, say, integral over k and UFD; this, if done, would be a full generalization of Lemma 3.1.

3.4. COROLLARY. *If $k \subset A \subset k^{[m]}$ are factorial domains where m is an integer, and if A has transcendence degree at most one over k, then $A = k$ or $A = k^{[1]}$.*

Proof. We only need to show that $A \subset k[y] \approx k^{[1]}$ for some y and then apply Section 3.2.

We proceed by induction on m, the result being trivial for $m \leqslant 1$.

Write $k^{[m]} = k[X_1, ..., X_m]$.

Let $E = \mathrm{qt}\, k$, $B = A \otimes_k E$. Then it is well known that we can write $k^{[m]} = k[Y_1, ..., Y_m]$ with $(Y_m)\, E[Y_1, ..., Y_m] \cap B = (0)$.

It follows that $(Y_m)\, k[Y_1, ..., Y_m] \cap A = (0)$ and hence

$$k \subset A \subset k[Y_1, ..., Y_m]/(Y_m) \approx k^{[m-1]}.$$

Now we are finished by induction.

3.5. *Notation.* Let L be a field and A an affine domain over L. Let $A[T] \approx A^{[1]}$ and $F \in A[T]\backslash A$. Let $B = A[T]/(F)$. Then we can, after suitable identification, write

$$A \subset A[t] = B,$$

where t is the image of T modulo (F). We assume that $A[T] \approx L^{[3]}$ and $B \approx L^{[2]}$.

3.6. THEOREM. *Let the notation be as in Section* 3.5 *above. Let $x, y \in B$ such that $B = L[x, y]$ and*

$$L(x) \cap A \not\subset L[x^p]$$

where p is the characteristic of L.

Then we have:

(3.6.1) *If $A[T] = L[F]^{[2]}$, then $A \approx L^{[2]}$.*

Conversely:

(3.6.2) *If $p = 0$ and $A \approx L^{[2]}$, then $A[T] = L[F]^{[2]}$.*

Proof. In view of Theorem 3.1, to prove (3.6.1) we only need to show that A is geometrically factorial. But if E is any algebraic extension of L, then

$$\left(A \underset{L}{\otimes} E\right)[T] = E^{[3]}$$

and hence $A \otimes_L E$ is factorial.

Now we prove (3.6.2). In view of Theorem 3.1 we can write $A = L[u, v]$, where $L[u] = A \cap L[x]$.

Now $L(u) \subset L(u)[v, T]/(F) \approx L(x)[y]$. We apply Lemma 1.2.1 with L and B replaced by $L(u)$ and $L(x)$, respectively, and consequently find $H_1 \in A[T]$ such that

$$L(u)[v, T] = L(u)[H_1]^{[1]} \tag{1}$$

and moreover $F \in L(u)[H_i]$.

Thus we can write

$$gF = \sum_0^n g_i H_1{}^i, \tag{2}$$

where $g, g_1, ..., g_n \in L[u]$ and $g, g_0, ..., g_n$ have no common factor.

Now apply the modification lemma (Lemma 2.2) with $S = L[u]\setminus\{0\}$, $k = K = L[u]$ and $A = L[u, v, T]$. Since the conditions are obviously satisfied we get $a, b \in L[u]$ and $H \in L[u, v, T]$ such that

$$bH + a = H_1 \tag{2}$$

and moreover

$$F \in L[u][H]. \tag{3}$$

From (1) and (2) we get that

$$L(u)[v, T] = L(u)[H]^{[1]}. \tag{4}$$

Next we want to prove that $L[u, v, T] = L[u, H]^{[1]}$.

We apply Theorem 2.3.1 with $S = L[u]\setminus\{0\}$, $k = L[u] \subset K = L[u, H]$ and $A = L[u, v, T]$. In view of (4) it is sufficient to establish the following:

Let $q \in L[u]$ be prime. Moreover let $E = L[u]/(q)$ and denote by $\tilde{H}$ the image of H in $E[v, T] \approx L[u, v, T]/(q)$.

Then

$$E[v, T] = E[\tilde{H}]^{[1]}. \tag{5}$$

In fact, Theorem 2.3.1(i) is nothing but (4) and Theorem 2.3.1(ii) is obvious. Also H is transcendental mod q for each prime q in $L[u]$ and $L(u, H) \cap L[u, v, T] = L[u, H]$ follows from Lemma 2.6.1. Hence Lemma 1.2.1(i) holds. Since Lemma 1.2.1(ii)(a) is obvious and Lemma 1.2.1(ii)(b)(c) are immediate consequences of (5), we have established Theorem 2.3.1(iii).

To prove (5) we first

CLAIM. *The image of* $L[u, H]$ *modulo* (F) *generates* $L[x]$.

Assume the validity of this claim for the moment. Denote by F' the image of F in $E[v, T]$ and by $\tilde{x}, \tilde{u}, \tilde{H}$ the images of x, u, H in $E[v, T]/(F')$. Then $E[\tilde{x}] = L[\tilde{x}] = L[\tilde{u}, \tilde{H}] = E[\tilde{H}]$. Moreover

$$E[v, T]/(F') \approx L[x, y]/(q) = E[\tilde{x}]^{[1]}$$

and hence $E[v, T] = E[\tilde{H}]^{[1]}$ follows from Corollary 1.2.2(ii) and (3).

It only remains to prove the claim.

Let "$\overline{}$" denote images modulo (F). Then $\bar{u} = u$, $\bar{H} \in L[x]$ and from Corollary 1.2.2, (3), and (4) we deduce that

$$L(u)[\bar{H}] = L(u) \bigotimes_{L[u]} L[x] = L(x).$$

Thus $\operatorname{qt} L[u, \bar{H}] = L(x)$ and to establish the claim we only need to prove that $L[u, \bar{H}]$ is normal.

We use the Jacobian criterion. Since

$$L[u, v, T]/(F) \approx L^{[2]}$$

is regular, we get that

$$\overline{\frac{\partial F}{\partial u}} + \overline{\frac{\partial F}{\partial H}}\,\overline{\frac{\partial H}{\partial u}},\ \overline{\frac{\partial F}{\partial H}}\,\overline{\frac{\partial H}{\partial v}},\ \overline{\frac{\partial F}{\partial H}}\,\overline{\frac{\partial H}{\partial T}}$$

generate the unit ideal in $L[u, v, T]/(F) = L[x, y]$. Now $\overline{\partial F/\partial u}$ and $\overline{\partial F/\partial H} \in L[u, \bar{H}] \subset L[x]$. Hence

$$\left(\overline{\frac{\partial F}{\partial u}}, \overline{\frac{\partial F}{\partial H}}\right) L[x] = L[x]$$

and since $F[x]/L[u, \bar{H}]$ is integral we get that $(\overline{F\partial/\partial u}, \overline{\partial F/\partial H}) L[u, \bar{H}] = L[u, \bar{H}]$. Then $L[u, \bar{H}]$ is regular and hence normal, and the claim is established.

Now we have shown that

$$L[u, v, T] = L[u, H]^{[1]},$$

and in the course of the proof we have shown that

$$L[u, H]/(F) = L[x] = L^{[1]}.$$

Hence by the epimorphism theorem of Abhyankar and Moh (Theorem 1.1) we get that $L[u, H] = L[F]^{[1]}$.

Thus

$$L[u, v, T] = L[u, H]^{[1]} = L[F]^{[2]}.$$

3.7. COROLLARY (a case of the cancellation problem). *Let A be a domain over a UFD k. Let*

$$A[T] \approx A^{[1]} = k[X, Y, Z] \approx k^{[3]}.$$

Assume that there exists $a \in A$ such that

$$a = f(X) + Zh(X, Y, Z),$$

where $f(X) \in k[X]$ and $h(X, Y, Z) \in k[X, Y, Z]$. Moreover assume that either

(i) $k = L$, *a field, and $f(X)$ is separable or*

(ii) $k[f(X)] = k[X]$.

Then $A \approx k^{[2]}$.

Proof. Note that $A^{[1]} \approx k^{[3]}$ implies that A is an affine factorial k-algebra. Note that we may assume $Z \notin A$. In fact, if $Z \in A$, then we may assume that, say, $Y \notin A$ and put $a = Z$. Then $a = Z + Y \cdot 0$ and we have the required situation for $(X', Y', Z') = (Z, X, Y)$.

Now in case (i), we take $Z = F$ and the proof is finished by (3.6.1).

In case (ii), we identify A as a subring of $A[T]/(Z) = k[X, Y] = A[t]$, where t is the image of T modulo (Z). Then $k[X] \subset A \subset A[t] = k[X, Y]$ and A is clearly factorial. The proof is finished by taking $k[X]$ in place of k in Section 3.2.

3.8.1. LEMMA. *Let the notation be as in Section* 3.5 *and assume that A is factorial. Let F be written as*

$$F = \sum_0^n a_i T^i \qquad \text{with} \quad a_i \in A.$$

Then the following are equivalent:

(i) $B \cap \operatorname{qt} A \supsetneqq A$.

(ii) $a_1, \ldots, a_n$ *have a nonunit common factor* $a \in A$.

Proof. Let $b/a = \sum_0^m b_i T^i \in B \cap \operatorname{qt} A$, where $a, b \in A$ have no common factor and $a \notin A^*$. Then

$$a\left(\sum_0^m b_i T^i\right) - b = FG = \left(\sum_0^n a_i T^i\right) G$$

for some $G \in A[T]$. Hence $F \equiv -b$ modulo (a') for any irreducible factor a' of a, that is, a' divides $a_1, \ldots, a_n$ and not a_0.

Conversely, if a divides $a_1, \ldots, a_n$, then no factor of a divides a_0 since F is irreducible and

$$\frac{a_1}{a} T + \cdots + \frac{a_n}{a} T^n = -\frac{a_0}{a} \in B \cap \operatorname{qt} A, \ -\frac{a_0}{a} \notin A.$$

3.8.2. *A case of the cancellation and the embedded plane problem.* Let the notation be as in Section 3.5 and assume that the equivalent conditions of Lemma 3.8.1 are satisfied.

Also assume that L has characteristic zero. Then

$$A[T] = L[F]^{[2]}$$

if and only if $A \approx L^{[2]}$ and $A[T]/(F) = B \approx L^{[2]}$.

Proof. We check that the conditions of Theorem 3.6 are satisfied. Clearly, if $A[T] = L[F]^{[2]}$, then $B \approx L^{[2]}$ and A is an affine factorial L-algebra.

Now $A[T]/(a, F) = A[T]/(a, a_0) = A/(a, a_0)[T] \approx R^{[1]}$, where R is a zero-dimensional L-algebra. On the other hand $A[T]/(a, F) \approx B/(a) \approx L^{[2]}/(a)$. From the parallel lines lemma (Lemma 1.2) we deduce that $B = L[x, y]$ with $a \in L[x]$ for suitable x, y. Hence the hypothesis of Theorem 3.6 is satisfied and (3.6.1) and (3.6.2) give the desired conclusion.

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