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Volume Comparison and Its Applications in Riemann–Finsler Geometry

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We introduce a new geometric quantity, *the mean covariation* for Finsler metrics, and establish a volume comparison theorem. As an application, we obtain some precompactness and finiteness theorems for Finsler manifolds.

1. INTRODUCTION

Early work in conjunction with the global comparison geometry of Finsler manifolds was done by L. Auslander [A], who proved that if a complete Finsler n-manifold (M, F) satisfies $\mathrm{Ric}_M \geqslant (n-1)$, then the diameter $\mathrm{diam}_M \leqslant \pi$. In particular, $\pi_1(M)$ is finite. This is the Bonnet–Myers theorem if F is Riemannian. Further, he proved that if a complete Finsler manifold (M, F) has non-positive flag curvature, then the exponential map $\exp_x \colon T_x M \to M$ is a local C^1-diffeomorphism. This is the Cartan–Hadamard theorem if F is Riemannian. We point out that Auslander used the Cartan connection and stated the above theorems under slightly stronger conditions on curvatures, because he did not realize that the curvature terms in the second variation formula can be simplified to the flag curvature. See [AZ2, BCh] for simplified formulas. We remark that the flag curvature is independent of a particular choice of connections, and hence is of particular interest in the metric geometry of Finsler manifolds. See [L, AZ1, D1–D3, GKR, Mo1, Mo2, K1, K2, BCh], etc., for other interesting work on global Finsler geometry.

In Riemann geometry, the Bishop–Gromov volume comparison theorem [BCr, GLP] plays a very important role in the global differential geometry of Riemann manifolds. One of its applications is the Gromov precompactness theorem for the class $\mathfrak{C}(n, \lambda, d)$ with respect to the Gromov–Hausdorff distance d_{GH}, where $\mathfrak{C}(n, \lambda, d)$ denotes the set of all Riemann manifolds (M, g) satisfying $\mathrm{Ric}_M \geqslant (n-1)\lambda$ and $\mathrm{diam}_M \leqslant d$. For $\varepsilon > 0$, $R \geqslant 1$, consider

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a sub-class $\mathfrak{C}_{\varepsilon, R}(n, \lambda, d) \subset \mathfrak{C}(n, \lambda, d)$, which consists of (M, g) satisfying that any ε-ball $B(x, \varepsilon)$ is contractible in $B(x, R\varepsilon) \subset M$. It can be shown that if two manifolds $M_0, M_1 \in \mathfrak{C}_{\varepsilon, R}(n, \lambda, d)$ are sufficiently close with respect to d_{GH}, then they have to be homotopy equivalent. Combining this with the pre-compactness of $\mathfrak{C}(n, \lambda, d)$, one can conclude that there are only finitely many homotopy types in $\mathfrak{C}_{\varepsilon, R}(n, \lambda, d)$ ([P1]) (see also [GPW] for a homeomorphism version of this). This is a generalized version of Cheeger's finiteness theorem [C1, C2] and Grove–Petersen's finiteness theorem [GP]. See also [Y, Z] for some other interesting finiteness theorems. Here we shall not mention the references on the convergence theory of Riemann metrics, because we will only focus on the finiteness problem. The reader is referred to a recent paper of P. Petersen [P3] for a beautiful discussion and references on the convergence of Riemannian metrics.

Petersen [P1, P2] studied the finiteness problem for metric spaces in great generally. This enables us to find geometric (curvature) conditions and establish a finiteness theorem for a much larger class of manifolds equipped with "smooth" inner metrics.

First we shall briefly discuss inner metrics which naturally lead to Finsler metrics. Let d be an inner metric on a C^∞ n-manifold M. Suppose it satisfies the following Lipschitz condition: at each point $x \in M$, there is a coordinate neighborhood $\varphi\colon V \subset \mathbb{R}^n \to U \subset M$ such that

$$A^{-1}\,|x_0 - x_1| \leqslant d(\varphi(x_0), \varphi(x_1)) \leqslant A\,|x_0 - x_1|, \qquad \forall x_0, x_1 \in V; \qquad (1.1a)$$

$$\begin{aligned} &|d(\varphi(x_0), \varphi(x_0 + y) - d(\varphi(x_1), \varphi(x_1 + y)| \\ &\qquad \leqslant A\,|x_0 - x_1|\,|y|, \qquad \forall x_0, x_1 \in V, \quad y \in \mathbb{B}^n(\varepsilon). \end{aligned} \qquad (1.1b)$$

We can prove that (1.1) implies that the following limit exists:

$$F_d(x, y) := \lim_{r \to 0} \frac{d(\varphi(x), \varphi(x + ry))}{|r|} \geqslant 0, \qquad \forall x \in V, \quad y \in \mathbb{R}^n \qquad (1.2)$$

Further, $F(x, y) = F_d(x, y)$ has the following properties:

$$F(x, y) = |r|\,F(x, y), \qquad \forall r \in \mathbb{R}, \quad y \in \mathbb{R}^n; \qquad (1.3)$$

$$F(x, y) = 0 \qquad \text{if and only if} \quad y = 0; \qquad (1.4)$$

$$|F(x_0, y) - F(x_1, y)| \leqslant A\,|x_0 - x_1|\,|y|, \qquad \forall x_0, x_1 \in V, \quad y \in \mathbb{R}^n; \qquad (1.5)$$

$$F(x, y_1 + y_2) \leqslant F(x, y_1) + F(x, y_2), \qquad \forall y_1, y_2 \in \mathbb{R}^n. \qquad (1.6)$$

A function $F\colon TM \to [0, \infty)$ is called a *Lipschitz Finsler metric* if it satisfies (1.3)–(1.6) in local coordinates. One can show that if $d = d_F$ is the inner metric induced by a Lipschitz Finsler metric F, then d satisfies (1.1) and reproduces $F = F_d$ by (1.2) ([BM]; see also [V, DCP] for the study of inner metrics satisfying (1.1a) only). Thus there is an one-to-one correspondence between Lipschitz inner metrics and Lipschitz Finsler metrics.

To study the differential geometry of Finsler manifolds, we shall restrict ourselves to C^∞ Finsler metrics. A function $F: TM \to [0, \infty)$ is called a C^∞ *Finsler metric* if it satisfies (1.3) and the following convexity/regularity conditions:

(1.7) F is smooth on $TM \setminus \{0\}$;

(1.8) the Hessian $(g_{ij}(x, y))$, $y \neq 0$, is positive definite, where

$$g_{ij}(x, y) := \frac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j}(x, y).$$

Note that (1.7), (1.8) imply (1.4), (1.5) and (1.6). The inner metric $d = d_F$ induced by F reproduces $F = F_d$ by (1.2). By the homogeneity of F, one always has $F(x, y) = \sqrt{g_{ij}(x, y)\, y^i y^j}$. F is called *Riemannian* if $g_{ij}(x) = g_{ij}(x, y)$ are independent of y. F is called *locally Minkowskian* if $g_{ij}(x, y) = g_{ij}(y)$ are independent of x.

Let F_x denote the restriction of F onto $T_x M$. When F is Riemannian, $(T_x M, F_x)$ are all isometric to the Euclidean spaces $\mathbb{R}^n$. But, for general Finsler metric F, $(T_x M, F_x)$ may be not isometric to each other. Thus the geometry of Finsler manifolds becomes more complicated.

The Finsler metric F induces a familly of inner products g^v in $T_x M$ by

$$g^v(u, w) = g_{ij}(x, y)\, u^i w^j, \tag{1.9}$$

where $v = y^i (\partial/\partial x^i)|_x$, $u = u^i (\partial/\partial x^i)|_x$, $w = w^i (\partial/\partial x^i)|_x$.

Take an arbitrary basis $\{e_i\}_{i=1}^n$ for $T_x M$. Let $B_x(1) = \{ y = (y^i): F(y^i e_i) \leq 1\}$. Put $g^v_{ij} = g^v(e_i, e_j)$. Define the *mean distortion* $\mu: TM \setminus \{0\} \to (0, \infty)$ by

$$\mu(v) := \frac{\mathrm{vol}(B_x(1))}{\mathrm{vol}(\mathbb{B}^n(1))} \sqrt{\det(g^v_{ij})}. \tag{1.10}$$

Clearly, $\mu(v)$ is independent of a particular choice of $\{e_i\}_{i=1}^n$, and $\mu(\lambda v) = \mu(v)$, $\forall \lambda \neq 0$.

The *mean covariation* $\mathrm{H}: TM \setminus \{0\} \to \mathbb{R}$ is defined by

$$\mathrm{H}(v) := \frac{d}{dt} [\ln \mu(\dot{\gamma}_v)]|_{t=0}, \tag{1.11}$$

where γ_v is the geodesic with $\dot{\gamma}_v(0) = v$. Put $\mathrm{H}(0) = 0$. It is easy to see that

$$\mathrm{H}(\lambda v) = \lambda \mathrm{H}(v), \qquad \forall \lambda \in \mathbb{R}.$$

Roughly speaking, the mean covariation $\mathrm{H}(v)$ measures the *average rate of changes* of (T_xM, F_x) in the direction $v \in T_xM$. We say $|\mathrm{H}_M| \leqslant \mu$ if $|\mathrm{H}(v)| \leqslant \mu F(v)$, $\forall v \in TM \backslash \{0\}$.

An important property is that $\mathrm{H} = 0$ for Finsler manifolds modeled on a single Minkowski space. In particular, $\mathrm{H} = 0$ for Berwald spaces. Locally Minkowski spaces and Riemann spaces are all Berwald spaces.

In order to introduce other geometric quantities, one needs (linear) connections. In Finsler geometry, there are several important linear connections such as the Berwald connection [Be], the Cartan connection [C], and the Chern connection [Ch1, Ch2, BCh]. See [S] for a new interesting linear connection and its relationship with others. All of them are uniquely determined by two equations (torsion equation and metric equation) and reduce to the common one when F is Riemannian. Except for the Cartan connection, all of them are torsion-free. But the Cartan connection is metric-compatible. In a Finsler manifold (M, F), one cannot get a connection which is torsion-free and metric-compatible, unless F is Riemannian. Using any of these connections, one can define three curvatures R, P, Q ($Q = 0$ for torsion free connections). These curvature terms depend on a particular choice of linear connections. The flag curvature tensor (defined by R only) is independent of a particular choice of these connections, that is, the term appears in the second variation of length, thus is of particular interest to us. We remark that if F is Riemannian, then $P = Q = 0$, and the flag curvature tensor is the Riemannian curvature tensor. The average of the flag curvature tensor is the Ricci curvature $\mathrm{Ric}\colon TM \to \mathbb{R}$. It has the property that $\mathrm{Ric}(rv) = r^2\,\mathrm{Ric}(v)$, $r > 0$, $v \in TM$. We say $\mathrm{Ric}_M \geqslant (n-1)\lambda$ if $\mathrm{Ric}(v) \geqslant (n-1)\lambda F(v)^2$, $\forall v \in TM$. More details will be given in Section 2.

For a constant $\lambda \in \mathbb{R}$ and $\mu \geqslant 0$, put

$$V_{\lambda,\mu}(r) := \mathrm{Vol}(\mathbb{S}^{n-1}(1)) \int_0^r e^{\mu t} s_\lambda(t)^{n-1}\, dt, \tag{1.12}$$

where $s_\lambda(t)$ denotes the unique solution to $y'' + \lambda y = 0$ with $y(0) = 0$, $y'(0) = 1$. Note that $V_{\lambda,\mu}(r) = \mathrm{vol}(\mathbb{B}^n(r))(1 + o(r))$ as $r \to 0^+$. We shall show that $\mathrm{vol}(B(x, r)) = \mathrm{vol}(\mathbb{B}^n(r))(1 + o(r))$ as $r \to 0^+$. Thus

$$\lim_{r \to 0^+} \frac{\mathrm{vol}(B(p, r))}{V_{\lambda,\mu}(r)} = 1. \tag{1.13}$$

The following is our main theorem.

Theorem 1.1. *Let (M, F) be complete Finsler manifold. Suppose that*

$$\mathrm{Ric}_M \geqslant (n-1)\lambda, \qquad |H_M| \leqslant \mu. \tag{1.14}$$

Then for any $0<r<R$,

$$\frac{\mathrm{vol}(B(x,R))}{V_{\lambda,\mu}(R)} \leqslant \frac{\mathrm{vol}(B(x,r))}{V_{\lambda,\mu}(r)}. \tag{1.15}$$

In particular,

$$\mathrm{vol}(B(x,r)) \leqslant V_{\lambda,\mu}(r). \tag{1.16}$$

In Section 6, we shall discuss the case when the equality in (1.15) holds.

Given $n, \lambda, \mu \geqslant 1$, let $\mathfrak{M}(n, \lambda, \mu)$ denote the class of pointed complete Finsler n-manifold (M, p, F) satisfying the bounds (1.14). As a direct consequence of Theorem 1.1, we have the following

COROLLARY 1.2. *The class* $\mathfrak{M}(n, \lambda, \mu)$ *is precompact in the pointed Gromov–Hausdorff topology.*

This corollary follows from Theorem 1.1 and [GLP, P2].

A function $\rho: [0, r) \to [0, \infty)$ is called a contractibility function if it satisfies (i) $\rho(0)=0$, (ii) $\rho(\varepsilon)\geqslant \varepsilon$, (iii) $\rho(\varepsilon)\to 0$ as $\varepsilon \to 0$, (iv) ρ is nondecreasing. Given a contractibility function ρ, a metric space X is said to be LGC(ρ) if for every $\varepsilon \in [0, r]$ and $x \in X$, the ball $B(x, \varepsilon)$ is contractible inside $B(x, \rho(\varepsilon))$. For a number $r>0$, if every ball $B(x, \varepsilon)$ is contractible inside $B(x, \varepsilon)$, $x\in X$, $0<\varepsilon<r$, namely, X is LGC(ρ) for $\rho(\varepsilon)=\varepsilon: [0, r)\to [0, r)$, then we say X has contractibility radius $c(X)\geqslant r$. Given n, λ, μ, ρ, d, let $\mathfrak{M}(n, \lambda, \mu, d, \rho)$ denote the class of compact LGC(ρ) Finsler n-manifolds satisfying

$$\mathrm{Ric}_M \geqslant (n-1)\lambda, \qquad |\mathrm{H}_M| \leqslant \mu, \qquad \mathrm{diam}_M \leqslant d. \tag{1.17}$$

COROLLARY 1.3. *The class* $\mathfrak{M}(n, \lambda, \mu, d, \rho)$ *contains only finitely many homotopy types.*

In [C1, C2], J. Cheeger proved that if a compact Riemannian manifold M^n satisfies the bound $\Lambda \geqslant K_M \geqslant \lambda$, $\mathrm{diam}_M \leqslant d$, $\mathrm{vol}_M \geqslant v$, then the injectivity radius $\mathrm{inj}_M \geqslant i_o(n, \lambda, \Lambda, d, v)>0$. Thus M is LGC(ρ) space for $\rho(\varepsilon)=\varepsilon$: $[0, i_o]\to[0,\infty)$. In [GP], Grove and Petersen proved that if a compact Riemannian manifold M^n satisfies the bounds $K_M\geqslant \lambda$, $\mathrm{diam}_M \leqslant d$, $\mathrm{vol}_M \geqslant v$, then every ball $B(x,\varepsilon)$ is contractible in $B(x, R\varepsilon)$ for $\forall \varepsilon \leqslant \varepsilon_o(n, \lambda, d, v)$ and $R=R(n, \lambda, d, v)$. Thus M is a LGC(ρ) space for $\rho(\varepsilon)=R\varepsilon: [0, \varepsilon_o]\to[0,\infty)$. It is an interesting question under what curvature bounds is a compact Finsler manifold M^n satisfying the bounds $\mathrm{diam}_M \leqslant d$, $\mathrm{vol}_M\geqslant v$ a LGC(ρ) space for some ρ depending only on those bounds. This problem will be discussed somewhere else. We remark that both arguments in [C1] and [GP] were carried out using Toponogov's comparison theorem. However, we can show that, in a Finsler manifold, Topogonov's triangle comparison theorem does not hold, unless it is Riemannian. In other words, a Finsler manifold is not curved from below in the sense of Alexandrov [A1, A2] unless it is Riemannian.

By the same argument as in [GLP], one can obtain the following

COROLLARY 1.4. *Let (M, F) be a complete Finsler n-manifold satisfying the bounds* (1.17). *Then the first Betti number $b_1(M) \leqslant c(n, \lambda, \mu, d)$.*

There are many other important theorems in Riemannian geometry are false for Finsler manifolds. For example, the Cheeger–Gromoll splitting theorem [CG] is no longer true, even for flat Finsler manifolds (say Minkowski spaces). Nevertheless, Milnor's theorem still holds for certain Finsler manifolds.

COROLLARY 1.5. *Let (M, F) be a complete Finsler n-manifold with*

$$\mathrm{Ric}_M \geqslant 0, \qquad H_M = 0.$$

Then any finitely generated subgroup Γ of the fundamental group $\pi_1(M)$ has polynomial growth of order $\leqslant n$.

2. PRELIMINARIES

In this section we shall recall some basic facts of Riemann–Finsler geometry. See [M, R, BCh] for more details.

We begin with the simplest Finsler manifolds. Let V be a vector space, and let $F_o\colon V \to [0, \infty)$ be a function satisfying (i) $F_o(\lambda y) = |\lambda|\, F(y)$, (ii) F_o is C^∞ on $V \backslash \{0\}$, and (iii) $g_{ij}(y) := \frac{1}{2}(\partial F_o^2/\partial y^i\, \partial y^j)(y)$ is positive definite for $y \neq 0$. F_o is called a *Minkowski norm* on V, and (V, F_o) is called a *Minkowski space*. Each $T_x V$ is naurally identified with V. Thus F_o induces a Finsler metric F on V.

There are lots of Minkowski spaces, but there is a unique Euclidean space, up to an isometry. Let (M, F) be an arbitrary Finsler manifold. By definition, the restriction F_x of F to $T_x M$ is a Minkowski norm in $T_x M$. $(T_x M, F_x)$ is called a *Minkowski tangent space* at $x \in M$. In general, $(T_x M, F_x)$ are not linearly isometric to each other. It is possible that on (M, F), we have infinitely many distinct Minkowski tangent spaces.

There is no notion of angles between two tangent vectors in a Finsler manifold. Nevertheless, the volume form is well-defined [B1]. Let $\{e_i\}_{i=1}^n$ be a local basis for TM and $\{\eta^i\}_{i=1}^n$ be its dual basis for T^*M. Put

$B_x(1) := \{y = (y^i) \colon F(y^i e_i) \leqslant 1\}$. $B_x(1)$ is a strictly convex open subset in $\mathbb{R}^n$. Let $\mathbb{B}^n(1)$ denote the standard unit ball in $\mathbb{R}^n$. The volume form dv is defined by

$$dv = \frac{\mathrm{vol}(\mathbb{B}^n(1))}{\mathrm{vol}(B_x(1))} \eta^1 \wedge \cdots \wedge \eta^n, \tag{2.1}$$

where $\mathrm{vol}(A)$ denotes the Euclidean volume of a subset $A \subset \mathbb{R}^n$. dv is independent of a particular choice of *positive* basis $\{e_i\}_{i=1}^n$! As usual, the volume $\mathrm{vol}(U)$ of an open subset $U \subset M$ is defined by $\mathrm{vol}(B(x, r)) = \int_U dv$. It is easy to see that the r-ball $B(x, r)$ in a Minkowski space has the same volume of $\mathbb{B}^n(r)$ in the Euclidean space $\mathbb{R}^n$. Busemann [B1] proved that for any bounded open subset $U \subset M$, $\mathrm{vol}(U) = H_d(U)$, where $H_d(U)$ denotes the Hausdorff measure of U with respect to $d = d_F$. This fact might be true for Lipschitz inner/Finsler metrics (compare [V]).

In order to define curvatures, it is more convenient to consider the pull-back tangent bundle than the tangent bundle, because our geometric quantities depend on directions.

Let $TM_o = TM \backslash \{0\}$ and let $\pi^* TM$ denote the pull-back of the tangent bundle TM by $\pi \colon TM_o \to M$. Denote vectors in $\pi^* TM$ by $(v; w)$, $v \in TM_o$, $w \in T_{\pi(v)} M$. For the sake of simplicity, we denote by $\partial_i|_v = (v; \partial/\partial x^i|_x)$, $v \in T_x M$ the natural local basis for $\pi^* TM$. The Finsler metric F defines two tensors g and A in $\pi^* TM$ by

$$g(\partial_{i|v}, \partial_{j|v}) = g_{ij}(x, y), \qquad A(\partial_{i|v}, \partial_{j|v}, \partial_{k|v}) = \frac{1}{2} F(x, y) \frac{\partial g_{ij}}{\partial y^k}(x, y),$$

where $v = y^i (\partial/\partial x^i)|_x$. g and A are called the *fundamental* and *Cartan* tensors, respectively. Note that $(\pi^* TM, g)$ is a Riemannian vector bundle. A trivial fact is that F is Riemannian if and only if $A = 0$.

In Finsler geometry, we study connections and curvatures in $(\pi^* TM, g)$, rather than in (TM, F). The pull-back tangent bundle $\pi^* TM$ is a very special vector bundle. It has a unit vector l defined by

$$l_v = \frac{1}{F(v)} (v; v).$$

It is easy to see that $A(X_1, X_2, X_3) = 0$ whenever $X_i = l$ for some $i = 1, 2, 3$.

Let $\{E_i\}_{i=1}^n$ be a local frame for $\pi^* TM$. Define the dual co-frame $\{\omega^i\}_{i=1}^n$ on TM_o by

$$(v; \pi_*(\hat{X})) = \omega^i(\hat{X}) E_i, \qquad \hat{X} \in T(TM_o).$$

Put $l = l^i E_i$, $A_{ijk} = A(E_i, E_j, E_k)$, $g_{ij} = g(E_i, E_j)$.

We have the following

THEOREM 2.1 (Chern). *There is a unique set of local* 1*-forms* $\{\omega_j^{\ i}\}_{1 \leqslant i, j \leqslant n}$ *on* TM_o *such that*

$$d\omega^i = \omega^j \wedge \omega_j^{\ i} \tag{2.2}$$

$$dg_{ij} = g_{kj}\omega_i^{\ k} + g_{ik}\omega_j^{\ k} + 2A_{ijk}\omega^{n+k}, \tag{2.3}$$

where $\omega^{n+k} := dl^k + l^l\omega_l^{\ k} + l^k d(\ln F)$. *Further,* $\{\omega^i; \omega^{n+i}\}$ *is a local coframe for* $T^*(TM_o)$.

In a standard local coordinate system $(x^i; y^i)$ in TM, take a natural basis $\{E_i = \partial_i\}_{i=1}^n$ for π^*TM. We have

$$\omega_j^{\ i} = \Gamma^i_{jk}(x, y)\, dx^k$$

where

$$\Gamma^i_{jk} = \gamma^i_{jk} + \frac{1}{F} g^{il}\{A_{jks}N^s_l - A_{ljs}N^s_k - A_{kls}N^s_j\}$$

$$N^s_l = \gamma^s_{la}y^a - \frac{1}{F} g^{sj}A_{jlk}\gamma^k_{ab}y^a y^b \tag{2.4}$$

$$\gamma^i_{jk} = \frac{1}{2}\, g^{il}\left\{\frac{\partial}{\partial x^j} g_{lk} + \frac{\partial}{\partial x^k} g_{lj} - \frac{\partial}{\partial x^l} g_{jk}\right\}.$$

We call $\{\omega_j^{\ i}\}$ the set of *local connection forms*. It defines a linear connection ∇ in π^*TM by

$$\nabla_{\hat{X}} Y = \{\hat{X}Y^i + Y^j\omega_j^{\ i}(\hat{X})\}\, E_i,$$
$$\hat{X} \in T(TM_o), \quad Y = Y^iE_i \in C^\infty(\pi^*TM). \tag{2.5}$$

We also get two bundle maps ρ, μ: $T(TM_o) \to \pi^*TM$, defined by

$$\rho := \omega^i \otimes E_i, \qquad \mu := F\omega^{n+i} \otimes E_i \tag{2.6}$$

Note that the $VTM := \ker \rho$ is the *vertical* tangent bundle of TM_o. Put $HTM := \ker \mu$. We have the direct composition $T(TM_o) = HTM \oplus VTM$. Tangent vectors in HTM are called *horizontal*, and tangent vectors in VTM are called *vertical*. An important fact is that $\rho|_{HTM}$ and $\mu|_{VTM}$ are bundle isomorphisms.

Define the set of local curvature forms $\Omega_j^{\ i}$ by

$$\Omega_j^{\ i} := d\omega_j^{\ i} - \omega_j^{\ k} \wedge \omega_k^{\ i}.$$

By (2.2), one can show that $\Omega_j^{\ i}$ does not have vertical part, that is, one can write

$$\Omega_j^{\ i} = \tfrac{1}{2} R_{j\ kl}^{\ i} \omega^k \wedge \omega^l + P_{j\ kl}^{\ i} \omega^k \wedge \omega^{n+l}.$$

Define the curvature tensors R, P in π^*TM by

$$R(U, V)W = u^k v^l w^j R_{j\ kl}^{\ i} E_i, \qquad P(U, V)W = u^k v^l w^j P_{j\ kl}^{\ i} E_i,$$

where $U = u^i E_i$, $V = v^i E_i$, $W = w^i E_i \in \pi^*TM$. A Finsler manifold (M, F) is called a *Berwald space* if $P = 0$.

Let $\sigma = \mathrm{span}\{u, v\} \subset T_xM$ be a two-dimensional section. The flag curvature $K(\sigma; v)$ of the *flag* $\{\sigma, v\}$ is defined by

$$K(\sigma; v) := \frac{g(R(U, V)V, U)}{g(V, V)\, g(U, U) - g(U, V)^2},$$

where $U = (v; u)$, $V := (v; v) \in \pi^*TM$. When F is a Riemannian, $K(\sigma) = K(\sigma; v)$ is independent of $v \in \sigma$, that is, the *sectional curvature* in Riemannian geometry. Further, it is independent of the above-mentioned linear connections. Thus it really does not matter which connection should be used in the metric geometry of Finsler manifolds.

Fixing a unit vector $v \in T_xM$, let $\{e_i\}_{i=1}^n$, $e_n = v$, be a basis for T_xM such that $\{(v; e_i)\}_{i=1}^n$ is an orthonormal basis for π^*TM. Let $\sigma_i = \mathrm{span}\{e_i, v\}$, $1 \leqslant i \leqslant n-1$. The Ricci curvature $\mathrm{Ric}(v)$ is defined by

$$\mathrm{Ric}(v) := \sum_{i=1}^{n-1} K(\sigma_i; v) F(v)^2.$$

The linear connection ∇ in (2.5) defines the covariant derivative $D_v u$ of a vector field u on M in the direction $v \in T_xM$ as follows. Let c be a curve in M with $\dot{c}(0) = v$. Let $\hat{c} = dc/dt$ be the canonical lift of c in TM_o. Let $u(t) = u|_{c(t)}$ and $U(t) := (\hat{c}; u(t)) \in \pi^*TM$. Define $D_v u$ by

$$(v; D_v u) := \nabla_{d\hat{c}/dt} U(0). \tag{2.7}$$

Note that D satisfies all properties of linear connections in TM, except for the linearity in v, that is, $D_{v_1+v_2} u \neq D_{v_1} u + D_{v_2} u$. Thus D is not a linear connection in TM in a usual sense. A vector field $u = u(t)$ along c is called *parallel* if $D_{dc/dt} u = 0$.

A curve γ: $[0, a] \to M$ is a geodesic if and only if $\dot{\gamma}$ is parallel along γ, i.e.,

$$D_{\dot{\gamma}} \dot{\gamma} = 0. \tag{2.8}$$

In this case, γ must be parametrized proportional to arc-length.

The exponential map $\exp_x\colon T_xM \to M$ is defined as usual, that is, $\exp_x(v) = \gamma_v(1)$, where γ_v is a geodesic with $\gamma_v(0) = x$ and $\dot\gamma(0) = v$. The Hopf–Rinow theorem says that if (M, d_F) is complete, then $\exp_x$ is defined on all of T_xM for all $x \in M$. This implies that any two points $x_0, x_1 \in M$ can be joined by a minimizing geodesic. From now on, we always assume that (M, d_F) is complete.

It is a well-known fact that the exponential map $\exp_x$ is C^∞ away from the origin in T_xM and only C^1 at the origin with $\exp_x|_0 = \text{identity}$ [W]. In [B2], Busemann proved that if $\exp_x$ is C^2 at the origin for all $x \in M$, then all (T_xM, F_x) are isometric to each other. On the other hand, Ichijyō [I] proved that in a Berwald space, all (T_xM, F_x) are isometric to each other. Finally, Akbar–Zadeh [AZ3] proved that $\exp_x$ is C^∞ all over T_xM for all $x \in M$, if and only if the Finsler metric is Berwald.

Using the exponential map $\exp_x$, one can easily show that $d_x^2 = d(x, \cdot)^2$ is C^∞ near x and C^1 at x. We remind the reader that $(F_x)^2$ is only C^1 at the origin, although $\exp_x$ is C^∞ in a Berwald space.

Proposition 2.2. *If d_x^2 is C^2 at x, then F is Riemannian at x.*

Proof. Let $\varphi\colon V \subset \mathbb{R}^n \to U \subset M$ be a local coordinate system at x with $\varphi(0) = x$. Let $h(z) = d_x^2 \circ \varphi(z)$, $z \in V$. Note that $h(z) \geqslant h(0) = 0$, $z \in V$. By assumption h is C^2 in V. Thus

$$h(z) = \frac{1}{2} \frac{\partial^2 h}{\partial z^i\, \partial z^j}(0)\, z^i z^j + o(|z|).$$

As we have pointed in Section 1, $d = d_F$ reproduces $F = F_d$ by (1.2), that is,

$$F(x, y)^2 = \lim_{r \to 0} \frac{h(ry)^2}{r^2} = \frac{1}{2} \frac{\partial^2 h}{\partial z^i\, \partial z^j}(0)\, y^i y^j.$$

By definition, F is Riemannian at x. ∎

The cut-value t_v of a vector $v \in T_xM$ is defined to be the largest number $r > 0$ such that γ_v is minimizing on $[0, r]$. Let $I_x := \{v \in T_xM,\ F(v) = 1\}$. The map $v \in I_x \to t_v \in [0, \infty)$ is continuous. The cut-locus $C_x = \{\exp_p t_v v\colon v \in I_x\}$ has zero Hausdorff measure in M. The injectivity radius inj_x at x is defined by $\mathrm{inj}_x = \inf_{v \in I_x} t_v$. Let $\Omega_x := M \backslash C_x$, and $O_x = \{(t, x)\colon t < t_v\}$. Then $\exp_x\colon O_x \to \Omega_x$ is a diffeomorphism (C^1 at the origin).

For $v \in T_xM$, define $R^v\colon T_xM \to T_xM$ by

$$R^v(u) = R(U, V)\, V, \tag{2.9}$$

where $U = (v : u)$, $V = (v; V) \in \pi^*TM$.

Fix $v \in T_xM$, and let γ_v be a normal geodesic from x with $\dot{\gamma}_v(0) = v$. Along γ_v, we have a family of inner products $g^t = g^{\dot{\gamma}_v(t)}$ in $T_{\gamma_v(t)}M$ (see (1.9)) and flag curvatures $R^t = R^{\dot{\gamma}_v(t)}: T_{\gamma_v(t)}M \to T_{\gamma_u(t)}M$. For the sake of simplicity, we shall denote $D_t = D_{\dot{\gamma}_v}$, if no confusion is caused. (2.3) implies that for any vector fields $u = u(t)$, $w = w(t)$ along γ_v,

$$\frac{d}{dt}(g^t(u(t), w(t)) = g^t(D_t u(t), w(t)) + g^t(u(t), D_t v(t)). \tag{2.10}$$

A vector field $J = J(t)$ along γ_v is called a Jacoby field if it satisfies

$$D_t D_t J(t) + R^t J(t) = 0. \tag{2.11}$$

Lemma 2.3. *A vector field J_u along γ_v with $J_u(0) = 0$ and $D_t J_u(0) = u$ is a Jacobi field if and only if*

$$J_u(t) = d(\exp_x)_{|tv}(tu). \tag{2.12}$$

In particular, a vector field J given by (2.12) *is smooth along γ_v.*

We shall always denote by $\dot{\gamma}_v$ the geodesic with $\dot{\gamma}_v(0) = v \in T_xM$ and by J_u the Jacobi field along $\dot{\gamma}_v$ defined by (2.12).

Lemma 2.4 (Gauss Lemma). *For every Jacobi field J_u, $J_u(t)$ is perpendicular to $\gamma_v(t)$ for all t with respect to g^t, if and only if u is perpendicular to v with respect to g^v.*

From (2.12), we can see that $\exp_x$ is singular at $rv \in T_xM$ if and only if there is $0 \neq u \in T_xM$, such that the Jacobi field J_u satisfies $J_u(r) = 0$. In this case we call $\gamma_v(r)$ a *conjugate point* with respect to x. By the standard argument, one can show that if the flag curvature $K(\sigma; v) \leqslant \lambda$, then there are no conjugate points on $\gamma|_{[0,r)}$, for $r = \pi/\sqrt{\lambda}$ ($= \infty$ if $\lambda \leqslant 0$). This is the so-called Cartan–Hadamard theorem in Finsler geometry [A].

The index lemma is still true. For a vector field $W = W(t)$ along γ_v with $W(0) = 0$, define

$$I(W, W) = \int_0^r \{g^t(D_t W(t), D_t W(t)) - g^t(R^t W(t), W(t))\}\, dt.$$

Lemma 2.5 (Index Lemma). *Suppose that γ_v does not contain conjugate points on* $[0, r]$. *Let J be a Jacobi field along γ_v with $J(0) = 0$. Then for any vector field W along γ with $W(0) = 0$ and $W(r) = J(r)$,*

$$I(J, J) \leqslant I(W, W),$$

the equality holds if and only if $W = J$.

For a Jacobi field J_u, we have

$$I(J_u, J_u) = g^r(D_t J_u(r), J_u(r)).$$

By a standard argument and the index lemma, one can easily prove Bonnet and Myers' theorem for Finsler manifolds ([A]).

A Finsler space (M, F) is said to be modeled on a *single* Minkowski space if for every geodesic γ, the parallel translation $P_{t_0, t_1}: (T_{\gamma(t_0)}M, F_{\gamma(t_0)}) \to (T_{\gamma(t_1)}M, F_{\gamma(t_1)})$ is an isometry for all t_0, t_1. In this case, all (T_xM, F_x) are linearly isometric to each other. The class of such manifolds contains all Riemann manifolds and locally minkowski manifolds. Define $F_\lambda: T\mathbb{R}^n \to [0, \infty)$ by

$$F_\lambda\left(y^i \frac{\partial}{\partial x^i}\bigg|_x\right) = \sqrt{\sum_{i=1}^n (y^i)^2 + \lambda \sqrt{\sum_{i=1}^n (y^i)^4}}, \tag{2.13}$$

where $\lambda: \mathbb{R}^n \to [0, \infty)$ is a C^∞ function. Clearly, $(\mathbb{R}^n, F_\lambda)$ in (2.13) is not modeled on a single Minkowski space, if $\lambda: \mathbb{R}^n \to [0, \infty)$ is not constant. See [AIM] for more interesting Finsler metrics from physics.

PROPOSITION 2.6. *If (M, F) is modeled on a single Minkowski space, then $H = 0$.*

Proof. By definition, if $u = u(t)$ is a parallel vector field along a geodesic γ_v, then $F(u(t)) = \text{constant}$. Let $\{e_i(t)\}_{i=1}^n$ be a parallel basis for $T_{\gamma_v(t)}M$ with $e_i(0) = e_i$, $1 \leqslant i \leqslant n$. We have

$$B_{\gamma_v(t)}(1) := \{(y^i): F(y^i e_i(t)) \leqslant 1\} = B_x(1).$$

We also have $\det(g^t(e_i(t), e_j(t))) = \det(g^v(e_i, e_j))$. Thus $\mu(\dot\gamma_v(t)) = \text{constant}$. This implies $\mathrm{H}(v) = 0$. ∎

PROPOSITION 2.7 ([I]). *Any Berwald space is modeled on a single Minkowski space.*

3. THE SINGULAR RIEMANN METRICS $\hat{g}^x$ AND g^x

The Finsler metric F induces a singular Riemann metric $\hat{g}^x$ on $T_xM \backslash \{0\}$ by

$$\hat{g}^x(u, w) := g^v(u, w), \qquad \forall u, w \in T_{tv}(T_xM) \sim T_xM, \quad v \in T_xM. \tag{3.1}$$

Let $I_x = \{v \in T_xM: F(v) = 1\}$. Let $\dot{g}^x$ be the induced Riemann metric of $\hat{g}^x$ on $I_x \subset T_xM$. Regard T_xM as the cone $C(I_x)$ over I_x. By the homogeneity of F, we have

$$\hat{g}^x = dt^2 \oplus t^2 \dot{g}^x. \tag{3.2}$$

Let $\hat{\eta}^x$ denote the volume form of $\hat{g}^x$ on $T_xM \backslash \{0\} = C(I_x) \backslash \{o\}$. Let $\{e_i\}_{i=1}^n$ be a basis for $T_0(T_xM) \sim T_xM$. Extend it to a global basis for $T(T_xM)$. Let $\{\theta^i\}_{i=1}^n$ be the dual basis for $T^*(T_xM)$. We have

$$\begin{aligned} \hat{\eta}^x_{|tv} &= \sqrt{\det(\hat{g}^x(e_i, e_j))_{|tv}}\, \theta^1 \wedge \cdots \wedge \theta^n \\ &= \sqrt{\det(g^v(e_i, e_j))}\, \theta^1 \wedge \cdots \wedge \theta^n. \end{aligned} \tag{3.3}$$

Let $\dot{\eta}^x$ denote the volume form of $\dot{g}^x$ on I_x. By (3.2) we have

$$\hat{\eta}^x_{|tv} = t^{n-1} \dot{\eta}^x|_v \wedge dt, \qquad \forall (t, v) \in C(I_x). \tag{3.4}$$

Define the density Θ_x at $x \in M$ by

$$\Theta_x = \frac{\mathrm{vol}(I_x, \dot{\eta}^x)}{\mathrm{vol}(\mathbb{S}^{n-1}(1))}. \tag{3.5}$$

When F is Riemannian, $\Theta_x = 1$, $\forall x \in M$. In general, Θ_x does not have to be constant unless F is a weak Landsberg space (see [BS]). It is still an open problem whether or not there are two universal constants $0 < a_n < b_n$, $n = \dim M$ such that

$$a_n \leqslant \Theta_x \leqslant b_n, \qquad \forall x \in M. \tag{3.6}$$

If the norm of the Cartan tensor A is small enough, say $\|A\| \leqslant \sqrt{3/10}$, then (3.6) holds for some a_n, b_n.

Define a smooth Riemannian metric g^x inside $\Omega_x \backslash \{x\}$ by

$$g^x|_{\gamma_v(t)} = g^{\dot{\gamma}_v(t)} \tag{3.7}$$

(see (1.9)). Put

$$\tilde{g}^x := (\exp_x)^* g^x. \tag{3.8}$$

In general, g^x is singular at the origin, unless F is Riemannian. Regard $\exp_x$ as a map $C(I_x) \to M$. By the Gauss Lemma, we have

$$\tilde{g}^x = dt^2 \oplus h_t, \tag{3.9}$$

where h_t is a family of Riemannian metrics on I_x.

LEMMA 3.1. *The metric* $\tilde{g}^x$ *satisfies*

$$\frac{1}{t^2} h_t \to \dot{g}^x, \qquad \frac{1}{2t}\frac{\partial h_t}{\partial t} \to \dot{g}^x. \tag{3.10}$$

Proof. Let $u, w \in T_v(I_x M) \subset T_{tv}(T_x M) \sim T_x M$. Let $g^t = g^{\dot{\gamma}(t)}$. Clearly,

$$g^t(u(t), w(t)) \to \dot{g}^x(u, w), \tag{3.11}$$

where $u(t)$, $w(t)$ are parallel vector fields along γ with $u(0) = u$, $w(0) = w$. For $u \in T_x M$, let $J_u(t) = d(\exp_x)_{|tv}(tu)$. We have

$$h_t(u, w) = g^x(d(\exp_x)_{|tv}(tu), d(\exp_x)_{|tv}(tw)) = g^t(J_u(t), J_w(t)). \tag{3.12}$$

Thus by (2.10)

$$\frac{\partial h_t}{\partial t}(u, w) = g^t(D_t J_u(t), J_w(t)) + g^t(J_u(t), D_t J_w(t)). \tag{3.13}$$

Put $\tilde{J}_u = (1/t)\, J_u(t)$, for $t \neq 0$. Since J is smooth along γ and $J(0) = 0$, then $\tilde{J}_u := (1/t) J_u(t)$ converges, as $t \to 0$. Put $\tilde{J}(0) = \lim_{t\to 0} \tilde{J}_u(t)$. Then $\tilde{J}_u$ is smooth. Thus $D_t \tilde{J}_u$ is bounded. Observe that $D_t J_u(t) = \tilde{J}_u(t) + t D_t \tilde{J}_u(t)$. We can conclude that

$$\frac{1}{t} J_u(t) \to D_t J_u(0) = u. \tag{3.14}$$

Now (3.10) follows from (3.11)–(3.13). ▮

The following proposition is important in computing the flag curvature.

PROPOSITION 3.2. (i) *Let* D^x *be the Levi–Civita connection of* g^x *in* Ω_x. *The along any normal geodesic* γ_v, $v \in I_x$,

$$D^x_t = D_t. \tag{3.15}$$

(ii) *Let* $u \in T_v(I_x)$, *and let* $\sigma_t = \operatorname{span}\{d(\exp_x)_{|tv}(u), \dot{\gamma}_v(t)\} \subset T_{\dot{\gamma}_v(t)} M$. *Let* $K^x(\sigma_t)$ *denote the sectional curvature of* g^x. *Then the flag curvature* $K(\sigma_t, \dot{\gamma}_v(t))$ *of* F *satisfies*

$$K(\sigma_t; \dot{\gamma}_v(t)) = K^x(\sigma_t) = \frac{-\frac{1}{2}\frac{\partial^2}{\partial t^2}(h_t)_{uu} + \frac{1}{4}\frac{\partial}{\partial t}(h_t)_{u\alpha}(h_t)^{\alpha\beta}\frac{\partial}{\partial t}(h_t)_{\mu\beta}}{(h_t)_{uu}}, \tag{3.16}$$

where $\{e_\alpha\}_{\alpha=1}^{n-1}$ *is a basis for* $T_v(I_x)$, $(h_t)_{uu} = h_t(u, u)$, $(h_t)_{u\alpha} = h_t(u, e_\alpha)$, *and* $(h_t)_{\alpha\beta} = h_t(e_\alpha, e_\beta)$.

Proof. Let $(t, \bar{x})$ be the conical (or polar) coordinate system in $\Omega_x = \exp_x(O_x)$, regarding T_xM as $C(I_x)$. Let $(t, \bar{x}; s, \bar{y})$ be the standard coordinate system in $T\Omega_x$. By the Gauss lemma or (3.9), we have

$$\begin{pmatrix} 1 & 0 \\ 0 & (g^x)_{\alpha\beta}(t, \bar{x}) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & g_{\alpha\beta}(t, \bar{x}; 1, 0) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & (h_t)_{\alpha\beta}(t, \bar{x}) \end{pmatrix}.$$

Put $D^x_{\partial/\partial t}(\partial/\partial x^\alpha) = \gamma^\beta_{t\alpha}(t, \bar{x})(\partial/\partial x^\beta)$. We have

$$\gamma^\beta_{t\alpha}(t, \bar{x}) = \frac{1}{2}(h_t)^{\beta\nu}(t, \bar{x}) \frac{\partial}{\partial t}(h_t)_{\alpha\nu\beta}(t, \bar{x}).$$

We can put $D_{(\partial/\partial t)}(\partial/\partial x^\alpha) = \Gamma^\beta_{t\alpha}(t, \bar{x}; 1, 0)(\partial/\partial x^\beta)$. By (2.4),

$$\Gamma^\beta_{t\alpha}(t, \bar{x}; 1, 0) = \gamma^\beta_{t\alpha}(t, \bar{x}) - (h_t)^{\beta\nu}(t, \bar{x}) A_{\alpha\nu\tau}(t, \bar{x}; 1, 0)\, \gamma^\tau_{tt}(t, \bar{x}).$$

An easy computation yields that $\gamma^\tau_{tt}(t, \bar{x}) = 0$. Thus

$$\Gamma^\beta_{t\alpha}(t, \bar{x}; 1, 0) = \gamma^\beta_{t\alpha}(t, \bar{x}).$$

This implies (3.15).

Let R^x denote the Riemann curvature tensor of g^x on Ω_x. Note that $J_\alpha(t) := d\exp_x|_{(t, \bar{x})}(t(\partial/\partial x^\alpha) = t(\partial/\partial x^\infty)|_{(t, \bar{x})}$ is a Jacobi field of both F and g^x. By the Jacobi equation (2.11), one obtains

$$(R^x)^t \frac{\partial}{\partial x^\alpha} = -\frac{1}{t} D^x_{\partial/\partial t} D^x_{\partial/\partial t} J_\alpha(t) = -\frac{1}{t} D_{\partial/\partial t} D_{\partial/\partial t} J_\alpha(t) = R^t \frac{\partial}{\partial x^\alpha}.$$

By an easy computation, one obtain that for $\sigma_t = \operatorname{span}\{\partial/\partial t, \partial/\partial x^\alpha\}$,

$$K^x(\sigma_t) = \frac{-\dfrac{\partial}{\partial t}[g^t(D_{\partial/\partial t} J_\alpha(t), J_\alpha(t))] - g^t(D_{\partial/\partial t} J_\alpha(t), D_{\partial/\partial t} J_\alpha(t))}{(g^t)(J_\alpha(t), J_\alpha(t))}$$

$$= \frac{-\dfrac{1}{2}\dfrac{\partial^2}{\partial t^2}(h_t)_{\alpha\alpha} + \dfrac{1}{4}\dfrac{\partial}{\partial t}(h_t)_{\alpha\beta}(h_t)^{\beta\nu}\dfrac{\partial}{\partial t}(h_t)_{\nu\alpha}}{(h_t)_{\alpha\alpha}}. \quad \blacksquare$$

See [LS] for further discussions on Riemann metrics with conical singularities.

4. THE VOLUME FORMS OF $\hat{g}^x$ and g^x

Let $v \in I_x$. Let η^x denote the volume form of g^x and put

$$\tilde{\eta}^x = (\exp_x)^* \eta^x. \tag{4.1}$$

Note that $\tilde{\eta}^x$ is the volume form of $\tilde{g}^x$. Define $\Theta(v, t)$, $t < t_v$, by

$$\tilde{\eta}^x_{|tv} = \Theta(v, t)\, t^{-(n-1)}\, \hat{\eta}^x_{|tv}.$$

Let $\gamma_v(t) = \exp_x(tv)$, $t \geqslant 0$, and $\{e_\alpha\}_{\alpha=1}^{n-1}$ be an orthonormal basis for $v^\perp \subset T_x M$ with respect to g^v. Extend $\{e_\alpha\}_{\alpha=1}^{n-1}$ to a global set for $T(T_x M)$. Let $J_\alpha(t) = d(\exp_x)_{|tv}(te_\alpha)$. We have

$$\Theta(v, t) := \sqrt{\frac{\det(h_t(e_\alpha, e_\beta))}{\det(\dot{g}^x(e_\alpha, e_\beta))}} = \sqrt{\frac{\det(g^t[J_\alpha(t), J_\beta(t)])}{\det(g^v(e_\alpha, \beta))}}, \qquad t < t_v. \tag{4.2}$$

Clearly, $\Theta(v, t)$ is independent of a particular choice of $\{e_\alpha\}_{\alpha=1}^{n-1}$. Lemma 3.1 implies

$$\lim_{t \to 0^+} \frac{\Theta(v, t)}{s_\lambda(t)^{n-1}} = 1. \tag{4.3}$$

Lemma 4.1. *For $t < t_v$, the function $t \to \Theta(v, t)/s_\lambda(t)^{n-1}$ is monotone decreasing. In particular,*

$$\Theta(v, t) \leqslant s_\lambda(t)^{n-1}. \tag{4.4}$$

Proof. For $r < t_v$, one can choose $\{e_\alpha\}_{\alpha=1}^{n-1}$ such that $g^r(J_\alpha(r), J_\beta(r)) = 0$, $\alpha \neq \beta$. Note that $J_v(t) = t\dot{\gamma}_v(t)$ and is orthogonal to $J_\alpha(t)$ for all $t > 0$, $\alpha = 1, ..., n-1$. Let $e_\alpha(t)$ be the parallel vector fields along γ such that $e_\alpha(0) = e_\alpha$ and $e_\alpha(r) = J_\alpha(r)/g^r(J_\alpha(r), J_\alpha(r))^{1/2}$.

Let $W_\alpha(t) = (s_\lambda(t)/s_\lambda(r))\, e_\alpha(t)$. By the index lemma and (2.10),

$$\begin{aligned}
\frac{\Theta'(v, r)}{\Theta(v, r)} &= \sum_{\alpha=1}^{n-1} \frac{g^r(D_t J_\alpha(r), J_\alpha(r))}{g^r(J_\alpha(r), J_\alpha(r))} \leqslant \sum_{\alpha=1}^{n-1} I(W_\alpha, W_\alpha) \\
&= \frac{1}{s_\lambda(r)^2} \int_o^r \{(n-1) s'_\lambda(t)^2 - \mathrm{Ric}(\dot{\gamma}(t))\, s_\lambda(t)^2\}\, dt \\
&\leqslant \frac{1}{s_\lambda(r)^2} \int_0^r \{(n-1) s'_\lambda(t)^2 - (n-1)\lambda s_\lambda(t)^2\}\, dt \\
&= (n-1) \frac{s'_\lambda(r)}{s_\lambda(r)}.
\end{aligned}$$

This implies

$$\frac{d}{dr}\ln\frac{\Theta(v,r)}{s_\lambda(r)^{n-1}}\leqslant 0.$$

Therefore $\Theta(v,r)/s_\lambda(r)^{n-1}$ is monotone decreasing in $r\in(0,t_v)$. This together with (4.3) implies (4.4). ∎

Put $\Theta(v,t)=0$ for $t\geqslant t_v$. The pointed volume of a metric ball $B(x,r)$ is defined by

$$V_x(r):=\mathrm{vol}(B(x,r),g^x).$$

Let $V_\lambda(r)=V_{\lambda,0}(r)$. It is easy to see that

$$V_x(r)=\int_0^r\left(\int_{I_x}\Theta(v,t)\dot{\mu}_x\right)dt. \tag{4.5}$$

By the standard argument, one can show that $r\to V_x(r)/V_\lambda(r)$ is decreasing. By (4.3) we have

$$\lim_{t\to 0^+}\frac{V_x(r)}{V_\lambda(r)}=\Theta_x.$$

Therefore we have

THEOREM 4.2. *Suppose that* (M,F) *satisfies* $\mathrm{Ric}_M\geqslant(n-1)\lambda$. *Then for every* $x\in M$, *the function* $r\to V_x(r)/V_\lambda(r)$ *is monotone decreasing. In particular,*

$$V_x(r)\leqslant\Theta_x V_\lambda(r).$$

5. PROOF OF THEOREM 1.1

Fix a basis $\{e_i\}_{i=1}^n$ for $T_0(T_xM)\sim T_xM$. Extend $\{e_i\}_{i=1}^n$ to be a global basis for $T(T_xM)$. Let $\{\theta^i\}_{i=1}^n$ be a dual basis for $T^*(T_xM)$.

Let γ_v: $[0,\infty)\to M$ be a normal geodesic with $\dot{\gamma}_v(0)=v\in T_xM$. Let $J_i(t)=d(\exp_x)_{|tv}(te_i)$ and $\tilde{J}_i(t)=d(\exp_x)_{|tv}e_i$. Put $\tilde{B}^t=\{(y^i):F(y^i\tilde{J}_i(t))\leqslant 1\}$.

Let $\{\tilde{\eta}^i\}_{i=1}^n$ be the co-frame dual to $\{\tilde{J}_i(t)\}_{i=1}^n$. The volume form dv has the form

$$dv_{|\exp_v(tv)}=\frac{\mathrm{vol}(\mathbb{B}^n(1))}{\mathrm{vol}(\tilde{B}^t(1))}\tilde{\eta}^1\wedge\cdots\wedge\tilde{\eta}^n.$$

We have

$$\theta^i_{|tv} = (\exp_x)^*_{|tv}\,\tilde{\eta}^i.$$

Thus

$$(\exp_x)^*_{|tv}\,dv = \frac{\mathrm{vol}(\mathbb{B}^n(1))}{\mathrm{vol}(\tilde{B}^t(1))}\,\theta^1 \wedge \cdots \wedge \theta^n.$$

Since the mean distortion is independent of a particular choice of $\{e_i\}_{i=1}^n$, we have

$$\mu(\dot{\gamma}_v(t)) = \frac{\mathrm{vol}(\tilde{B}^t(1))\,\sqrt{\det(g^t(\tilde{J}_i(t), \tilde{J}_j(t))}}{\mathrm{vol}(\mathbb{B}^n(1))}.$$

Thus

$$(\exp_x)^*_{|tv}\,dv = \frac{1}{\mu(\dot{\gamma}(t))}\sqrt{\det(g^t(J_i(t), J_j(t)))}\,t^{-n}\theta^1 \wedge \cdots \wedge \theta^n. \tag{5.1}$$

Choose $\{e_i\}_{i=1}^n$ with $e_n = v$ such that e_α, $1 \leqslant \alpha \leqslant n-1$, are orthogonal to v with respect to g^v. By the Gauss lemma, all $J_\alpha(t)$, $1 \leqslant \alpha \leqslant n-1$, are orthogonal to $J_n(t) = t\dot{\gamma}_v(t)$. Therefore

$$\begin{aligned}(\exp_x)^*_{|tv}\,dv &= \frac{1}{\mu(\dot{\gamma}(t))}\sqrt{\det(g^t(J_\alpha(t), J_\beta(t)))}\,t^{-(n-1)}\theta^1 \wedge \cdots \wedge \theta^n \\ &= \frac{\Theta(v,t)}{\mu(\dot{\gamma}_v(t))}\,t^{-(n-1)}\sqrt{\det(g^v(e_\alpha, e_\beta))}\,\theta^1 \wedge \cdots \wedge \theta^n \\ &= \frac{\Theta(v,t)}{\mu(\dot{\gamma}_v(t))}\,t^{-n(n-1)}\hat{\eta}^x \\ &= \frac{\Theta(v,t)}{\mu(\dot{\gamma}_v(t))}\,\dot{\eta}^x \wedge dt.\end{aligned}$$

Here $\dot{\eta}^x$ is the volume form of $(I_x M, \dot{g}^x)$.

To prove Theorem 1.1, one needs the following

LEMMA 5.1. *Suppose that $|H_M| \leqslant \mu$. Then the function $t \to e^{-\mu t}/\mu(\dot{\gamma}_v(t))$ is monotone decreasing.*

Proof. By the definition of H, we have

$$\mathrm{H}(\dot\gamma(t)) = \frac{d}{dt}[\ln \mu(\dot\gamma_v(t))] \geqslant -\mu = \frac{d}{dt}[\ln e^{-\mu t}]. \tag{5.2}$$

This implies that $t \to (e^{-\mu t}/\mu(\dot\gamma_v(t)))$ is monotone decreasing. ∎

Let

$$\sigma_x(r) = \int_{I_x} \frac{\Theta(v, r)}{\mu(\dot\gamma_v(r))}\, \dot\eta^x, \qquad \sigma_{\lambda,\mu}(r) = \mathrm{vol}(\mathbb{S}^{n-1}(1))\, e^{\mu r} s_\lambda(r)^{n-1}. \tag{5.3}$$

We have

$$\mathrm{vol}(B(x, r)) = \int_0^r \sigma_x(t)\, dt, \qquad V_{\lambda,\mu}(r) = \int_0^r \sigma_{\lambda,\mu}(t)\, dt.$$

By Lemma 4.1 and Lemma 5.1, we conclude that $h(r) := \sigma_x(r)/\sigma_{\lambda,\mu}(r)$ is monotone decreasing. By the standard argument [G], the function

$$\bar h(r) := \frac{\int_0^r \sigma_x(t)\, dt}{\int_0^r \sigma_{\lambda,\mu}(t)\, dt}$$

is also monotone decreasing.

Next we are going to prove that $\lim_{r\to 0^+} \bar h(r) = 1$.

LEMMA 5.2. *For every $x \in M$,*

$$\lim_{\varepsilon\to 0^+} \frac{\mathrm{vol}(B(x, \varepsilon))}{V_{\lambda,\mu}(\varepsilon)} = 1. \tag{5.4}$$

Proof. The proof is standard. Take the normal coordinates (x^i) at x. Clearly, $x' \to \mathrm{vol}(B_{x'}(r))$ is continuous. For small $\varepsilon > 0$, the ε-ball $B(x, \varepsilon) \subset M$ is mapped onto $B_x(\varepsilon) \subset \mathbb{R}^n$ by $\exp_x^{-1}$. Thus as $\varepsilon \to 0^+$,

$$\frac{\mathrm{vol}(B(x, \varepsilon))}{\mathrm{vol}(\mathbb{B}^n(\varepsilon))} = \frac{1}{\mathrm{vol}(\mathbb{B}^n(\varepsilon))} \int_{B_x(\varepsilon)} \frac{\mathrm{vol}(\mathbb{B}^n(1))}{\mathrm{vol}(B_{x'}(1))}\, dx' = \frac{\mathrm{vol}(B_x(\varepsilon))}{\mathrm{vol}(B_{x'_\varepsilon}(\varepsilon))} \to 1. \quad ∎$$

Lemma 5.2 implies that

$$\int_{I_x} \frac{1}{\mu(v)}\, \dot\eta_x = 1.$$

Suppose the mean distortion satisfies

$$\mu^{-1} \leqslant \mu(v) \leqslant \mu \qquad \forall v \in TM. \tag{5.5}$$

By (5.3), we have

$$\mu^{-1}V_x(r) \leqslant \mathrm{vol}(B(x, r)) \leqslant \mu V_x(r).$$

We obtain the following

THEOREM 5.3. *Let (M, F) be a complete Finsler n-manifold satisfying* $\mathrm{Ric}_M \geqslant (n-1)\lambda$ *and* (5.5). *Then for all $r < R$,*

$$\frac{\mathrm{vol}(B(x, R))}{V_\lambda(R)} \leqslant \mu^2 \frac{\mathrm{vol}(B(x, r))}{V_\lambda(r)}.$$

In particular,

$$\mathrm{vol}(B(x, R)) \leqslant \mu^2 V_\lambda(R).$$

This theorem is sharp for Riemann manifolds, since $\mu(v) = 1$, but it is not sharp for Finsler manifolds (say Minkowski spaces). Nevertheless, the precompactness and finiteness theorems can be established for $\mu(v)$ instead of $\mathrm{H}(v)$.

6. THE RIGIDITY PROBLEM

In this section we shall study the case when

$$\frac{\mathrm{vol}(B(x, r))}{V_{\lambda, \mu}(r)} = 1. \tag{6.1}$$

By the proof of Theorem 1.1, one can see that $\mathrm{inj}_x \geqslant r$. Further,

$$\Theta(v, t) = s_\lambda(t)^{n-1}, \qquad 0 \leqslant t \leqslant r. \tag{6.2}$$

and

$$\mathrm{H}(\dot{\gamma}_v(t)) = \mu, \qquad 0 \leqslant t \leqslant r. \tag{6.3}$$

We have $\mathrm{H}(v) = \mathrm{H}(-v)$, $\forall v \in I_x$, since (6.3) holds for all $v \in I_x$ and $t = 0$. On the other hand, by the definition of $\mathrm{H}(v)$, it is easy to see that $\mathrm{H}(-v) = -\mathrm{H}(v)$, $\forall v \in T_x M$. Thus $\mu = 0$ and

$$\mathrm{H}(\dot{\gamma}_v(t)) = 0, \qquad 0 \leqslant t \leqslant r. \tag{6.4}$$

Lemma 4.1 and (6.2) imply that any Jacobi field $J_u(t)$ along γ_v has the form

$$J_u(t) = s_\lambda(t) u(t), \qquad 0 \leqslant t \leqslant r, \tag{6.5}$$

where $u = u(t)$ is a parallel vector field along γ_v. By the Jacobi field equation, one has

$$R^t u(t) = u(t). \tag{6.6}$$

This means the flag curvature for the flag $\{\sigma_t, \dot{\gamma}_v(t)\}$ is constant 1, where $\sigma_t = \text{span}\{\dot{\gamma}_v(t), u\}$. We remark that this does not say the flag curvature at $\gamma_v(t)$ is constant. We have

PROPOSITION 6.1. *If J_u satisfies* (6.5) *for all $u \in T_x M$, then*

$$\tilde{g}^x_{|tv} = (\exp_x)^* g^x_{|tv} = dt^2 \oplus s_\lambda(t)^2 \dot{g}^x.$$

Proof. By (3.9),

$$\tilde{g}^x_{|tv} = (\exp_x)^* g^x_{|tv} = dt^2 \oplus h_t$$

where h_t is a family of Riemann metrics on I_x. By Proposition 3.2 and (6.6), for any basis $\{e_\alpha\}_{\alpha=1}^{n-1}$ for $T_v(I_x)$,

$$(h_t)_{uu} = -\frac{1}{2}\frac{\partial^2}{\partial t^2}(h_t)_{uu} + \frac{1}{4}\frac{\partial}{\partial t}(h_t)_{u\alpha}(h_t)^{\alpha\beta}\frac{\partial}{\partial t}(h_t)_{u\beta}.$$

Lemma 3.1 says that h_t satisfies the initial condition (3.10). By the rigidity theorem in [LSY], one can conclude that $h_t = s_\lambda(t)^2 \dot{g}^x$. ▮

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